Applied Mathematics E-Notes, 25(2025), 297-305 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Exponential Stabilization Of A Coupled Axially Moving Beam With Nonlinear Tension*

Billal Lekdim[†‡], Ammar Khemmoudj[§]

Received 1 September 2024

Abstract

This paper investigates an axially moving beam that includes the coupling of longitudinal and transversal vibrations, as well as nonlinear tension. By employing an appropriate boundary control technique, we prove the exponential stability result using the Lyapunov method.

1 Introduction

Consider a nonlinear coupling of the longitudinal and transverse beam displacements under axial transport of mass, with a mass fixed at its end:

$$\begin{cases} \rho\left(v_{tt}+2\gamma v_{tx}+\gamma^2 v_{xx}\right)+c_v\left(v_t+\gamma v_x\right)+EIv_{xxxx}=\left\{Pv_x+EAv_x\left(u_x+\frac{1}{2}v_x^2\right)\right\}_x, \\ \rho\left(u_{tt}+2\gamma u_{tx}+\gamma^2 u_{xx}\right)+c_u\left(u_t+\gamma u_x\right)-EA\left(u_x+\frac{1}{2}v_x^2\right)_x=0, \quad \text{in } (0,L)\times\mathbb{R}_+, \\ v(x,0)=v^0(x),\ v_t(x,0)=v^1(x),\ u(x,0)=u^0(x),\ u_t(x,0)=u^1(x), \qquad x\in(0,L), \end{cases} \tag{1}$$

subject to

$$\begin{cases} mv_{tt}(L,t)=U_v(t)+EIv_{xxx}(L,t)-\left\{Pv_x+EAv_x\left(u_x+\frac{1}{2}v_x^2\right)\right\}(L,t), \\ mu_{tt}(L,t)=U_u(t)-EA\left(u_x+\frac{1}{2}v_x^2\right)(L,t), \\ v(0,t)=v_x(0,t)=v_{xx}(L,t)=0, \qquad u(0,t)=0, \qquad \forall t\in\mathbb{R}_+, \end{cases} \tag{2}$$

where $v(x,t)$ and $u(x,t)$ are the transversal and longitudinal displacements of the beam at the position x for time t, the subscripts mean partial derivatives, $\rho,\ L,\ \gamma,\ P,\ EI$ and EA are the volumetric mass, length, transport speed of mass, axial tension, bending stiffness and axial stiffness of the beam, and m is mass fixed at $x=L$. c_u and c_v are the structural damping coefficients, $U_v(t)$ and $U_u(t)$ are the control applied at $x=L$.

Recently, the technique of boundary control has seen widespread use in various fields, including the control of vibrations in flexible structures. In studying these systems, they are modeled using second-order partial differential equations (for strings and cables) and fourth-order partial differential equations (for beams and plates). Where many results of stability/stabilisation have been established in this regard (see [1, 4, 5, 13, 16, 17, 18, 19, 20, 23]) and for an axially moving system, see [2, 7, 8, 12, 14].

In [9], an axially moving Kirchhoff string is controlled by a boundary viscoelastic term. For high gain adaptive output feedback type, we can refer to the work in [10], for a distributed delay in internal feedback to [11]. The authors in [6], studied system (1) without axial motion, i.e., $\gamma=0$. They proved an exponential stability result for the riser system under robust boundary control. The same result was reached in [15], where the riser system was considered by replacing frictional dissipation with viscoelastic dissipation.

Our goal in this work is to select the appropriate boundary control that enables us to achieve exponential stability for the system without imposing any conditions on the transport speed γ.

*Mathematics Subject Classifications: 93D23, 93D05, 74H45.

†Faculty of Exact Sciences and Computer Science, University Ziane Achour of Djelfa, PO Box 3117, Djelfa, Algeria

‡SD Laboratory, Faculty of Mathematics, University of Science and Technology Houari Boumediene, P.O. Box 32, El-Alia 16111, Bab Ezzouar, Algiers, Algeria

§SD Laboratory, Faculty of Mathematics, University of Science and Technology Houari Boumediene, P.O. Box 32, El-Alia 16111, Bab Ezzouar, Algiers, Algeria

The rest of the paper is organized as follows: In the next section, we will present some basic tools that are essential for our work. In the final section, we will present a result on exponential stability and demonstrate a result regarding the uniform boundedness of solutions.

