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Linearizing Third-Order Quasilinear Delay Difference Equations for Establishing Oscillation Criteria*

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Abstract

This work investigates the oscillatory behavior of third-order quasilinear delay difference equations. By linearizing the equation and comparing it to first-order delay difference equations, new oscillation criteria are derived. The obtained results are new and improve the existing results reported in the literature. Examples are offered to demonstrate the significance of the key findings.

1 Introduction

This work focuses on the oscillatory behavior of solutions of third-order delay difference equation

$$\Delta\left(\mathcal{A}(\upsilon)(\Delta^2\mu(\upsilon))^{\alpha}\right)+\mathcal{B}(\upsilon)\mu^{\beta}(\sigma(\upsilon))=0,\ \upsilon\geq\upsilon_0\geq 0, \tag{E}$$

where υ_0 is a positive integer and $\alpha\geq 1$ and β are the ratio of odd positive integers. In the sequel, we assume that:

(i) $\{\mathcal{A}(\upsilon)\}$ and $\{\mathcal{B}(\upsilon)\}$ are positive real sequences for all $\upsilon\geq\upsilon_0$;

(ii) $\{\sigma(\upsilon)\}$ is a sequence of integers with $\sigma(\upsilon)\leq\upsilon-1$ and $\sigma(\upsilon)\to\infty$ as $\upsilon\to\infty$;

(iii) $\mathcal{C}(\upsilon,\upsilon_*)=\sum_{s=\upsilon_*}^{\upsilon-1}\frac{1}{\mathcal{A}^{1/\alpha}(s)}\to\infty$ as $\upsilon\to\infty$, where υ_* is an integer with $\upsilon_*\geq\upsilon_0$.

Let $\theta=\min_{\upsilon\geq\upsilon_0}\{\sigma(\upsilon)\}$. By a solution of (E), we mean a real sequence $\{\mu(\upsilon)\}$ defined for all $\upsilon\geq\theta$ and satisfying (E) for all $\upsilon\geq\upsilon_0$. We consider only those solutions of (E) that satisfy $\sup\{|\mu(\upsilon)|:\upsilon>\mathcal{V}\}>0$ for all $\upsilon\geq\upsilon_0$ and we tacitly assume that (E) has such solutions. A solution of (E) is said to be oscillatory if it is neither eventually positive nor eventually negative and it is called nonoscillatory otherwise.

Following the publication of Hartman's seminal paper [16], there has been a growing interest in studying the oscillatory features of different classes of difference equations; see, for instance, the monographs [2, 3] and the references discussed therein. Though such equations naturally emerge in the social sciences and engineering (see,[7]), the subject of finding oscillation and asymptotic behavior of solutions to nonlinear third order difference equations has received substantially less attention among scholars. The reader may consult [4, 6, 8, 10, 11, 12, 13, 14, 17, 20, 21, 23, 24, 25, 26, 28, 29, 30, 31, 33, 34, 35, 36] and the references therein for some classical and current results on third-order ordinary, delay, and neutral difference equations.

Note that the oscillatory and asymptotic behavior of solutions of canonical and non-canonical types of delay, neutral and functional differential equations can be found in [9, 15, 18, 19, 22] and the references contained therein.

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In [20, 21], the authors considered the equation of the form

$$\Delta\left(\mathcal{A}(\upsilon)(\Delta(b(\upsilon)\Delta\mu(\upsilon))^{\alpha})\right)+\mathcal{B}(\upsilon)\mu^{\alpha}(\upsilon-\sigma)=0,\ \upsilon\geq\upsilon_0, \tag{E1}$$

under the condition

$$\sum_{\upsilon=\upsilon_0}^{\infty}\mathcal{A}^{-\alpha}(\upsilon)=\sum_{\upsilon=\upsilon_0}^{\infty}1/b(\upsilon)=\infty.$$

They developed a number of conditions to ensure that every (E1) solution is either oscillatory or monotonically approaches zero.

In [14], the authors analyzed the equation (E1) and established various oscillation results that enhanced those found in [20, 21], in the sense that every solution is oscillatory for the situation $\alpha\leq 1$.

