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On L^γ Inequalities For Polar Derivative Of Polynomials*

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Abstract

In this paper, we have extended an inequality concerning the integral analogue of an inequality for ordinary derivative recently proved by Reingachan et al. [Int. J. App. Math. 53(1)(2022)] into polar derivative setting which, in addition, generalizes as well as sharpens some other earlier known results in this direction.

1 Introduction

Let $p(z)$ be a polynomial of degree n. We define

$$\|p\|_\gamma = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}}, \quad 0 < \gamma < \infty. \tag{1}$$

If we let $\gamma \to \infty$ in the above equality and make use of the well-known fact from analysis [11] that

$$\lim_{\gamma\to\infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta \right\}^{\frac{1}{\gamma}} = \max_{|z|=1} |p(z)|,$$

we can suitably denote

$$\|p\|_\infty = \max_{|z|=1} |p(z)|.$$

Similarly, one can define $\|p\|_0 = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{i\theta})| d\theta \right\}$ and show that $\lim_{\gamma\to 0^+} \|p\|_\gamma = \|p\|_0$. It would be of further interest that by taking limit as $\gamma \to 0^+$ that the stated results holding for $\gamma > 0$, also hold for $\gamma = 0$ as well. For $r > 0$, we denote

$$M(p, r) = \max_{|z|=r} |p(z)|.$$

The famous inequality due to Bernstein [9, 12], states that if $p(z)$ is a polynomial of degree n, then

$$\|p'\|_\infty \le n\|p\|_\infty. \tag{2}$$

Restricting to the class of polynomials having no zero in $|z| < 1$, inequality (2) can be improved by

$$\|p'\|_\infty \le \frac{n}{2}\|p\|_\infty. \tag{3}$$

Inequality (3) was conjectured by Erdös and later verified by Lax [6]. Another generalization of (3), Malik [7] proved that if $p(z)$ does not vanish in $|z| < k$, $k \ge 1$, then

$$\|p'\|_\infty \le \frac{n}{1+k}\|p\|_\infty. \tag{4}$$

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Further, as a generalization of (4), Bidkham and Dewan [13] proved that

$$\|p'(rz)\|_\infty \leq \frac{n(r+k)^{n-1}}{(1+k)^n}\|p\|_\infty, \quad \text{for } 1 \leq r \leq k. \tag{5}$$

As another generalization of (5), Aziz and Zargar [4] proved that if $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then for $0 < r \leq R \leq k$

$$\|p'(Rz)\|_\infty \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}}\|p(rz)\|_\infty. \tag{6}$$

Equality holds in (6) for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ.

As an improvement and generalization of (5), Aziz and Shah [14] proved that if $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$

$$\|p'(Rz)\|_\infty \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}}\left\{\|p(rz)\|_\infty - m\right\}. \tag{7}$$

The result is best possible and equality in (7) holds for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ.

Extensions of (6) and (7) into L^γ norm were done very recently by Chanam et al. [5] by proving the following two results.

Theorem 1 *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$ and $\gamma > 0$*

$$\|p'(Rz)\|_\gamma \leq \frac{n}{R}F_\gamma\left[\int_0^{2\pi}\left\{|p(re^{i\theta})| + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu}M(p,t)dt\right\}^\gamma d\theta\right]^{\frac{1}{\gamma}},$$

where

$$M(p,t) = \max_{|z|=t}|p(z)| \quad \text{and} \quad F_\gamma = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\left(\frac{k}{R}\right)^\mu + e^{i\alpha}\right|^\gamma d\alpha\right\}^{\frac{-1}{\gamma}}.$$

Theorem 2 *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$ and $\gamma > 0$*

$$\|p'(Rz)\|_\gamma \leq \frac{n}{R}F_\gamma\left[\int_0^{2\pi}\left\{|p(re^{i\theta})| + n\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu}\left(M(p,t) - m\right)dt - m\right\}^\gamma d\theta\right]^{\frac{1}{\gamma}}, \tag{8}$$

where F_γ and $M(p,t)$ are as defined in Theorem 1 and $m = \min_{|z|=k}|p(z)|$.