2 Preliminary

In this section, we will provide the fundamental materials necessary to demonstrate our results. Below, we will denote the inner product and norm in $L^2(0,L)$ by $(\cdot,\cdot)$ and $\|\cdot\|_2$, respectively.

To stabilize the beam (1)–(2), we propose the following control:

$$\begin{cases} U_v(t) = -k_1 v_t(L,t) - k_2 v_{xt}(L,t) - k_3 v_x(L,t) - k_4 v(L,t), \\ U_u(t) = -k_5 u_t(L,t) - k_6 u_{xt}(L,t) - k_7 u_x(L,t) - k_8 u(L,t), \end{cases} \tag{3}$$

where k_i, $i = 1, ..., 8$, are positive constants.

Because the beam is moving with a constant speed γ, the total derivative operator with respect to time is defined by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial x}.$$

For more details see [22]. The energy of system (1)–(2) is defined by

$$E(t) = \frac{\rho}{2}\left[\|v_t + \gamma v_x\|_2^2 + \|u_t + \gamma u_x\|_2^2\right] + E_L(t) + \frac{EA}{2}\left\|u_x + \frac{1}{2}v_x^2\right\|_2^2 + \frac{EI}{2}\|v_{xx}\|_2^2 + \frac{P}{2}\|v_x\|_2^2, \tag{4}$$

where $E_L(t) = \frac{m}{2}\left[u_t^2 + v_t^2\right](L,t)$.

Lemma 1 *The total derivative of energy (4) is given by*

$$\begin{aligned} \frac{d}{dt}E(t) \quad = \quad & -c_v \|v_t + \gamma v_x\|_2^2 - \gamma m v_{tt} v_x(L,t) - \gamma EI v_{xx}^2(0,t) + U_v(t)(v_t + \gamma v_x)(L,t) \\ & -c_u \|u_t + \gamma u_x\|_2^2 - \gamma m u_{tt} u_x(L,t) - \gamma EA u_x^2(0,t) + U_u(t)(u_t + \gamma u_x)(L,t). \end{aligned} \tag{5}$$

Proof. The $L^2(0,L)$ inner product of $(v_t + \gamma v_x)$ and $(v_t + \gamma v_x)$ with first and second equations in (1), respectively, and integrating by parts, leads to

$$\rho\left(\left(v_{tt} + 2\gamma v_{tx} + \gamma^2 v_{xx}\right) + c_v\left(v_t + \gamma v_x\right), \left(v_t + \gamma v_x\right)\right) = \frac{\rho}{2}\frac{d}{dt}\|v_t + \gamma v_x\|_2^2 + c_v\|v_t + \gamma v_x\|_2^2, \tag{6}$$

$$(EI v_{xxxx}, v_t + \gamma v_x) = EI v_{xxx}(v_t + \gamma v_x)(L,t) + \gamma EI v_{xx}^2(0,t) + \frac{EI}{2}\frac{d}{dt}\|v_{xx}\|_2^2,$$

$$-P(v_{xx}, v_t + \gamma v_x) = -P v_x (v_t + \gamma v_x)(L,t) + \frac{P}{2}\frac{d}{dt}\|v_x\|_2^2,$$

$$\begin{aligned} -EA(\left\{v_x\left(u_x + \frac{1}{2}v_x^2\right)\right\}_x, (v_t + \gamma v_x)) \quad = \quad & EA\int_0^L \left(u_x + \frac{1}{2}v_x^2\right) v_x (v_t + \gamma v_x)_x \, dx \\ & -EA v_x \left(u_x + \frac{1}{2}v_x^2\right)(v_t + \gamma v_x)(L,t), \end{aligned}$$

$$\rho\left(\left(u_{tt} + 2\gamma u_{tx} + \gamma^2 u_{xx}\right) + c_u\left(u_t + \gamma u_x\right), \left(u_t + \gamma u_x\right)\right) = \frac{\rho}{2}\frac{d}{dt}\|u_t + \gamma u_x\|_2^2 + c_u\|u_t + \gamma u_x\|_2^2,$$

$$\begin{aligned} -EA(\left(u_x + \frac{1}{2}v_x^2\right)_x, (u_t + \gamma u_x)) \quad = \quad & \gamma EA u_x(0,t) - EA\left(u_x + \frac{1}{2}v_x^2\right)(u_t + \gamma u_x)(L,t) \\ & + EA\int_0^L \left(u_x + \frac{1}{2}v_x^2\right)(u_t + \gamma u_x)_x \, dx. \end{aligned}$$