The authors of [30] considered the equation of the type

$$\Delta(\mathcal{A}(\upsilon)\Delta(b(\upsilon)(\Delta\mu(\upsilon))^{\alpha}))+\mathcal{B}(\upsilon)\mu^{\alpha}(\upsilon-\sigma+1)=0,\ \upsilon\geq\upsilon_0, \tag{E2}$$

under the condition

$$\sum_{\upsilon=\upsilon_0}^{\infty}\mathcal{A}^{-1}(\upsilon)=\sum_{\upsilon=\upsilon_0}^{\infty}(1/b(\upsilon))^{\frac{1}{\alpha}}=\infty,$$

and used the Riccati transformation method and the summation averaging technique to offer some oscillation criteria.

By taking $a_1(n)=1$, $\alpha_1=1$ and $f(u)=u^{\beta}$ in equation (1.1; 1) considered in [4], then it becomes equation (E) and the authors established a number of oscillation results for the situation $\alpha\geq\beta$ using the summation averaging approach and comparison method.

The research data discussed above demonstrates that, with the exception of [4], few authors have examined equations of the kind (E). Indeed, the authors in [4] ($a_1(n)=1$, $\alpha_1=1$ and $f(u)=u^{\beta}$) obtained several criteria for the oscillation of all solutions under different assumptions.

This motivated us to consider the equation(E) and to present some new sufficient conditions under which every solution of (E) is oscillatory for all α and β. To obtain the results, we first linearize the equation (E) and then transform the resulting equation to a first-order delay difference inequality. We then obtain oscillation results by comparing this inequality to some first-order delay difference equations. Another result is obtained by comparison with a second-order linear difference equation. From the form of equation (E), it is possible to extend our results to more general equations. In view of this, it is our hope that the results established in this paper will create interest in research on higher odd-order difference equations. Examples are provided to illustrate the main results.

2 Main Results

First note that dealing with only the positive solutions of (E) suffices to consider nonoscillatory solutions of (E). We begin with the following theorem.

Theorem 1 *Assume that $a=1-\alpha+\beta\geq 0$ and there exist positive nondecreasing sequences $\{\xi(\upsilon)\}$ and $\{\eta(\upsilon)\}$ of integers such that*

$$\sigma(\upsilon)<\xi(\upsilon)<\eta(\upsilon)\leq\upsilon-1, \tag{1}$$

for $\upsilon\geq\upsilon_1\geq\upsilon_0$. If the first order delay difference equations

$$\Delta W(\upsilon)+\frac{1}{\alpha}\mathcal{B}(\upsilon)\left(\sum_{s=\upsilon_1}^{\sigma(\upsilon)-1}\mathcal{C}(s,\upsilon_1)\right)^{\beta}W^{a}(\sigma(\upsilon))=0 \tag{2}$$

and

$$\Delta Y(\upsilon)+\mathcal{B}(\upsilon)\mathcal{C}^{\beta}(\eta(\upsilon),\xi(\upsilon))(\xi(\upsilon)-\sigma(\upsilon))^{\beta}Y^{\frac{\beta}{\alpha}}(\eta(\upsilon))=0 \tag{3}$$

are oscillatory, then every solution of (E) is oscillatory.

Proof. Let $\{\mu(\upsilon)\}$ be a nonoscillatory solution of (E), say $\mu(\upsilon) > 0$, and $\mu(\sigma(\upsilon)) > 0$ for all $\upsilon \geq \upsilon_1$ for some integer $\upsilon_1 \geq \upsilon_0$. From (E), it follows that

$$\Delta(\mathcal{A}(\upsilon)(\Delta^2\mu(\upsilon))^\alpha) = -\mathcal{B}(\upsilon)\mu^\beta(\sigma(\upsilon)) \leq 0,\ \upsilon \geq \upsilon_1, \tag{4}$$

and so $\{\mathcal{A}(\upsilon)(\Delta^2\mu(\upsilon))^\alpha\}$ is eventually of one sign. We shall have two possible cases:

(I) $\Delta\mu(\upsilon) > 0,\ \mathcal{A}(\upsilon)(\Delta^2\mu(\upsilon))^\alpha > 0$, or

(II) $\Delta\mu(\upsilon) < 0,\ \mathcal{A}(\upsilon)(\Delta^2\mu(\upsilon))^\alpha > 0$,

for all $\upsilon \geq \upsilon_1$. From (E) by taking Δ-derivative, we see that

$$\begin{aligned}\Delta\left(\mathcal{A}(\upsilon)(\Delta^2\mu(\upsilon))^\alpha\right) &= \Delta\left((\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon))^\alpha\right)\\ &\geq \alpha\left(\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon)\right)^{\alpha-1}\Delta\left(\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon)\right),\end{aligned}$$

and so

$$\Delta\left(\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon)\right) + \frac{1}{\alpha}\left(\mathcal{A}^{1/\alpha}(n)\Delta^2\mu(\upsilon)\right)^{\alpha-1}\mathcal{B}(\upsilon)\mu^\beta(\sigma(\upsilon)) \leq 0. \tag{5}$$