For a polynomial $p(z)$ of degree n, we now define the polar derivative of $p(z)$ with respect to a real or complex number α as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

This polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p'(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha} = p'(z),$$

uniformly with respect to z for $|z| \leq R$, $R > 0$.

A variety of results on the polar derivative of a polynomial can be found in the comprehensive books of Milovanović et al. [9], Marden [15] and Rahman and Schmeisser [16], where some methods for deriving polynomial inequalities are based on using techniques of the geometric function theory and findings.

Among those who extended some of the above inequalities to polar versions, Aziz [1] was the first who extended inequality (4) to polar derivative of a polynomial by proving that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k \geq 1$, then for any complex number α with $|\alpha| \geq 1$

$$\max_{|z|=1}|D_\alpha p(z)| \leq n\left(\frac{|\alpha| + k}{1+k}\right)\max_{|z|=1}|p(z)|. \tag{9}$$

2 Lemmas

We need the following lemmas to prove the theorem. The first lemma is due to Qazi [10].

Lemma 1 *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then on $|z| = 1$*

$$|q'(z)| \geq k^{\mu+1} \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu-1} + 1}{1 + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}} |p'(z)|, \tag{10}$$

where $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$.

The next lemma is due to Govil and Kumar [3].

Lemma 2 *Let p, q be any two positive real numbers such that $p \geq qx$, where $x \geq 1$. If ξ is any real such that $0 \leq \xi < 2\pi$, then for any $y \geq 1$*

$$\frac{p + qy}{x + y} \leq \left|\frac{p + qe^{i\xi}}{x + e^{i\xi}}\right|. \tag{11}$$

The next lemma is due to Govil and Kumar [3].

Lemma 3 *Let z_1 and z_2 be any two complex numbers not depending on β, where β is real. Then for each $\gamma > 0$*

$$\int_0^{2\pi} \left|z_1 + z_2 e^{i\beta}\right|^\gamma d\beta = \int_0^{2\pi} \left||z_1| + |z_2|e^{i\beta}\right|^\gamma d\beta. \tag{12}$$

The following result is due to Aziz and Rather [2].

Lemma 4 *Let $p(z)$ be a polynomial of degree n. Then for every ξ with $0 \leq \xi < 2\pi$ and $\gamma > 0$*

$$\int_0^{2\pi}\int_0^{2\pi} |q'(e^{i\theta}) + e^{i\xi}p'(e^{i\theta})|^\gamma d\xi d\theta \leq 2\pi n^\gamma \int_0^{2\pi} |p(e^{i\theta})|^\gamma d\theta, \tag{13}$$

where $q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$.

The next three lemmas are due to Reingachan et al. [8].

Lemma 5 *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $0 < r \leq R \leq k$*

$$\begin{aligned} |p(Re^{i\theta})| \quad &\leq \quad |p(re^{i\theta})| + n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}\left(k^{\mu+1}t^\mu + k^{2\mu}t\right) + k^{\mu+1}} \\ &\quad \times \{M(p,t) - m\}\, dt, \end{aligned} \tag{14}$$

where $m = \min_{|z|=k}|p(z)|$ and $M(p,t) = \max_{|z|=t}|p(z)|$.

Lemma 6 *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $0 < r \leq R \leq k$*

$$\begin{aligned} &\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + k^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}\left(k^{\mu+1}t^\mu + k^{2\mu}t\right)} \{M(p,t) - m\}\, dt \\ \leq \quad &\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu}\{M(p,t) - m\}\, dt, \end{aligned} \tag{15}$$

where $M(p,t) = \max_{|z|=t}|p(z)|$ *and* $m = \min_{|z|=k}|p(z)|$.

Lemma 7 *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k, k > 0$, then for $0 < R \le k$*

$$\frac{\frac{\mu}{n}\frac{|a_\mu|R}{|a_0|-m}k^{2\mu} + k^{\mu+1}}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^\mu + R^{\mu+1}} \ge 1. \tag{16}$$

Lemma 8 *The function*

$$g(x) = k^{\mu+1}\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{x}k^{\mu-1} + 1}{\frac{\mu}{n}\frac{|a_\mu|}{x}k^{\mu+1} + 1}\right\} \tag{17}$$

where $k \ge 1, \mu > 0$ and $n \in \mathbb{N}$, is a non-decreasing function of $x > 0$.