Taking into account the fact that:

$$EA\int_0^L \left[\left(u_x+\frac{1}{2}v_x^2\right)v_x\left(v_t+\gamma v_x\right)_x+\left(u_x+\frac{1}{2}v_x^2\right)\left(u_t+\gamma u_x\right)_x\right]dx=\frac{EA}{2}\frac{d}{dt}\left\|u_x+\frac{1}{2}v_x^2\right\|_2^2. \tag{7}$$

Combining the results (6)–(7), we get (5). ■

Lemma 2 *Let u be a function defined on $[0,L]\times\mathbb{R}_+$ satisfies $u(0,t)=u_x(0,t)=0$. Then*

$$u^2(x,t)\le L\left\|u_x(t)\right\|_2^2, \quad \text{and} \quad \left\|u(t)\right\|_2^2\le L^2\left\|u_x(t)\right\|_2^2\le L^4\left\|u_{xx}(t)\right\|_2^2, \quad \forall t\ge 0.$$

Lemma 3 *We have*

$$ab\le\delta a^2+\frac{1}{4\delta}b^2, \quad \forall \quad a,b\in\mathbb{R}, \quad \delta>0.$$

Lemma 4 ([3]) *Let $u\in C^1([0,L])$ satisfying $u(0,t)=0$. Then the following inequality hold:*

$$\left\|u^2(t)\right\|_\infty\le 2\left\|u(t)\right\|_2\left\|u_x(t)\right\|_2, \qquad \forall t\ge 0,$$

where $\|.\|_\infty$ is the norm of $L^\infty([0,L])$.

3 Stability

Now, we define the Lyapunov function by

$$\mathcal{L}(t)=\beta E(t)+\sum\nolimits_{i=1}^5 V_i(t),$$

where β is a positive constant, $E(t)$ is the energy given by (4) and

$$V_1(t)=V_{11}(t)+V_{12}(t)+V_{13}(t),$$

$$V_{11}(t)=\rho\int_0^L \frac{1}{2}v\left(v_t+\gamma v_x\right)+u\left(u_t+\gamma u_x\right)dx,$$

$$V_{12}(t)=\int_0^L \frac{c_v}{4}v^2+\frac{c_u}{2}u^2dx,$$

$$V_{13}(t)=m\left[\frac{1}{2}v_tv+u_tu\right](L,t),$$

$$V_2(t)=m\beta\gamma\left[v_tv_x+u_tu_x\right](L,t), \tag{8}$$

$$V_3(t)=\frac{\beta\gamma}{2}\left[k_2v_x^2+k_6u_x^2\right](L,t),$$

$$V_4(t)=\left[\frac{k_1/2+\beta k_4}{2}v^2+\frac{k_5+\beta k_8}{2}u^2\right](L,t),$$

$$V_5(t)=\frac{k_2}{2}vv_x+k_6uu_x. \tag{9}$$

Now we aim to prove the exponential decay of the Lyapunov function. Using the following proportions, we will analyze the energy decay.

Proposition 1 *If $k_2=\gamma m$, $k_6=\gamma m$, $\frac{k_1/2+\beta k_4}{m}\ge\frac{1}{2\beta}$ and $\frac{k_5+\beta k_8}{m}\ge\frac{1}{\beta}$, then*

$$\beta E_L(t)+V_{13}(t)+\sum_{i=2}^5 V_i(t)\ge 0, \quad t\ge 0. \tag{10}$$

Proof. We take the sum of $E_L(t)$ and the five functions $V_{13}(t)$ and $V_i(t), i = 2, ..., 5,$, we have