First, we consider Case (I). Now

$$\Delta\mu(\upsilon) \geq \Delta\mu(\upsilon) - \Delta\mu(\upsilon_1) = \sum_{s=\upsilon_1}^{\upsilon-1}\mathcal{A}^{-1/\alpha}(s)\mathcal{A}^{1/\alpha}(s)\Delta^2\mu(s) \geq \mathcal{C}(\upsilon,\upsilon_1)\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon).$$

Summing the last inequality from υ_1 to $\upsilon - 1$, we get

$$\mu(\upsilon) \geq \left(\sum_{s=\upsilon_1}^{\upsilon-1}\mathcal{C}(s,\upsilon_1)\right)\left(\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon)\right)$$

or

$$\mu(\sigma(\upsilon)) \geq \left(\sum_{s=\upsilon_1}^{\sigma(\upsilon)-1}\mathcal{C}(s,\upsilon_1)\right)\left(\mathcal{A}^{1/\alpha}(\sigma(\upsilon))\Delta^2\mu(\sigma(\upsilon))\right), \tag{6}$$

for $\upsilon > \upsilon_2 \geq \upsilon_1$. Since $\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon)$ is nonincreasing and $\alpha \geq 1$, we have

$$\left(\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon)\right)^{1-\alpha} \geq \left(\mathcal{A}^{1/\alpha}(\sigma(\upsilon))\Delta^2\mu(\sigma(\upsilon))\right)^{1-\alpha}. \tag{7}$$

Using (7) in (5) implies that

$$\Delta(\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon)) + \frac{1}{\alpha}(\mathcal{A}^{1/\alpha}(\sigma(\upsilon))\Delta^2\mu(\sigma(\upsilon)))^{1-\alpha}\mathcal{B}(\upsilon)\mu^\beta(\sigma(\upsilon)) \leq 0. \tag{8}$$

In view of (6) and (8), we see that

$$\begin{aligned}\Delta(\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon)) &\leq -\frac{1}{\alpha}(\mathcal{A}^{1/\alpha}(\sigma(\upsilon))\Delta^2\mu(\sigma(\upsilon)))^{1-\alpha}\mathcal{B}(\upsilon)\mu^\beta(\sigma(\upsilon))\\ &\leq -\frac{1}{\alpha}(\mathcal{A}^{1/\alpha}(\sigma(\upsilon))\Delta^2\mu(\sigma(\upsilon)))^{1-\alpha+\beta}\mathcal{B}(\upsilon)\left(\sum_{s=\upsilon_1}^{\sigma(\upsilon)-1}\mathcal{C}(s,\upsilon_1)\right)^\beta.\end{aligned} \tag{9}$$

Letting $W(\upsilon) = \mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon)$, in (9), we see that $\{W(\upsilon)\}$ is a positive solution of the first order delay difference inequality

$$\Delta W(\upsilon) + \frac{1}{\alpha}\mathcal{B}(\upsilon)\left(\sum_{s=\upsilon_1}^{\sigma(\upsilon)-1}\mathcal{C}(s,\upsilon_1)\right)^\beta W^a(\sigma(\upsilon)) \leq 0.$$

Since the sequence $\{W(\upsilon)\}$ is decreasing for all $\upsilon \geq \upsilon_1$, so by [36, Lemma 1.1], the associated difference equation (2) also has a positive solution, which contradicts the fact that (2) is oscillatory.

Next, we consider Case (II). Since $\Delta\mu(\upsilon) < 0$, we see that

$$-\Delta\mu(\xi(\upsilon)) \geq \sum_{s=\xi(\upsilon)}^{\eta(\upsilon)} \mathcal{A}^{-1/\alpha}(s)\mathcal{A}^{1/\alpha}(s)\Delta^2\mu(s) \geq \mathcal{C}(\eta(\upsilon),\xi(\upsilon)) \left(\mathcal{A}^{1/\alpha}(\eta(\upsilon))\Delta^2\mu(\eta(\upsilon))\right). \tag{10}$$

