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Investigation Of A Coupled System Of Caputo Sequential Fractional Differential Equations With Closed Coupled Boundary Conditions*

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Abstract

A new class of nonlinear boundary value problems consisting of Caputo type coupled sequential fractional differential equations subject to closed coupled boundary conditions is investigated in this paper. The existence and uniqueness results for the given problem are proved with the aid of the standard fixed point theorems. Examples illustrating the abstract results are constructed. Our results significantly contribute to the literature on boundary value problems involving nonlinear sequential fractional differential equations, and specialize to several new results as special cases.

1 introduction

We study the existence of solutions for a coupled system of nonlinear sequential fractional differential equations:

$$\begin{cases} ({}^C D^q + k_1\ {}^C D^{q-1})x(t) = f(t, x(t), y(t)), & t \in J := [0,T],\ T > 0, \\ ({}^C D^p + k_2\ {}^C D^{p-1})y(t) = g(t, x(t), y(t)), & t \in J := [0,T],\ T > 0, \end{cases} \tag{1}$$

subject to the closed coupled boundary conditions given by

$$\begin{cases} x(T) = \alpha_1 y(0) + \beta_1 T y'(0), \quad T x'(T) = \gamma_1 y(0) + \delta_1 T y'(0), \\ y(T) = \alpha_2 x(0) + \beta_2 T x'(0), \quad T y'(T) = \gamma_2 x(0) + \delta_2 T x'(0), \end{cases} \tag{2}$$

where ${}^C D^q$, ${}^C D^p$ denote the Caputo fractional derivative operators of order $q \in (1,2]$ and $p \in (1,2]$, respectively, $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$, and $f, g : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

Many researchers have shown a keen interest in investigating the fractional-order nonlinear boundary value problems. It has been mainly due to the application of fractional-order operators (nonlocal in nature) in the mathematical modeling of several real world phenomena occurring in physical and technical sciences. In fact, the mathematical models based on fractional-order differential and integral operators are found to be more informative and practical than their associated integer-order counterparts. Some examples of such models include synchronization of chaotic systems [1, 2], anomalous diffusion [3], disease models [4, 5, 6], ecological models [7], etc. For further applications of fractional calculus, see the books [8, 9], while the theoretical background of this branch of mathematical analysis can be found in [10]. For some recent works

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on boundary value problems involving a variety of fractional differential equations and boundary conditions, we refer the reader to the articles [11]–[18].

In [19], the authors studied a nonlinear coupled system of Caputo-type fractional differential equations equipped with coupled closed boundary conditions. Keeping in mind the application of closed boundary conditions in the real world phenomena, such as deblurring problems [20], honeycomb lattice [21], wavefield decomposition [22], magneto-electro-elastic panel [23], etc., we investigate a coupled system of nonlinear sequential fractional differential equations of different orders (1) complemented with closed coupled boundary conditions (2). For some recent works on sequential fractional differential equations, for instance, see [24]–[27].

The objective of the present work is to enrich the literature on boundary value problems involving sequential fractional differential equations with a new class of boundary conditions. Precisely, we apply Leray-Schauder's alternative and Banach's contraction mapping principle to develop the criteria ensuring the existence and uniqueness of solutions for the system (1)–(2). Our results are new and specialize to several new results by fixing the parameters involved in the closed coupled boundary conditions.

The composition of the rest of the paper is as follows. In Section 2, some fundamental ideas of fractional calculus are recalled. We also prove an auxiliary lemma for the linear version of the system (1)–(2). Section 3 contains the main results, while the examples illustrating these results are presented in Section 4. The paper concludes with some interesting observations.

2 Preliminaries

Let us first recall some definitions from fractional calculus [10].

Definition 1 *For $\sigma \in L_1[0,T]$, we define the (left) Riemann-Liouville fractional integral of order $p > 0$ as*

$$I^p\sigma(t) = \int_0^t \frac{(t-\bar{t})^{p-1}}{\Gamma(p)}\sigma(\bar{t})d\bar{t}.$$

Definition 2 *The (left) Caputo fractional derivative for a function $\sigma \in AC^m[0,T]$ of order $p \in (m-1,m]$, $m \in \mathbb{N}$ is defined by*

$${}^C D^p\sigma(t) = \int_0^t \frac{(t-\bar{t})^{m-p-1}}{\Gamma(m-p)}\sigma^{(m)}(\bar{t})d\bar{t}.$$