$$\begin{aligned}
&\beta E_L(t) + V_{13}(t) + \sum_{i=2}^{5} V_i(t) \\
= &\frac{\beta m}{2}\left[v_t^2 + 2\gamma v_t v_x + \gamma^2 v_x^2 + \frac{1}{\beta} v_t v + \frac{k_1/2 + \beta k_4}{\beta m} v^2 + \frac{\gamma}{\beta} v v_x\right] \\
&+ \frac{\beta m}{2}\left[u_t^2 + 2\gamma u_t u_x + \gamma^2 u_x^2 + \frac{2}{\beta} u_t u + \frac{k_5 + \beta k_8}{\beta m} u^2 + \frac{2}{\beta}\gamma u u_x\right] \\
= &\frac{\beta m}{2}\left[(v_t + \gamma v_x)^2 + \frac{k_1/2 + \beta k_4}{\beta m} v^2 + \frac{1}{\beta} v (v_t + \gamma v_x) + (u_t + \gamma u_x)^2 + \frac{k_5 + \beta k_8}{\beta m} u^2 + \frac{2}{\beta} u (u_t + \gamma u_x)\right] \\
= &\frac{\beta m}{2}\left[\left(v_t + \gamma v_x - \frac{1}{2\beta} v^2\right)^2 + \left[\frac{k_1/2 + \beta k_4}{\beta m} - \frac{1}{4\beta^2}\right] v^2 + \left(u_t + \gamma u_x + \frac{1}{\beta} u\right)^2 + \left[\frac{k_5 + \beta k_8}{\beta m} - \frac{1}{\beta^2}\right] u^2\right].
\end{aligned}$$

Since $\frac{k_1/2+\beta k_4}{m} \geq \frac{1}{2\beta}$ and $\frac{k_5+\beta k_8}{m} \geq \frac{1}{\beta}$, we have (10). ■

Proposition 2 *There exist $\alpha_i > 0$, $i = 1, 2$, such that*

$$\alpha_1 \left(\beta E(t) + V_{13}(t) + \sum_{i=2}^{5} V_i(t)\right) \leq \mathcal{L}(t) \leq \alpha_2 \left(\beta E(t) + V_{13}(t) + \sum_{i=2}^{5} V_i(t)\right), \quad \forall t \geq 0. \tag{11}$$

Remark 1 ([21]) *From the practical points of view, the slope of the beam v_x in the vibration never goes to infinity. Hence, we assume that there exists $c \in \mathbb{R}_+$ such that $\forall t \geq 0$ and $x \in [0, L]$, $|v_x| \leq c$.*

Proof. By using Young's inequality we obtain

$$V_{11}(t) + V_{12}(t) \leq \frac{\rho}{2}\left[\|v_t + \gamma v_x\|_2^2 + \|u_t + \gamma u_x\|_2^2\right] + L \int_0^L \frac{2\rho + c_v}{4} v_x^2 + \frac{\rho + c_u}{2} u_x^2 dx.$$

On the other hand

$$\int_0^L u_x^2 dx \leq \left\|u_x + \frac{1}{2} v_x^2\right\|_2^2 + \frac{1}{4}\left\|v_x^2\right\|_2^2 \leq \left\|u_x + \frac{1}{2} v_x^2\right\|_2^2 + \frac{1}{4}\|v_x\|_\infty^2 \|v_x\|_2^2. \tag{12}$$

Combining the inequalities (10), (12) and Remark 1, we get

$$|V_{11}(t) + V_{12}(t)| \leq \max\left\{1, \frac{2(\rho + c_u) L}{EA}, \frac{(2\rho + c_v) L + (\rho + c_u) c}{2P}\right\} E(t) = \lambda E(t).$$

Choosing β properly, for all $t \geq 0$, we deduce (11), with $\alpha_1 = \beta - \lambda > 0$ and $\alpha_2 = \beta + \lambda$. ■

Lemma 5 *The total derivative of $V_1(t)$ yields*

$$\begin{aligned}
\frac{dV_1}{dt}(t) \quad = \quad &\frac{\rho}{2}\|v_t + \gamma v_x\|_2^2 + \rho\|u_t + \gamma u_x\|_2^2 - \frac{EI}{2}\|v_{xx}\|_2^2 - EA\left\|u_x + \frac{1}{2} v_x^2\right\|_2^2 \\
&- \frac{P}{2}\|v_x\|_2^2 + \frac{m}{2} v_t^2(L, t) + \frac{1}{2} U_v(t) v(L, t) + m u_t^2(L, t) + U_u u(L, t).
\end{aligned} \tag{13}$$

Proof. We write the functional $V_1(t)$ as

$$V_1(t) = V_v(t) + V_u(t),$$

where

$$V_v(t) = \int_0^L \frac{\rho}{2} v\left(v_t + \gamma v_x\right) + \frac{c_v}{4} v^2 dx + \frac{m}{2} v_t v(L,t) \quad \text{and} \quad V_u(t) = \int_0^L \rho u\left(u_t + \gamma u_x\right) + \frac{c_u}{2} u^2 dx + m u_t u(L,t).$$