Proof. The proof follows simply by the first derivative test. ■

3 Main Results

The present paper is mainly motivated by the desire to establish an improved and generalized version in polar derivative of Theorems 1 and 2. Our result also extends an inequality recently proved by Reingachan et al. [8] to its polar derivative version. In fact, we prove

Theorem 3 *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for $\gamma > 0$, $0 < r \le R \le k$ and for any complex number α with $|\alpha| \ge R$, β with $|\beta| < 1$*

$$\begin{aligned}\|D_\alpha p(Rz)| + n\beta m\|_\gamma \quad &\le \quad \frac{n\left(A + \frac{|\alpha|}{R}\right)}{\|A+z\|_\gamma}\Bigg\||p(re^{i\theta})| + n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}\left(k^{\mu+1}t^\mu + k^{2\mu}t\right) + k^{\mu+1}} \\ &\quad \times \{M(p,t) - m\}\,dt - |\beta|m\Bigg\|_\gamma, \end{aligned} \tag{18}$$

where

$$A = \left\{\frac{\frac{\mu}{n}\frac{|a_\mu|R}{|a_0|-m}k^{2\mu} + k^{\mu+1}}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}k^{\mu+1}R^\mu + R^{\mu+1}}\right\}, \quad M(p,t) = \max_{|z|=t}|p(z)| \quad \text{and} \quad m = \min_{|z|=k}|p(z)|.$$

Proof. Since $p(z)$ has no zero in $|z| < k$, $k > 0$, for every real or complex number β with $|\beta| < 1$, by Rouche's theorem, the polynomial $p(z) + \beta m$, where $m = \min_{|z|=k}|p(z)|$, has no zero in $|z| < k$, $k > 0$. Let $0 < r \le R \le k$. Then the polynomial $P(z) = p(Rz) + \beta m$ has no zero in $|z| < \frac{k}{R}$, $\frac{k}{R} \ge 1$.

Applying Lemma 1 to $P(z)$, we have for $0 \le \theta < 2\pi$

$$A_0|P'(e^{i\theta})| \le |Q'(e^{i\theta})|, \tag{19}$$

where

$$A_0 = \left\{\frac{\frac{\mu}{n}\frac{|a_\mu|R}{|a_0+\beta m|}k^{2\mu} + k^{\mu+1}}{\frac{\mu}{n}\frac{|a_\mu|R^\mu}{|a_0+\beta m|}k^{\mu+1} + R^{\mu+1}}\right\} \quad \text{and} \quad Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}.$$

By Lemma 8, we have $A_0 \ge A$, where

$$A = \left\{\frac{\frac{\mu}{n}\frac{|a_\mu|R}{|a_0|-m}k^{2\mu} + k^{\mu+1}}{\frac{\mu}{n}\frac{|a_\mu|R^\mu}{|a_0|-m}k^{\mu+1} + R^{\mu+1}}\right\}.$$

Therefore, inequality (19) gives

$$A|P'(e^{i\theta})| \le |Q'(e^{i\theta})|. \tag{20}$$

Now, it can be easily verified that for $0 \leq \theta < 2\pi$

$$nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta}) = e^{i(n-1)\theta}\overline{Q'(e^{i\theta})}. \tag{21}$$

For any complex number δ and $0 \leq \theta < 2\pi$, we have

$$D_\delta P(e^{i\theta}) = nP(e^{i\theta}) + (\delta - e^{i\theta})P'(e^{i\theta}),$$

which on using (21) yields

$$D_\delta P(e^{i\theta}) \leq |nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| + |\delta||P'(e^{i\theta})| = |Q'(e^{i\theta})| + |\delta||P'(e^{i\theta})|. \tag{22}$$