Now, for $t > s \geq \upsilon_0$, we have

$$\mu(s) \geq (t-s)(-\Delta\mu(t)).$$

Replacing s and t by $\sigma(\upsilon)$ and $\xi(\upsilon)$ respectively in the above inequality, we obtain

$$\mu(\sigma(\upsilon)) \geq (\xi(\upsilon)-\sigma(\upsilon))(-\Delta\mu(\xi(\upsilon))).$$

Combining the last inequality with (10) and (E) and letting $Y(\upsilon) = \mathcal{A}(\upsilon)(\Delta^2\mu(\upsilon))^\alpha$, we have

$$\Delta Y(\upsilon) + \mathcal{B}(\upsilon)(\xi(\upsilon)-\sigma(\upsilon))^\beta \mathcal{C}^\beta(\eta(\upsilon),\xi(\upsilon))Y^{\beta/\alpha}(\eta(\upsilon)) \leq 0. \tag{11}$$

Rest of the proof is similar to that of Case (I) and hence is omitted. This completes the proof of the theorem. ■

Corollary 1 *Let $a = 1$, $\sigma(\upsilon) = \upsilon - k$, $\xi(\upsilon) = \upsilon - \ell$ and $\eta(\upsilon) = \upsilon - m$ where k, ℓ and m are positive integers with $k > \ell > m$. If*

$$\liminf_{\upsilon\to\infty} \sum_{s=\upsilon-k}^{\upsilon-1} \mathcal{B}(s) \left(\sum_{t=\upsilon_1}^{s-k-1} \mathcal{C}(t,\upsilon_1)\right)^\beta > \alpha \left(\frac{k}{k+1}\right)^{k+1} \tag{12}$$

and

$$\liminf_{\upsilon\to\infty} \sum_{s=\upsilon-m}^{\upsilon-1} \mathcal{B}(s)\mathcal{C}^\beta(\eta(s),\xi(s)) > \frac{1}{(\ell-m)^\beta} \left(\frac{m}{m+1}\right)^{m+1} \tag{13}$$

hold for all $\upsilon \geq \upsilon_1$, then equation (E) is oscillatory.

Proof. First note that $a = 1$ implies that $\alpha = \beta$. In view of Theorem 6.20.5 of [1], conditions (12) and (13) ensure equations (2) and (3) are oscillatory. An application of Theorem 1 completes the proof. ■

Corollary 2 *Let $a < 1$, $\sigma(\upsilon) = \upsilon - k$, $\xi(\upsilon) = \upsilon - \ell$ and $\eta(\upsilon) = \upsilon - m$ where k, ℓ and m are positive integers with $k > \ell > m$. If*

$$\sum_{\upsilon=\upsilon_0}^{\infty} \mathcal{B}(\upsilon) \left(\sum_{s=\upsilon_1}^{\upsilon-k-1} \mathcal{C}(s,\upsilon_1)\right)^\beta = \infty \tag{14}$$

and

$$\sum_{\upsilon=\upsilon_0}^{\infty} \mathcal{B}(\upsilon)\mathcal{C}^\beta(\eta(\upsilon),\xi(\upsilon)) = \infty \tag{15}$$

holds, for all $\upsilon \geq \upsilon_1 \geq \upsilon_0$, then equation (E) is oscillatory.

Proof. First note that $a < 1$ gives that $\alpha > \beta$. By Theorem 1 of [29], conditions (14) and (15) ensure equations (2) and (3) are oscillatory. An application of Theorem 1 completes the proof. ■

Corollary 3 *Let $a > 1$, $\sigma(\upsilon) = \upsilon - k$, $\xi(\upsilon) = \upsilon - \ell$ and $\eta(\upsilon) = \upsilon - m$ where k, ℓ and m are positive integers with $k > \ell > m$. If there exists $\lambda > \frac{1}{k}\ln a$ such that*

$$\liminf_{\upsilon\to\infty} \left[\mathcal{B}(\upsilon) \left(\sum_{s=\upsilon_1}^{\upsilon-k-1} \mathcal{C}(s,\upsilon_1)\right)^\beta exp(-e^{\lambda\upsilon})\right] > 0, \tag{16}$$

and there exists $\delta > \frac{1}{m}\ln a$ *such that*

$$\lim_{\upsilon\to\infty}\inf\left[\mathcal{B}(\upsilon)\mathcal{C}^{\beta}(\eta(\upsilon),\xi(\upsilon))exp(-e^{\delta\upsilon})\right] > 0 \tag{17}$$

holds, for all $\upsilon \geq \upsilon_1 \geq \upsilon_0$*, then equation (E) is oscillatory.*

Proof. First note that $a > 1$ gives that $\alpha < \beta$. By Theorem 2 of [29], conditions (16) and (17) ensure equations (2) and (3) are oscillatory. An application of Theorem 1 completes the proof. ∎

In our next result we provide one more oscillation criterion for (E) when $\alpha = \beta$.