A total derivative of $V_v(t)$ yield

$$\frac{dV_v}{dt}(t) = \frac{\rho}{2}\left\|v_t + \gamma v_x\right\|_2^2 + \frac{\rho}{2}\int v\left(v_{tt} + 2\gamma v_{tx} + \gamma^2 v_{xx}\right)dx + \frac{c_v}{2}\int v\left(v_t + \gamma v_x\right)dx + \frac{m}{2}\left[v_{tt}v + v_t^2\right](L,t). \quad (14)$$

Substituting the first equation of (1) in (14), and integrating by parts, we have

$$\left(\frac{P}{2}v_{xx} - \frac{EI}{2}v_{xxxx}, v\right) = -\frac{EI}{2}v_{xxx}v(L,t) - \frac{EI}{2}\left\|v_{xx}\right\|_2^2 + \frac{P}{2}v_x v(L,t) - \frac{P}{2}\left\|v_x\right\|_2^2, \quad (15)$$

$$\frac{EA}{2}\left(\left\{v_x\left(u_x + \frac{1}{2}v_x^2\right)\right\}_x, v\right) = \frac{EA}{2}\left(u_x + \frac{1}{2}v_x^2\right)v_x v(L,t) - \frac{EA}{2}\int_0^L\left(u_x + \frac{1}{2}v_x^2\right)v_x^2 dx. \quad (16)$$

Substituting (15)–(16) into (14), we have

$$\frac{dV_v}{dt}(t) = \frac{\rho}{2}\left\|v_t + \gamma v_x\right\|_2^2 - \frac{EI}{2}\left\|v_{xx}\right\|_2^2 - \frac{P}{2}\left\|v_x\right\|_2^2 + \frac{m}{2}v_t^2(L,t) + \frac{1}{2}U_v(t)v(L,t) - \frac{EA}{2}\int_0^L\left(u_x + \frac{1}{2}v_x^2\right)v_x^2 dx.$$

In the same way, a total derivative of $V_u(t)$ yields

$$\frac{dV_u}{dt}(t) = \rho\left\|u_t + \gamma u_x\right\|_2^2 + m u_t^2(L,t) + U_u u(L,t) - EA\int_0^L\left(u_x + \frac{1}{2}v_x^2\right)u_x dx.$$

By adding the last two relations, we obtain (13). ■

Lemma 6 *The functions $V_i(t), i = 2, ..., 4,$ given by (8)–(9), respectively, satisfy*

$$\frac{d}{dt}V_2(t) = m\beta\gamma\left[v_{tt}v_x + v_t v_{tx} + u_{tt}u_x + u_{tt}u_{tx}\right](L,t), \quad (17)$$

$$\frac{d}{dt}V_3(t) = \beta\gamma\left[k_2 v_x v_{tx} + k_6 u_x u_{tx}\right](L,t),$$

$$\frac{d}{dt}V_4(t) = \left[\left(k_1/2 + \beta k_4\right)v v_t + \left(k_5 + \beta k_8\right)u u_t\right](L,t),$$

$$\frac{d}{dt}V_5(t) = \left[\frac{k_2}{2}v_t v_x + \frac{k_2}{2}v v_{tx} + k_6 u_t u_x + k_6 u u_{tx}\right](L,t). \quad (18)$$

Proof. A differentiation of (8)–(9) leads to (17)–(18), respectively. ■

Lemma 7 *Time derivative of $\mathcal{L}(t)$ satisfies*

$$\frac{d}{dt}\mathcal{L}(t) \leq -\alpha\mathcal{L}(t), \quad (19)$$

where α is positive constant.

Proof. We have

$$\frac{d}{dt}\mathcal{L}(t) = \beta\frac{d}{dt}E(t) + \sum_{i=1}^{4}\frac{d}{dt}V_1(t).$$