In the following lemma, we obtain an integral presentation of a solution of the linear variant of the problem (1)–(2). Before presenting this lemma, let us introduce the notation:

$$\Delta = \mu_1\mu_4 + \mu_2\mu_6 + \rho_3\rho_4 - T^2k_1k_2\rho_9\rho_{10} - \mu_7 e^{-(k_1+k_2)T} \neq 0, \tag{3}$$

$$\begin{aligned}
&\mu_1 = \gamma_2 e^{-k_1T},\ \mu_2 = \gamma_1 e^{-k_2T}, \mu_3 = Tk_2\rho_7 - \rho_1,\ \mu_4 = Tk_1\rho_5 - \rho_3,\ \mu_5 = Tk_1\rho_8 - \rho_2,\\
&\mu_6 = Tk_2\rho_6 - \rho_4,\ \mu_7 = T^2k_1k_2 - \rho_1\rho_2,\\
&\rho_1 = \gamma_1 + T\alpha_1k_1,\ \rho_2 = \gamma_2 + T\alpha_2k_2,\ \rho_3 = \gamma_1 - T\delta_1k_2,\ \rho_4 = \gamma_2 - T\delta_2k_1,\\
&\rho_5 = T\beta_1k_2 - \alpha_1, \rho_6 = T\beta_2k_1 - \alpha_2,\ \rho_7 = \delta_1 + T\beta_1k_1,\ \rho_8 = \delta_2 + T\beta_2k_2,\\
&\rho_9 = \alpha_1\delta_1 - \beta_1\gamma_1,\ \rho_{10} = \alpha_2\delta_2 - \beta_2\gamma_2,\\
&\omega_1 = \alpha_2\rho_1,\ \omega_2 = \alpha_1\rho_2,\ \omega_3 = \gamma_2\rho_1,\ \omega_4 = \gamma_1\rho_2,\ \omega_5 = \gamma_2\rho_7,\ \omega_6 = \gamma_1\rho_8,\ \omega_7 = \alpha_2\rho_7,\\
&\omega_8 = \alpha_1\rho_8,\ \omega_9 = \alpha_2\rho_9,\ \omega_{10} = \alpha_1\rho_{10},\ \omega_{11} = \gamma_2\rho_5,\ \omega_{12} = \gamma_1\rho_6.
\end{aligned} \tag{4}$$

Lemma 1 *Let $F, G \in C[0,T]$. If $\Delta \neq 0$ (Δ is given by (3)), then the solution of sequential fractional differential system:*

$$\begin{cases}
({}^C D^q + k_1\ {}^C D^{q-1})x(t) = F(t), \qquad t \in J := [0,T],\\
({}^C D^p + k_2\ {}^C D^{p-1})y(t) = G(t),\\
x(T) = \alpha_1 y(0) + \beta_1 Ty'(0),\quad Tx'(T) = \gamma_1 y(0) + \delta_1 Ty'(0),\\
y(T) = \alpha_2 x(0) + \beta_2 Tx'(0),\quad Ty'(T) = \gamma_2 x(0) + \delta_2 Tx'(0),
\end{cases} \tag{5}$$

is given by a pair of fractional integral equations:

$$\begin{aligned} x(t) &= \int_0^t e^{-k_1(t-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} F(\tau) d\tau ds + \nu_1(t) \int_0^T e^{-k_1(T-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} F(\tau) d\tau \ ds \\ &+\nu_2(t) \int_0^T e^{-k_2(T-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} G(\tau) d\tau \ ds + \nu_3(t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} F(s) \ ds \\ &+\nu_4(t) \int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)} G(s) \ ds, \end{aligned} \tag{6}$$

$$\begin{aligned} y(t) &= \int_0^t e^{-k_2(t-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} G(\tau) d\tau ds + \nu_5(t) \int_0^T e^{-k_1(T-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} F(\tau) d\tau \ ds \\ &+\nu_6(t) \int_0^T e^{-k_2(T-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} G(\tau) d\tau \ ds + \nu_7(t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} F(s) \ ds \\ &+\nu_8(t) \int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)} G(s) \ ds, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \nu_1(t) &= \frac{-1}{\Delta}\Big\{\mu_6\rho_1 e^{-k_2T} - \mu_3\rho_4 - [\omega_3 - Tk_2\omega_5 + \mu_7 \ e^{-k_2T}]e^{-k_1t}\Big\}, \\ \nu_2(t) &= \frac{-Tk_2}{\Delta}\Big\{\mu_4 e^{-k_1T} + \mu_5\rho_9 + (\rho_3 + \rho_9\rho_2)e^{-k_1t}\Big\}, \\ \nu_3(t) &= \frac{-T}{\Delta}\Big\{\rho_4\rho_5 + (\omega_2 - Te^{-k_1T}k_2 - Tk_1\omega_8)e^{-k_2T} - [\omega_{11} + (\omega_2 - Tk_2)e^{-k_2T}]e^{-k_1t}\Big\}, \\ \nu_4(t) &= \frac{T}{\Delta}\Big\{(\mu_3 + \rho_1 e^{-k_2T})e^{-k_1T} + Tk_2\rho_9\rho_6 + (\rho_3 - \mu_2 + Tk_2\omega_9)e^{-k_1t}\Big\}, \\ \nu_5(t) &= \frac{-Tk_1}{\Delta}\Big\{\mu_6 e^{-k_2T} + \mu_3\rho_{10} - (\rho_4 - \rho_1\rho_{10})e^{-k_2t}\Big\}, \\ \nu_6(t) &= \frac{-1}{\Delta}\Big\{-\mu_5\rho_3 + \mu_4\rho_2 e^{-k_1T} - (\omega_4 - Tk_1\omega_6 + \mu_7 e^{-k_1T})e^{-k_2t}\Big\}, \\ \nu_7(t) &= \frac{T}{\Delta}\Big\{(\rho_2 e^{-k_1T} + \mu_5)e^{-k_2T} + Tk_1\rho_5\rho_{10} + (\rho_4 - \mu_1 + Tk_1\omega_{10})e^{-k_2t}\Big\}, \\ \nu_8(t) &= \frac{-T}{\Delta}\Big\{\rho_3\rho_6 + (\omega_1 - Tk_2\omega_7 - Tk_1e^{-K_2T})e^{-k_1T} - [\omega_{12} + (\omega_1 - Tk_1)e^{-k_1T}]e^{-k_2t}\Big\}, \end{aligned}$$