Theorem 2 *Let* $\alpha = \beta$ *and assume that there exist positive nondecreasing sequences* $\{\xi(\upsilon)\}$ *and* $\{\eta(\upsilon)\}$ *of integers such that* (1) *holds. If* (3) *and the equation*

$$\Delta(\mathcal{A}^{1/\alpha}(\upsilon)\Delta z(\upsilon)) + \frac{1}{\alpha}\frac{\mathcal{C}_1^{\alpha}(\sigma(\upsilon),\upsilon_1)}{\mathcal{C}(\sigma(\upsilon),\upsilon_1)}\mathcal{B}(\upsilon)z(\sigma(\upsilon)) = 0, \tag{18}$$

where $\mathcal{C}_1(\upsilon,\upsilon_1) = \sum_{\upsilon=\upsilon_1}^{\upsilon-1}\mathcal{C}(s,\upsilon_1)$ *are oscillatory then equation (E) is oscillatory.*

Proof. Let $\{\mu(\upsilon)\}$ be a nonoscillatory solution of (E), say $\mu(\upsilon) > 0$ and $\mu(\sigma(\upsilon)) > 0$ for all $\upsilon \geq \upsilon_1$ for some integer $\upsilon_1 \geq \upsilon_0$. Proceeding as in the proof of Theorem 1, we have two cases (I) and (II) to consider and arrive at (5), (6) and (10).

First consider Case (I), then it follows from (6), and the fact that $\mathcal{A}^{1/\alpha}(n)\Delta^2\mu(\upsilon)$ is nonincreasing that

$$\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon) \leq \mathcal{A}^{1/\alpha}(\sigma(\upsilon))\Delta^2\mu(\sigma(\upsilon)) \leq \left(\sum_{s=\upsilon_1}^{\sigma(\upsilon)-1}\mathcal{C}(s,\upsilon_1)\right)^{-1}\mu(\sigma(\upsilon)),$$

and so

$$\left(\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon)\right)^{1-\alpha} \leq \left(\sum_{s=\upsilon_1}^{\sigma(\upsilon)-1}\mathcal{C}(s,\upsilon_1)\right)^{\alpha-1}(\mu(\sigma(\upsilon)))^{1-\alpha}.$$

Using the last inequality in (5) gives

$$\Delta(\mathcal{A}^{1/\alpha}(\upsilon)\Delta^2\mu(\upsilon)) + \frac{1}{\alpha}\left(\sum_{s=\upsilon_1}^{\sigma(\upsilon)-1}\mathcal{C}(s,\upsilon_1)\right)^{\alpha-1}\mathcal{B}(\upsilon)\mu(\sigma(\upsilon)) \leq 0.$$

By condition (18), we get the desired contradiction.

Case (II) is similar to that of Theorem 1. This completes the proof of the theorem. ∎

Corollary 4 *Let* $\alpha = \beta$ *and assume that* $\sigma(\upsilon) = \upsilon - k$*,* $\xi(\upsilon) = \upsilon - \ell$ *and* $\eta(\upsilon) = \upsilon - m$ *where* k*,* ℓ *and* m *are positive integers with* $k > \ell > m$*. If* (13) *and*

$$\mathcal{C}(\upsilon - k,\upsilon_1)\sum_{s=\upsilon}^{\infty}\frac{\mathcal{C}_1^{\alpha}(s-k,\upsilon_1)}{\mathcal{C}(s-k,\upsilon_1)}\mathcal{B}(s) > \frac{\alpha}{4} \tag{19}$$

are hold, then (E) is oscillatory.

Proof. Proceeding as in Theorem 2, we have two cases(I) and (II) to consider. First case can be proved by using Theorem 2.3 of [37] in the equation(18) and the second case follows from Corollary 1. This completes the proof. ∎

We conclude this paper with the following examples.