Now, for every complex number δ with $|\delta| \geq 1$, $\gamma > 0$ and ξ real on using (22), we have

$$\int_0^{2\pi} |A + e^{i\xi}|^\gamma d\xi \int_0^{2\pi} |D_\delta P(e^{i\theta})|^\gamma d\theta \leq \int_0^{2\pi}\int_0^{2\pi} |A + e^{i\xi}|^\gamma \left\{|Q'(e^{i\theta})| + |\delta||P'(e^{i\theta})|\right\}^\gamma d\xi d\theta. \tag{23}$$

By Lemma 7, $A \geq 1$ and by taking $p = |Q'(e^{i\theta})|$, $q = |P'(e^{i\theta})|$, $x = A$ and $y = |\delta| \geq 1$ in Lemma 2 so that $p \geq qx$ satisfied by (20), we get

$$|A + e^{i\xi}| \left\{|Q'(e^{i\theta})| + |\delta||P'(e^{i\theta})|\right\} \leq (A + |\delta|) \left||Q'(e^{i\theta})| + e^{i\xi}|P'(e^{i\theta})|\right|. \tag{24}$$

By Lemma 3, for every $\gamma > 0$ and $z_1, z_2 \in \mathbb{C}$ with ξ real, we have

$$\int_0^{2\pi} \left|z_1 + z_2 e^{i\xi}\right|^\gamma d\xi = \int_0^{2\pi} \left||z_1| + |z_2|e^{i\xi}\right|^\gamma d\xi. \tag{25}$$

With the help of (24) and (25), inequality (23) implies

$$\begin{aligned}
\int_0^{2\pi} |A + e^{i\xi}|^\gamma d\xi \int_0^{2\pi} |D_\delta P(e^{i\theta})|^\gamma d\theta &\leq (A + |\delta|)^\gamma \int_0^{2\pi}\int_0^{2\pi} \left||Q'(e^{i\theta})| + e^{i\xi}|P'(e^{i\theta})|\right|^\gamma d\xi d\theta \\
&= (A + |\delta|)^\gamma \int_0^{2\pi}\int_0^{2\pi} \left|Q'(e^{i\theta}) + e^{i\xi}P'(e^{i\theta})\right|^\gamma d\xi d\theta \\
&\leq (A + |\delta|)^\gamma 2\pi n^\gamma \int_0^{2\pi} \left|P(e^{i\theta})\right|^\gamma d\theta, \qquad (26)
\end{aligned}$$

where the last inequality follows by Lemma 4.

The above inequality becomes

$$\int_0^{2\pi} |D_\delta P(e^{i\theta})|^\gamma d\theta \leq \frac{n^\gamma (A + |\delta|)^\gamma}{\left(\frac{1}{2\pi}\int_0^{2\pi} |A + e^{i\xi}|^\gamma d\xi\right)} \int_0^{2\pi} \left|P(e^{i\theta})\right|^\gamma d\theta.$$

Note that $P(z) = p(Rz) + \beta m$ and putting $\delta = \frac{\alpha}{R}$ such that $\frac{|\alpha|}{R} \geq 1$, we have

$$\int_0^{2\pi} |D_{\frac{\alpha}{R}}\{p(Re^{i\theta}) + \beta m\}|^\gamma d\theta \leq \frac{n^\gamma \left(A + \frac{|\alpha|}{R}\right)^\gamma}{\left(\frac{1}{2\pi}\int_0^{2\pi} |A + e^{i\xi}|^\gamma d\xi\right)} \int_0^{2\pi} \left|p(Re^{i\theta}) + \beta m\right|^\gamma d\theta. \tag{27}$$

Since

$$\begin{aligned}
D_{\frac{\alpha}{R}}\{p(Re^{i\theta}) + \beta m\} &= n\{p(Re^{i\theta}) + \beta m\} + \left(\frac{\alpha}{R} - e^{i\theta}\right) Rp'(Re^{i\theta}) \\
&= D_\alpha p(Re^{i\theta}) + n\beta m,
\end{aligned}$$

inequality (27) is equivalent to

$$\int_0^{2\pi} |D_\alpha p(Re^{i\theta}) + n\beta m|^\gamma d\theta \leq \frac{n^\gamma \left(A + \frac{|\alpha|}{R}\right)^\gamma}{\left(\frac{1}{2\pi}\int_0^{2\pi} |A + e^{i\xi}|^\gamma d\xi\right)} \int_0^{2\pi} \left|p(Re^{i\theta}) + \beta m\right|^\gamma d\theta. \tag{28}$$