and μ_i $(i = 1, \dots, 7)$, ρ_j $(j = 1, \dots, 10)$ *and* ω_k $(k = 1, \dots, 12)$ *are given in* (4).

Proof. As argued in [10], the general solution of sequential fractional equations in (5) can be written as

$$x(t) = -A_1 + A_0 e^{-k_1t} + \int_0^t e^{-k_1(t-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} F(\tau) \ d\tau \ ds, \tag{8}$$

$$y(t) = -B_1 + B_0 e^{-k_2t} + \int_0^t e^{-k_2(t-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} G(\tau) \ d\tau \ ds, \tag{9}$$

where A_1, A_0, B_1 and B_0 are unknown arbitrary constants. Differentiating (8) and (9) with respect to t, we obtain

$$x'(t) = -k_1 A_0 e^{-k_1 t} - k_1 \int_0^t e^{-k_1(t-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} F(\tau)\, d\tau\, ds + \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)} F(s) ds, \tag{10}$$

and

$$y'(t) = -k_2 B_0 e^{-k_2 t} - k_2 \int_0^t e^{-k_2(t-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} G(\tau)\, d\tau\, ds + \int_0^t \frac{(t-s)^{p-2}}{\Gamma(p-1)} G(s) ds. \tag{11}$$

Inserting (8)–(11) in the boundary conditions of the system (5), we obtain

$$\begin{aligned} A_1 - \alpha_1 B_1 - A_0 e^{-k_1 T} + (\alpha_1 - T\beta_1 k_2) B_0 &= \int_0^T e^{-k_1(T-s)} I^{q-1} F(s)\, ds, \\ -\gamma_1 B_1 + k_1 T A_0 e^{-k_1 T} + (\gamma_1 - T\delta_1 k_2) B_0 &= -k_1 T \int_0^T e^{-k_1(T-s)} I^{q-1} F(s) ds + T I^{q-1} F(T), \\ -\alpha_2 A_1 + B_1 + (\alpha_2 - k_1 \beta_2 T) A_0 - B_0 e^{-k_2 T} &= \int_0^T e^{-k_2(T-s)} I^{p-1} G(s) ds, \\ -\gamma_2 A_1 + (\gamma_2 - T\delta_2 k_1) A_0 + T k_2 B_0 e^{-k_2 T} &= -k_2 T \int_0^T e^{-k_2(T-s)} I^{p-1} G(s) ds + T I^{p-1} G(T). \end{aligned} \tag{12}$$

Solving the system (12) for A_1, B_1, A_0, and B_0, we find that

$$\begin{aligned} A_1 &= \frac{1}{\Delta}\bigg\{ \Big[-\rho_4\mu_3 + \rho_1\mu_6 e^{-k_2 T} \Big] \int_0^T e^{-k_1(T-s)} I^{q-1} F(s)\, ds \\ &\quad + \Big[T k_2 \rho_9 \mu_5 + T k_2 \mu_4 e^{-k_1 T} \Big] \int_0^T e^{-k_2(T-s)} I^{p-1} G(s) ds \\ &\quad + \Big[\rho_4 \rho_5 + (\omega_2 - T k_1 \omega_8) e^{-k_2 T} - T k_2 e^{-(k_1+k_2)T} \Big] T I^{q-1} F(T) \\ &\quad - \Big[T k_2 \rho_6 \rho_9 + \mu_3 e^{-k_1 T} + \rho_1 e^{-(k_1+k_2)T} \Big] T I^{p-1} G(T) \bigg\}, \\ B_1 &= \frac{1}{\Delta}\bigg\{ \Big[k_1 \mu_6 e^{-k_2 T} + k_1 \rho_{10} \mu_3 \Big] T \int_0^T e^{-k_1(T-s)} I^{q-1} F(s)\, ds \\ &\quad + \Big[-\rho_3 \mu_5 + \rho_2 \mu_4 e^{-k_1 T} \Big] \int_0^T e^{-k_2(T-s)} I^{p-1} G(s) ds \\ &\quad - \Big[T k_1 \rho_5 \rho_{10} + \mu_5 e^{-k_2 T} + \rho_2 e^{-(k_1+k_2)T} \Big] T I^{q-1} F(T) \\ &\quad + \Big[\rho_3 \rho_6 + (\omega_1 - T k_2 \omega_7) e^{-k_1 T} + T k_1 e^{-(k_1+k_2)T} \Big] T I^{p-1} G(T) \bigg\}, \\ A_0 &= \frac{1}{\Delta}\bigg\{ \Big[\omega_3 - T k_2 \omega_5 + \mu_7 e^{-k_2 T} \Big] \int_0^T e^{-k_1(T-s)} I^{q-1} F(s)\, ds \\ &\quad - \Big[k_2 \rho_3 + k_2 \rho_2 \rho_9 \Big] T \int_0^T e^{-k_2(T-s)} I^{p-1} G(s) ds + \Big[\omega_{11} + (\omega_2 - T k_2) e^{-k_2 T} \Big] T I^{q-1} F(T) \\ &\quad + \Big[\rho_3 - \mu_2 + T k_2 \omega_9 \Big] T I^{p-1} G(T) \bigg\}, \\ B_0 &= \frac{1}{\Delta}\bigg\{ \Big[-\rho_1 \rho_{10} + \rho_4 \Big] T k_1 \int_0^T e^{-k_1(T-s)} I^{q-1} F(s)\, ds \end{aligned}$$