Taking estimates of $\frac{d}{dt}E(t)$ and $\frac{d}{dt}V_i(t), i = 1, ..., 4$, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \quad = \quad & -\left[\beta c_v - \frac{\rho}{2}\right]\|v_t + \gamma v_x\|_2^2 - [\beta c_u - \rho]\|u_t + \gamma u_x\|_2^2 - \frac{EI}{2}\|v_{xx}\|_2^2 - \frac{P}{2}\|v_x\|_2^2 - EA\left\|u_x + \frac{1}{2}v_x^2\right\|_2^2 \\ & + U_v(t)\left(\beta v_t + \beta\gamma v_x + \frac{1}{2}v\right)(L,t) + U_u(t)\left(\beta u_t + \beta\gamma u_x + u\right)(L,t) + \beta\gamma m v_t v_{tx}(L,t) + \frac{m}{2}v_t^2(L,t) \\ & + m u_t^2(L,t) + \beta\gamma m u_t u_{tx}(L,t) - \beta\gamma EA u_x^2(0,t) + \left[\frac{k_1/2 + \beta k_4}{2}vv_t + \frac{k_5 + \beta k_8}{2}uu_t\right](L,t) \\ & - \beta\gamma EI v_{xx}^2(0,t) + \left[\frac{k_2}{2}v_t v_x + \frac{k_2}{2}vv_{tx} + k_6 u_t u_x + k_6 u u_{tx}\right](L,t) + \beta\gamma\left[k_2 v_x v_{tx} + k_6 u_x u_{tx}\right](L,t).\end{aligned}$$

Using the control law defined by (3), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \quad = \quad & -\left[\beta c_v - \frac{\rho}{2}\right]\|v_t + \gamma v_x\|_2^2 - [\beta c_u - \rho]\|u_t + \gamma u_x\|_2^2 - \frac{EI}{2}\|v_{xx}\|_2^2 - \frac{P}{2}\|v_x\|_2^2 - EA\left\|u_x + \frac{1}{2}v_x^2\right\|_2^2 \\ & - \left[\beta k_1 - \frac{m}{2}\right]v_t^2(L,t) - \left(\beta k_3 + \gamma\beta k_1 - \frac{k_2}{2}\right)v_t v_x(L,t) - \left(\gamma\beta k_4 + \frac{k_3}{2}\right)vv_x(L,t) - \gamma\beta k_3 v_x^2(L,t) \\ & + [\beta\gamma m - \beta k_2]v_t v_{tx}(L,t) - \frac{k_4}{2}v^2(L,t) - [\beta k_5 - m]u_t^2(L,t) - \gamma\beta k_7 u_x^2(L,t) - (\gamma\beta k_8 + k_7)uu_x(L,t) \\ & - k_8 u^2(L,t) - (\beta k_7 + \gamma\beta k_5 - k_6)u_t u_x(L,t) + [\beta\gamma m - \beta k_6]u_t u_{tx}(L,t) - \beta\gamma(EIv_{xx}^2 + EAu_x^2)(0,t).\end{aligned}$$

By applying Young's inequality to the non-constant sign term and fixed $k_2 = k_6 = \gamma m$, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) \quad \leq \quad & -\left[\beta c_v - \frac{\rho}{2}\right]\|v_t + \gamma v_x\|_2^2 - [\beta c_u - \rho]\|u_t + \gamma u_x\|_2^2 - \frac{EI}{2}\|v_{xx}\|_2^2 - \frac{P}{2}\|v_x\|_2^2 - EA\left\|u_x + \frac{1}{2}v_x^2\right\|_2^2 \\ & - \left[\beta k_1 - \frac{m}{2} - \frac{|\beta k_3 + \gamma\beta k_1 - k_2/2|}{\delta_1}\right]v_t^2(L,t) - \left[\frac{k_4}{2} - \frac{\gamma\beta k_4 + k_3/2}{\delta_2}\right]v^2(L,t) - \beta\gamma EA u_x^2(0,t) \\ & - \left[\beta k_5 - m - \frac{|\beta k_7 + \gamma\beta k_5 - k_6|}{\delta_3}\right]u_t^2(L,t) - \left[k_8 - \frac{\gamma\beta k_8 + k_7}{\delta_4}\right]u^2(L,t) - \beta\gamma EI v_{xx}^2(0,t) \\ & - \left[\gamma\beta k_3 - \delta_1|\beta k_3 + \gamma\beta k_1 - k_2/2| - \delta_2(\gamma\beta k_4 - k_3/2)\right]v_x^2(L,t) \\ & - \left[\gamma\beta k_7 - \delta_3|\beta k_7 + \gamma\beta k_5 - k_6| - \delta_4(\gamma\beta k_8 + k_7)\right]u_x^2(L,t).\end{aligned}$$

where δ_1, δ_2, δ_3 and δ_4 are positive constants.