Now, we choose the argument of β suitably such that

$$|p(Re^{i\theta}) + \beta m| = |p(Re^{i\theta})| - |\beta| m. \tag{29}$$

Using equality (29) in the right side of (28), we get

$$\int_0^{2\pi} |D_\alpha p(Re^{i\theta}) + n\beta m|^\gamma d\theta \leq \frac{n^\gamma \left(A + \frac{|\alpha|}{R}\right)^\gamma}{\left(\frac{1}{2\pi}\int_0^{2\pi} |A + e^{i\xi}|^\gamma d\xi\right)} \int_0^{2\pi} \{|p(Re^{i\theta})| - |\beta| m\}^\gamma d\theta. \tag{30}$$

Applying Lemma 5 in (30), we have for $\gamma > 0$

$$\begin{aligned} &\left\{\int_0^{2\pi} |D_\alpha p(Re^{i\theta}) + n\beta m|^\gamma d\theta\right\}^{1/\gamma} \\ \leq\ & \frac{n\left(A + \frac{|\alpha|}{R}\right)}{\left(\frac{1}{2\pi}\int_0^{2\pi} |A + e^{i\xi}|^\gamma d\xi\right)^{1/\gamma}} \left[\int_0^{2\pi} \left\{|p(re^{i\theta})|\right.\right. \\ & \left.\left. + n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu + k^{2\mu} t) + k^{\mu+1}} \{M(p,t) - m\}\, dt - |\beta| m\right\}^\gamma d\theta\right]^{1/\gamma}, \end{aligned} \tag{31}$$

which is equivalent to (18). This completes the proof of Theorem 3. ■

Remark 1 *Dividing both sides of (18) of Theorem 3 by $|\alpha|$ and letting $|\alpha| \to \infty$ and $|\beta| \to 1$, we obtain a result due to Reingachan et al. [8]. Also since $|p(re^{i\theta})| \leq \max_{|z|=r} |p(z)|$, Theorem 3 reduces to the following interesting result.*

Corollary 1 *If $p(z) = a_0 + \sum_{\nu=\mu}^n a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zero in $|z| < k$, $k > 0$, then for $0 < r \leq R \leq k$, any complex numbers α, β with $|\alpha| \geq R$, $|\beta| < 1$ and $\gamma > 0$,*

$$\begin{aligned} & \left\| D_\alpha p(Rz)| + n\beta m \right\|_\gamma \\ \leq\ & \frac{n\left(A + \frac{|\alpha|}{R}\right)}{\|A + z\|_\gamma} \left\| |M(p,r)| + n\int_r^R \frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m} k^{\mu+1} t^{\mu-1} + t^\mu}{t^{\mu+1} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m}(k^{\mu+1}t^\mu + k^{2\mu} t) + k^{\mu+1}} \right. \\ & \left. \times \{M(p,t) - m\}\, dt - |\beta| m \right\|_\gamma, \end{aligned} \tag{32}$$

where

$$A = \left\{\frac{\frac{\mu}{n}\frac{|a_\mu| R}{|a_0|-m} k^{2\mu} + k^{\mu+1}}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-m} k^{\mu+1} R^\mu + R^{\mu+1}}\right\}, \quad M(p,t) = \max_{|z|=t} |p(z)|,$$

$$M(p,r) = \max_{|z|=r} |p(z)| \quad \text{and} \quad m = \min_{|z|=k} |p(z)|.$$

Remark 2 *Since $(\frac{k}{R})^{\mu} \leq A$, where A is as defined in Corollary 1, and by (15) of Lemma 6, the bound given by Corollary 1 is better than both the bounds given by Theorems 1 and 2 which were recently proved by Chanam et al. [5] and hence Theorem 3 is an improved and generalized version concerning polar derivative of the inequalities of Theorems 1 and 2.*

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