$$+\Big[\omega_4 - Tk_1\omega_6 + \mu_7 e^{-k_1 T}\Big]\int_0^T e^{-k_2(T-s)} I^{p-1}G(s)ds$$
$$+\Big[\rho_4 - \mu_1 + Tk_1\omega_{10}\Big]TI^{q-1}F(T) + \Big[\omega_{12} + (\omega_1 - Tk_1)e^{-k_1 T}\Big]TI^{p-1}G(T)\Bigg\}.$$

Substituting the above values of A_1, B_1, A_0 and B_0 in (8)-(9), we get the solution (6)–(7). The converse of the lemma can be established by direct computation. This completes the proof. ■

2.1 Main Result

Let us introduce the Banach space $X = \mathbb{C}([0,T],\mathbb{R})$ endowed with the usual norm $\|u\| = \max\{|u(t)|, t \in [0,T]\}$. Then, it is well-known that the product space $X \times X$ equipped with the norm $\|(u,v)\| = \|u\| + \|v\|$ is also a Banach space.

Next, we define an operator $\mathcal{V} : X \times X \to X \times X$ associated with the problem (1)–(2) as

$$\mathcal{V}(u,v)(t) = \Big(\mathcal{V}_1(u,v)(t), \mathcal{V}_2(u,v)(t)\Big), \tag{13}$$

where

$$\begin{aligned}
\mathcal{V}_1(u,v)(t) &= \int_0^t e^{-k_1(t-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} f(\tau,u(\tau),v(\tau))d\tau ds \\
&+\nu_1(t)\int_0^T e^{-k_1(T-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} f(\tau,u(\tau),v(\tau))d\tau\ ds \\
&+\nu_2(t)\int_0^T e^{-k_2(T-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} g(\tau,u(\tau),v(\tau))d\tau\ ds \\
&+\nu_3(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,u(s),v(s))\ ds \\
&+\nu_4(t)\int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)} g(s,u(s),v(s))\ ds,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{V}_2(u,v)(t) &= \int_0^t e^{-k_2(t-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} g(\tau,u(\tau),v(\tau))d\tau ds \\
&+\nu_5(t)\int_0^T e^{-k_1(T-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} f(\tau,u(\tau),v(\tau))d\tau\ ds \\
&+\nu_6(t)\int_0^T e^{-k_2(T-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} g(\tau,u(\tau),v(\tau))d\tau\ ds \\
&+\nu_7(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s,u(s),v(s))\ ds \\
&+\nu_8(t)\int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)} g(s,u(s),v(s))\ ds.
\end{aligned} \tag{14}$$

In the forthcoming analysis, we need the following assumptions.

(H_1) There exist real constants $a_i, b_i > 0,\ i = 1,2$, and $a_0, b_0 > 0$ such that

$$|f(t,x,y)| \le a_0 + a_1|x| + a_2|y|, \quad |g(t,x,y)| \le b_0 + b_1|x| + b_2|y|, \quad \forall t \in [0,T], \quad x,y \in \mathbb{R}.$$

(H_2) The functions $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the Lipschitz condition with Lipschitz constants $\kappa_i, \overline{\kappa}_i, i = 1,2$:

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le \kappa_1|u_1 - v_1| + \kappa_2|u_2 - v_2|, \forall\ u_i, v_i \in \mathbb{R},\ i = 1,2,$$

$$|g(t,u_1,u_2) - g(t,v_1,v_2)| \le \overline{\kappa}_1|u_1 - v_1| + \overline{\kappa}_2|u_2 - v_2|, \forall\ u_i, v_i \in \mathbb{R},\ i = 1,2.$$