Example 1 *Consider the third-order delay difference equation*

$$\Delta\left(\frac{1}{2^{\upsilon}}\left(\Delta^2\mu(\upsilon)\right)^3\right)+\frac{96}{2^{\upsilon}}\mu^3(\upsilon-4)=0,\ \upsilon\geq 1. \tag{20}$$

Here $\mathcal{A}(\upsilon)=\frac{1}{2^{\upsilon}}$, $\mathcal{B}(\upsilon)=\frac{96}{2^{\upsilon}}$, $\alpha=\beta=3$, $\sigma=n-4$. By taking $\ell=3$ and $m=2$, we see that condition (1) is satisfied. A simple computation show that $\mathcal{C}(\upsilon,1)\approx 2^{\frac{(\upsilon-2)}{3}}$ and $\mathcal{C}(\upsilon-2,\upsilon-3)=2^{\frac{(\upsilon-3)}{3}}$. It is clear that the conditions (12) and (13) are satisfied. Therefore by Corollary 1, equation (20) is oscillatory and in fact $\{(-1)^{\upsilon}\}$ is one such oscillatory solution of (20).

Note that using Corollary 2.13(A1) ($a_1(n)=1$, $\alpha_1=1$) of [4] we cannot get this conclusion since the condition (2.51) of [4] is not satisfied. Hence our result is superior than Corollary 2.13(A1) of [4].

Example 2 *Consider the third-order delay difference equation*

$$\Delta\left(\frac{1}{2^{\upsilon}}\Delta^2\mu(\upsilon)\right)+\frac{d}{2^{\upsilon/3}}\mu^{1/3}(n-3)=0,\ \upsilon\geq 1. \tag{21}$$

Here $\mathcal{A}(\upsilon)=\frac{1}{2^{\upsilon}}$, $\mathcal{B}(\upsilon)=\frac{d}{2^{\upsilon}}$, $\alpha=1$, and $\beta=1/3$. By letting $\ell=3$ and $m=2$ we see that condition (1) is satisfied. A simple computation shows that $\mathcal{C}(\upsilon,1)\approx 2^{\upsilon-2}$, and $\mathcal{C}(\upsilon-2,\upsilon-3)=2^{\upsilon-3}$. It is easy to see that conditions (14) and (15) are satisfied if $d>0$. Therefore by Corollary 2 equation (21) is oscillatory.

Note that Corollary 2.13(A2)($a_1(n)=1$, $\alpha_1=1$) of [4] cannot yield this conclusion since the condition (2.51) of [4] again not satisfied. Thus our result is better than Corollary 2.13(A2) of [4].

Example 3 *Consider the third-order delay difference equation*

$$\Delta\left(\frac{1}{\upsilon}\Delta^2\mu(\upsilon)\right)+(30)^{\upsilon}\mu^3(\upsilon-3)=0,\ \upsilon\geq 1. \tag{22}$$

Here $\mathcal{A}(\upsilon)=1/\upsilon$, $\mathcal{B}(\upsilon)=(30)^{\upsilon}$, $\alpha=1$, and $\beta=3$. A simple computation shows that $\mathcal{C}(\upsilon,1)\approx\frac{\upsilon^2}{2}$, and $\mathcal{C}(\upsilon-1,\upsilon-2)\approx(\upsilon-1)$. By taking $\lambda=1$, we see that condition (18) is clearly satisfied and taking $\ell=2$, $m=1$ and $\delta=1$ it is easy to see that condition (19) is satisfied. Therefore by Corollary 3 equation (22) is oscillatory.

3 Conclusion

Note that the results in [12, 14, 20, 21, 24, 33] cannot be applied to our equations since the form of our equations are different from the above mentioned papers for the case $\alpha\neq 1$ and $\beta\neq 1$. Further the results in [4] cannot be applied to this particular equation (22) since $\alpha<\beta$. Hence the results obtained here are new to the literature. By using the results in [20, 21] we get only every solution of (20) and (21) is either oscillatory or tends to zero as $\upsilon\to\infty$ for particular case $\alpha=\beta=1$, but our results yield that every solution is oscillatory. Thus our results improve that of in [20, 21].

The results obtained here can be extended to higher order quasilinear delay difference equations of the form

$$\Delta(\mathcal{A}(\upsilon)(\Delta^{m-1}\mu(\upsilon))^{\alpha})+\mathcal{B}(\upsilon)\mu^{\beta}(\sigma(\upsilon))=0,$$

where $m\geq 3$ is a positive integer, α and β are ratios of odd positive integers. We only considered the case $\alpha\geq 1$; so the results for $0<\alpha<1$ would be of interest. It is also interesting to extend the results to equations with advanced arguments and to neutral equations.

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