Now we choose β, k_i, $i = 1, ..., 8$, and δ_i, $i = 1, ..., 4$, so that all the coefficients in the previous inequality are strictly positive. Then, we have

$$\frac{d}{dt}\mathcal{L}(t) \leq -\alpha_3\left(\beta E(t) + V_{13}(t) + \sum_{i=2}^{4}V_i(t)\right), \tag{20}$$

where

$$\begin{aligned}\alpha_3 \quad = \quad & 2\min\left\{\frac{\beta c_v - \rho/2}{\beta\rho}, \frac{\beta c_u - \rho}{\beta\rho}, \frac{1}{2\beta}, \frac{\beta k_1 - m/2 - |\beta k_3 + \gamma\beta k_1 - k_2/2|/\delta_1}{\beta m},\right. \\ & \frac{\beta k_5 - m - |\beta k_7 + \gamma\beta k_5 - k_6|/\delta_3}{\beta m}, \frac{\gamma\beta k_3 - \delta_1|\beta k_3 + \gamma\beta k_1 - k_2/2| - \delta_2\gamma\beta k_4 - \delta_2 k_3/2}{\beta\gamma k_2}, \\ & \left.\frac{\gamma\beta k_7 - \delta_3|\beta k_7 + \gamma\beta k_5 - k_6| - \delta_4(\gamma\beta k_8 + k_7)}{\beta\gamma k_6}, \frac{\delta_2 k_4 - 2\gamma\beta k_4 + k_3}{2\delta_2(k_1/2 + \beta k_4)}, \frac{\delta_4 k_8 - \gamma\beta k_8 + k_7}{\delta_4(k_5 + \beta k_8)}\right\}.\end{aligned}$$

Now, combining the inequality (11) and (20), we have(19) with $\alpha = \frac{\alpha_3}{\alpha_2}$. ■

3.1 Uniform Boundedness

Theorem 1 *For the system (1)–(2), under the control (3), given that the initial conditions are bounded, we can conclude that uniform boundedness (UB), the state of the closed-loop system $v(x,t)$ and $u(x,t)$ will remain in the compact set*

$$\Omega = \left\{ (u(x,t), v(x,t)) \in \mathbb{R}^2 / \left| u(x,t) \right|, \left| v(x,t) \right| \leq D, \forall (x,t) \in [0,L] \times [0,+\infty) \right\}. \tag{21}$$

where constant

$$D = \max \left\{ \sqrt{\frac{2L}{P\alpha_1}\mathcal{L}(0)}, \sqrt{\frac{2L}{EA\alpha_1}\mathcal{L}(0)} \right\}.$$

Proof. Multiplying equation (19) by $e^{\alpha t}$, we obtain

$$\frac{d}{dt}\left(\mathcal{L}(t)e^{\alpha t}\right) \leq 0.$$

Integrating over $(0,t)$, we obtain

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\alpha t} \in L^\infty([0,+\infty)), \tag{22}$$

which implies that $\mathcal{L}$ is bounded. By utilizing Lemma 2 and Proposition 2, we have

$$\frac{P}{2L}v^2(x,t) \leq \frac{P}{2}\left\| v_x \right\|_2^2 \leq \frac{1}{\alpha_1}\mathcal{L}(t) \quad \text{and} \quad \frac{EA}{2L}u^2(x,t) \leq \frac{EA}{2}\left\| u_x \right\|_2^2 \leq \frac{1}{\alpha_1}\mathcal{L}(t). \tag{23}$$

Combining the inequalities (22) and (23), we obtain that $v(x,t)$ and $u(x,t)$ is uniformly bounded as follows:

$$|v(x,t)| \leq \sqrt{\frac{2L}{EA\alpha_1}\mathcal{L}(0)} \quad \text{and} \quad |u(x,t)| \leq \sqrt{\frac{2L}{P\alpha_1}\mathcal{L}(0)}.$$

■

3.2 Exponential Stability

Theorem 2 *The energy $E(t)$ satisfies*

$$E(t) \leq Ae^{-\alpha t}, \quad \forall t \geq 0,$$

where A and α are two positive constants.

Proof. Using the Proposition 2 into the inegality (22), we have

$$E(t) \leq \frac{\beta \mathcal{L}(0)}{\alpha_1}e^{-\alpha t},$$

with $A = \frac{\beta \mathcal{L}(0)}{\alpha_1}$. ■

Acknowledgment. The authors would like to thank the anonymous referees for their valuable comments and suggestions. and express their gratitude to DGRSDT for the financial support.