For computational convenience, we introduce the notation:

$$\begin{array}{l} S_1 = \max_{t\in[0,T]}\left\{\frac{t^{q-1}(1-e^{-k_1 t})}{k_1\Gamma(q)} + \frac{|\nu_1(t)|T^{q-1}(1-e^{-k_1 T})}{k_1\Gamma(q)} + \frac{|\nu_3(t)|T^{q-1}}{\Gamma(q)}\Big]\right\}, \\ S_2 = \max_{t\in[0,T]}\left\{\frac{|\nu_2(t)|T^{p-1}(1-e^{-k_2 T})}{k_2\Gamma(p)} + \frac{|\nu_4(t)|T^{p-1}}{\Gamma(p)}\right\}, \\ S_3 = \max_{t\in[0,T]}\left\{\frac{t^{p-1}(1-e^{-k_2 t})}{k_2\Gamma(p)} + \frac{|\nu_6(t)|T^{p-1}(1-e^{-k_2 T})}{k_2\Gamma(p)} + \frac{|\nu_8(t)|T^{p-1}}{\Gamma(p)}\right\}. \\ S_4 = \max_{t\in[0,T]}\left\{\frac{|\nu_5(t)|T^{q-1}(1-e^{-k_1 T})}{k_1\Gamma(q)} + \frac{|\nu_7(t)|T^{q-1}}{\Gamma(q)}\right\}, \end{array} \tag{15}$$

$$S_0 = \min\left\{1 - [a_1(S_1+S_4) + b_1(S_2+S_3)], 1 - [a_2(S_1+S_4) + b_2(S_2+S_3)]\right\}. \tag{16}$$

Now, the platform is set for presenting the the main results. In our first result, we establish the existence of at least one solution for the problem (1)–(2) by applying Leray-Schauder's alternative.

Lemma 2 (Leray-Schauder's alternative [28]) *Let $F : E \to E$ be a completely continuous operator (that is, a continuous map F restricted to any bounded set in E is compact). Let $\vartheta(F) = \{x \in E : x = \lambda F(x), 0 < \lambda < 1\}$. Then, either the set $\vartheta(F)$ is bounded or F has at least one fixed point.*

Theorem 1 *Let $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions satisfying the condition (H_1) and the following inequalities hold:*

$$a_1(S_1+S_4) + b_1(S_2+S_3)] < 1, \qquad a_2(S_1+S_4) + b_2(S_2+S_3) < 1,$$

where S_1, S_2, S_3 and S_4 are given in (15). Then, the problem (1)–(2) has at least one solution on $[0,T]$.

Proof. In the first step, we show that the operator $\mathcal{V} : X \times X \to X \times X$ defined by (13) is completely continuous. Observe that continuity of functions f and g implies that the operator $\mathcal{V}$ is continuous. Let us consider a bounded set

$$\Omega = \{(u,v) \in X \times X : \|(u,v)\| \le r\} \subset X \times X,$$

where r is a fixed number. Then, we have

$$\begin{aligned} |f(t,x,y) &\le a_0 + a_1|x| + a_2|y| \\ &\le a_0 + a_1(\|x\| + \|y\|) + a_2(\|x\| + \|y\|) \\ &\le a_0 + (a_1 + a_2)r = L_1. \end{aligned}$$

Likewise, we have $|g(t,x,y)| \le b_0 + (b_1 + b_2)r = L_2$. For any $(u,v) \in \Omega$, we have

$$\begin{aligned} \|\mathcal{V}_1(u,v)\| \le \max_{t\in[0,T]}\Bigg\{ &\int_0^t e^{-k_1(t-s)} \int_0^s \frac{(t-s)^{q-2}}{\Gamma(q-1)}|f(\tau,u(\tau),v(\tau))|\,d\tau\,ds \\ &+|\nu_1(t)|\int_0^T e^{-k_1(T-s)}\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,u(\tau),v(\tau))|d\tau\,ds \\ &+|\nu_2(t)|\int_0^T e^{-k_2(T-s)}\int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)}|g(\tau,u(\tau),v(\tau))|\,d\tau\,ds \\ &+|\nu_3(t)|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s,u(s),v(s))|\,ds + |\nu_4(t)|\int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)}|g(s,u(s),v(s))|\,ds\Big]\Bigg\} \end{aligned}$$

$$\begin{aligned}
&\leq L_1 \max_{t\in[0,T]} \left\{ \int_0^t e^{-k_1(t-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\, d\tau\, ds + |\nu_1(t)| \int_0^T e^{-k_1(T-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\, d\tau\, ds \right.\\
&\left. +\nu_3(t)| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} ds \right\} + L_2 \max_{t\in[0,T]} \left\{ |\nu_2(t)| \int_0^T e^{-k_2(T-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} d\tau\, ds \right.\\
&\left. +|\nu_4(t)| \int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)} ds \right\}\\
&\leq L_1 \max_{t\in[0,T]} \left\{ \frac{t^{q-1}(1-e^{-k_1 t})}{k_1\Gamma(q)} + \frac{|\nu_1(t)| T^{q-1}(1-e^{-k_1 T})}{k_1\Gamma(q)} + \frac{|\nu_3(t)| T^{q-1}}{\Gamma(q)} \right\}\\
&+L_2 \max_{t\in[0,T]} \left\{ \frac{|\nu_2(t)| T^{p-1}(1-e^{-k_2 T})}{k_2\Gamma(p)} + \frac{|\nu_4(t)| T^{p-1}}{\Gamma(p)} \right\}\\
&\leq L_1 S_1 + L_2 S_2,
\end{aligned}$$

where S_1 and S_2 are given in (15). Similarly, one can obtain

$$\|\mathcal{V}_2(u,v)\| \leq L_1 S_4 + L_2 S_3,$$

where S_3 and S_4 are given in (15). Thus, it follows from the above inequalities that the operator $\mathcal{V}(\Omega)$ in uniformly bounded.

Next, we show that $\mathcal{V}(\Omega)$ is equicontinuous. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then, we have

$$\begin{aligned}
&|\mathcal{V}_1(u(t_2), v(t_2)) - \mathcal{V}_1(u(t_1), v(t_1))|\\
\leq\ & \int_0^{t_1} \left| e^{-k_1(t_2-s)} - e^{-k_1(t_1-s)} \right| \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} |f(\tau, u(\tau), v(\tau))|\, d\tau\, ds\\
&+ \int_{t_1}^{t_2} e^{-k_1(t_2-t_1)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} |f(s, u(s), v(s)| ds\\
&+ \left| e^{-k_1 t_2} - e^{-k_1 t_1} \right| \left\{ \left| \omega_3 - T k_2 \omega_5 + \mu_7 e^{-k_2 T} \right| \int_0^T e^{-k_1(T-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} |f(\tau, u(\tau), v(\tau))|\, d\tau\, ds \right.\\
&+ \left| \rho_3 + \rho_9 \rho_2 \right| \int_0^T e^{-k_2(T-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} |g(\tau, u(\tau), v(\tau)) d\tau\, ds\\
&+ \left| \omega_{11} + (\omega_2 - T k_2) e^{-k_2 T} \right| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} |f(s, u(s), v(s)| ds\\
&\left. + \left| \rho_3 - \mu_2 + T k_2 \omega_9 \right| \int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)} |g(s, u(s), v(s)| ds \right\}\\
\leq\ & L_1 \left\{ \int_0^{t_1} \left| e^{-k_1(t_2-s)} - e^{-k_1(t_1-s)} \right| \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\, d\tau\, ds + \int_{t_1}^{t_2} e^{-k_1(t_2-t_1)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} ds \right\}\\
&+ \left| e^{-k_1 t_2} - e^{-k_1 t_1} \right| \left\{ L_1 \left(\left| \omega_3 - T k_2 \omega_5 + \mu_7 e^{-k_2 T} \right| \int_0^T e^{-k_1(T-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} |\, d\tau\, ds \right.\right.\\
&\left. + \left| \omega_{11} + (\omega_2 - T k_2) e^{-k_2 T} \right| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} ds \right)\\
&\left. + L_2 \left(\left| \rho_3 + \rho_9 \rho_2 \right| \int_0^T e^{-k_2(T-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} d\tau\, ds + \left| \rho_3 - \mu_2 + T k_2 \omega_9 \right| \int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)} ds \right) \right\},
\end{aligned}$$

and

$$
\begin{aligned}
& |\mathcal{V}_2(u(t_2), v(t_2)) - \mathcal{V}_2(u(t_1), v(t_1))| \\
\leq\ & L_2\bigg\{ \int_0^t \Big|e^{-k_2(t_2-s)} - e^{-k_2(t_1-s)}\Big| \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} d\tau\ ds + \int_{t_1}^{t_2} e^{-k_2(t_2-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} d\tau\ ds \bigg\} \\
& + \Big|e^{-k_2 t_2} - e^{-k_2 t_1}\Big| \bigg[L_2\bigg(\Big|\omega_4 - Tk_1\omega_6 + \mu_7 e^{-k_1 T}\Big| \int_0^T e^{-k_2(T-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} d\tau\ ds \\
& + \Big|\omega_{12} + (\omega_1 - Tk_1)e^{-k_1 T}\Big| \int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)}\ ds \bigg) \\
& + L_1\bigg(\Big|\rho_4 - \rho_1\rho_{10}\Big| \int_0^T e^{-k_1(T-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} d\tau\ ds + \Big|\rho_4 - \mu_1 + k_1\omega_{10}\Big| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} ds \bigg) \bigg].
\end{aligned}
$$

Clearly, the right-hand sides of the above two inequalities tend to zero as $t_2 - t_1 \to 0$ independent of $(u, v) \in \Omega$. Therefore, the operators $\mathcal{V}_1(\Omega)$ and $\mathcal{V}_2(\Omega)$ are equicontinuous and hence the operator $\mathcal{V}(\Omega)$ is equicontinuous. Thus, by Arzelá-Ascoli theorem, we deduce that the operator $\mathcal{V}(\Omega)$ is completely continuous.