References

[1] B. Basti and N. Benhamidouche. Global existence and blow-up of generalized self-similar solutions to nonlinear degenerate diffusion equation not in divergence form, Appl. Math. E-Notes, 20(2020), 367–387.

[2] J. R. Chang, W. J. Lin, C. J. Huang and S. T. Choi, Vibration and stability of an axially moving Rayleigh beam, Appl. Math. Model, 34(2010), 1482–1497.

[3] K. D. Do and J. Pan, Boundary control of transverse motion of marine risers with actuator dynamics, J. Sound Vib, 318(2008), 768–791.

[4] B. Feng, General decay for a viscoelastic wave equation with strong time-dependent delay, Bound. Value Probl., 2017(2017), 1–11.

[5] B. Feng, Asymptotic behavior of a semilinear non-autonomous wave equation with distributed delay and analytic nonlinearity, Nonlinearity, 37(2024).

[6] S. S. Ge, W. He, B. V. E. How and Y. S. Choo, Boundary control of a coupled nonlinear flexible marine riser, IEEE Trans. Control Syst. Technol., 18(2009), 1080–1091.

[7] M. H. Ghayesh, Stability and bifurcations of an axially moving beam with an intermediate spring support, Nonlinear Dyn., 69(2012), 193–210.

[8] A. Kelleche and N. E. Tatar, Existence and stabilization of a Kirchhoff moving string with a distributed delay in the boundary feedback, Math. Model. Nat. Phenom., 12(2017), 106–117.

[9] A. Kelleche, N. E. Tatar and A. Khemmoudj, Uniform stabilization of an axially moving Kirchhoff string by a boundary control of memory type, J. Dyn. Control Syst., 23(2017), 237–247.

[10] A. Kelleche and N. E. Tatar, Adaptive Stabilization of a Kirchhoff Moving String, J. Dyn. Control Syst., 26(2020), 255–263.

[11] A. Kelleche and N. E. Tatar, Adaptive boundary stabilization of a nonlinear axially moving string, ZAMM-Z Angew Math Me, 101(2021), 14 pp.

[12] A. Kelleche and S. Fardin, Stabilization of an axially moving Euler Bernoulli beam by an adaptive boundary control, J. Dyn. Control Syst., 29(2023), 1037–1054.

[13] A. Kelleche, Well-Posedness and a Blow up Result for a Fractionally Damped Coupled System, Bull. Malays. Math. Sci. Soc., 46(2023), 33 pp.

[14] C. W. Kim, H. Park and K. S. Hong, Boundary control of axially moving continua: application to a zinc galvanizing line, Int. J. Control. Autom., 3(2005), 601–611.

[15] B. Lekdim and A. Khemmoudj, General decay of energy to a nonlinear viscoelastic two-dimensional beam, Appl. Math. Mech. (English Ed.), 39(2018), 1661–1678.

[16] B. Lekdim and A. Khemmoudj, Existence and energy decay of solution to a nonlinear viscoelastic two-dimensional beam with a delay, Multidimens. Syst. Signal Process. 32(2021), 1–17.

[17] B. Lekdim and A. Khemmoudj, Existence and general decay of solution for nonlinear viscoelastic two-dimensional beam with a nonlinear delay, Ric. Mat., 73(2021), 1–22.

[18] B. Lekdim and A. Khemmoudj, Existence and Exponential Stabilization of an Axial Vibrations Cable with Time-Varying Length, J. Dyn. Control Syst., 29(2023), 2041–2053.

[19] S. Misra, G. Gorain and S. Kar, Stability of wave equation with a tip mass under unknown boundary external disturbance, Appl. Math. E-Notes, 19(2019), 128–140.

[20] N. Ouagueni and Y. Arioua, Existence and uniqueness of solution for a mixed-type fractional differential equation and Ulam-Hyers stability, Appl. Math. E-Notes, 22(2022), 476–495.

[21] R. F. Fung, J. W. Wu and S. L. Wu, Stabilization of an axially moving string by nonlinear boundary feedback, J. Dyn. Sys., Meas., Control, (1999), 117–121.

[22] B. Tabarrok, C. M. Leech and Y. I. Kim, On the dynamics of an axially moving beam, J. Franklin Inst., 297(1974), 201–220.

[23] V. M. Ungureanu, Uniform exponential stability for linear discrete time systems with stochastic perturbations in Hilbert spaces, Bollettino della Unione Matematica Italiana-B, 3(2004), 757–772.