Finally, it will be established that the set $\varepsilon(F) = \{x \in E : x = \lambda F(x), 0 < \lambda < 1\}$ is bounded. Let $(u, v) \in \varepsilon$, then $(u, v) = \lambda\mathcal{V}(u, v)$. For any $t \in [0, T]$, we have $u(t) = \lambda\mathcal{V}_1(u, v)(t),\ v(t) = \lambda\mathcal{V}_2(u, v)(t)$. Then, in view of the assumption (H_1), we obtain

$$
\begin{aligned}
|u(t)| \leq\ & |\mathcal{V}_1(u, v)| \\
\leq\ & \int_0^t e^{-k_1(t-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} \Big[a_0 + a_1|u| + a_2|v|\Big] d\tau\ ds \\
& + \nu_1(t)| \int_0^T e^{-k_1(T-s)} \int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)} \Big[a_0 + a_1|u| + a_2|v|\Big] d\tau\ ds \\
& + |\nu_2(t)| \int_0^t e^{-k_2(T-s)} \int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)} \Big[b_0 + b_1|u| + b_2|v|\Big] d\tau\ ds \\
& + |\nu_3(t)| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} \Big[a_0 + a_1|u| + a_2|v|\Big]\ ds \\
& + |\nu_4(t)| \int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)} \Big[b_0 + b_1|u| + b_2|v|\Big]\ ds,
\end{aligned}
$$

which, on taking maximum for $t \in [0, T]$ and using (15), yields

$$\|u\| \leq [a_0 + a_1\|u\| + a_2\|v\|]S_1 + [b_0 + b_1\|u\| + b_2\|v\|]S_2. \tag{17}$$

In a similar way, we can find that

$$\|v\| \leq [b_0 + b_1\|u\| + b_2\|v\|]S_3 + [a_0 + a_1\|u\| + a_2\|v\|]S_4. \tag{18}$$

From (17)–(18), we obtain

$$\|u\| + \|v\| \leq \frac{(a_0(S_1 + S_4) + b_0(S_2 + S_3)}{S_0},$$

where S_0 is defined by (16). Consequently, we get

$$\|(u, v)\| \leq \frac{(a_0(S_1 + S_4) + b_0(S_2 + S_3)}{S_0},$$

which shows that the set ε is bounded. Thus, by Lemma 2, the operator $\mathcal{V}$ has at least one solution on $[0, T]$. The proof is complete. ■

We make use of Banach's contraction mapping principle [28] to establish the existence of a unique solution to the problem (1)–(2).

Theorem 2 *Let $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions and (H_2) holds. Then, there exists a unique solution to the problem (1)–(2) on $[0,T]$, provided that*

$$(\kappa_1+\kappa_2)(S_1+S_4)+(\overline{\kappa}_1+\overline{\kappa}_2)(S_2+S_3)<1, \tag{19}$$

where S_1, S_2, S_3 and S_4 are given in (15).

Proof. Fixing $\sup_{t\in[0,T]} f(t,0,0) = N_1 < \infty$, $\sup_{t\in[0,T]} g(t,0,0) = N_2 < \infty$, and using the assumption (H_2), we obtain

$$\begin{aligned} |f(t,u(t),v(t)| &= |f(t,u(t),v(t)-f(t,0,0)+f(t,0,0)| \le \kappa_1\|u\|+\kappa_2\|v\|+N_1,\\ |g(t,u(t),v(t))| &= |g(t,u(t),v(t)-g(t,0,0)+g(t,0,0)| \le \overline{\kappa}_1\|u\|+\overline{\kappa}_2\|v\|+N_2. \end{aligned} \tag{20}$$

Now, we consider a closed ball $B_r = \{(u,v)\in X\times X : \|(u,v)\| \le r\}$, where

$$\frac{N_1(S_1+S_4)+N_2(S_2+S_3)}{1-[(\kappa_1+\kappa_2)(S_1+S_4)+(\overline{\kappa}_1+\overline{\kappa}_2)(S_2+S_3)]} \le r, \tag{21}$$

and show that $\mathcal{V}(B_r)\subset B_r$. For $(u,v)\in B_r$, it follows by using (20) that

$$\begin{aligned} |\mathcal{V}_1(u,v)(t)| \le\ & \int_0^t e^{-k_1(t-s)}\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\Big[\kappa_1|u|+\kappa_2|v|+N_1\Big]d\tau\ ds\\ &+|\nu_1(t)|\int_0^T e^{-k_1(T-s)}\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}\Big[\kappa_1|u|+\kappa_2|v|+N_1\Big]d\tau\ ds\\ &+|\nu_2(t)|\int_0^T e^{-k_2(T-s)}\int_0^s \frac{(s-\tau)^{p-1}}{\Gamma(p-1)}\Big[\overline{\kappa}_1|u|+\overline{\kappa}_2|v|+N_2\Big]\ ds\\ &+|\nu_3(t)|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\Big[\kappa_1|u|+\kappa_2|v|+N_1\Big]\ ds\\ &+|\nu_4(t)|\int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)}\Big[\overline{\kappa}_1|u|+\overline{\kappa}_2|v|+N_2\Big]\ ds\Bigg\}, \end{aligned}$$

which, on taking the norm for $t\in[0,T]$, yields

$$\|\mathcal{V}_1(u,v)\| \le [(\kappa_1+\kappa_2)r+N_1]S_1+[(\overline{\kappa}_1+\overline{\kappa}_2)r+N_2]S_2.$$

In the same way, we can find that

$$\|\mathcal{V}_2(u,v)\| \le [(\overline{\kappa}_1+\overline{\kappa}_2)r+N_2]S_3+[(\kappa_1+\kappa_2)r+N_1]S_4.$$

From the above two inequalities together with (21), we find that $\|\mathcal{V}(u,v)\|\le r$, that is $\mathcal{V}(u,v)\in B_r$. Hence, $\mathcal{V}(B_r)\subset B_r$.

Next, we show that the operator $\mathcal{V}$ is a contraction. For that, let $(u_2,v_2),(u_1,v_1)\in X\times X$. Then, for any $t\in[0,T]$, we get

$$\begin{aligned} &\|\mathcal{V}_1(u_2,v_2)-\mathcal{V}_1(u_1,v_1)\|\\ \le\ & \max_{t\in[0,T]}\Bigg\{\int_0^t e^{-k_1(t-s)}\int_0^s \frac{(t-s)^{q-2}}{\Gamma(q-1)}|f(\tau,u_2(\tau),v_2(\tau))-f(\tau,u_1(\tau),v_1(\tau))|d\tau\ ds\\ &+|\nu_1(t)|\int_0^T e^{-k_1(T-s)}\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}|f(\tau,u_2(\tau),v_2(\tau))-f(\tau,u_1(\tau),v_1(\tau))|d\tau\ \ ds\\ &+|\nu_2(t)|\int_0^T e^{-k_2(T-s)}\int_0^s \frac{(s-\tau)^{p-2}}{\Gamma(p-1)}|g(\tau,u_2(\tau),v_2(\tau))-g(\tau,u_1(\tau),v_1(\tau))|d\tau\ ds \end{aligned}$$

$$
\begin{aligned}
&+|\nu_3(t)|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s,u_2(s),v_2(s))-f(s,u_1(s),v_1(s))|\,ds\\
&+|\nu_4(t)|\int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)}|g(s,u_2(s),v_2(s))-g(s,u_1(s),v_1(s))|ds\Big]\Bigg\}\\
\leq\ & (\kappa_1\|u_2-u_1\|+\kappa_2\|v_2-v_1\|)\max_{t\in[0,T]}\Bigg\{\int_0^t e^{-k_1(t-s)}\int_0^s \frac{(t-s)^{q-2}}{\Gamma(q-1)}d\tau\,ds\\
&+|\nu_1(t)|\int_0^T e^{-k_1(T-s)}\int_0^s \frac{(s-\tau)^{q-2}}{\Gamma(q-1)}d\tau\,ds+|\nu_3(t)|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\,ds\Bigg\}\\
&+(\overline{\kappa_1}\|u_2-u_1\|+\overline{\kappa}_2\|v_2-v_1\|)\max_{t\in[0,T]}\Bigg\{|\nu_2(t)|\int_0^T e^{-k_2(T-s)}\int_0^s \frac{(T-s)^{p-2}}{\Gamma(p-1)}d\tau\,ds\\
&+|\nu_4(t)|\int_0^T \frac{(T-s)^{p-2}}{\Gamma(p-1)}\,ds\Bigg\}\\
\leq\ & (\kappa_1\|u_2-u_1\|+\kappa_2\|v_2-v_1\|)S_1+(\overline{\kappa}_1\|u_2-u_1\|+\overline{\kappa}_2\|v_2-v_1\|)S_2,
\end{aligned}
$$

which implies that

$$\|\mathcal{V}_1(u_2,v_2)-\mathcal{V}_1(u_1,v_1)\|\leq\Big[(\kappa_1+\kappa_2)S_1+(\overline{\kappa}_1+\overline{\kappa}_2)S_2\Big](\|u_2-u_1\|+\|v_2-v_1\|).$$

In a similar manner, one can obtain that

$$\|\mathcal{V}_2(u_2,v_2)-\mathcal{V}_2(u_1,v_1)\|\leq\Big[(\kappa_1+\kappa_2)S_4+(\overline{\kappa}_1+\overline{\kappa}_2)S_3\Big](\|u_2-u_1\|+\|v_2-v_1\|).$$

Therefore, we have

$$\|\mathcal{V}(u_2,v_2)-\mathcal{V}(u_1,v_1)\|\leq\Big[(\kappa_1+\kappa_2)(S_1+S_4)+(\overline{\kappa}_1+\overline{\kappa}_2)(S_2+S_3)\Big](\|u_2-u_1\|+\|v_2-v_1\|),$$

which, in view of (19), shows that the operator $\mathcal{V}$ is a contraction. Hence, it follows by Banach's contraction mapping principle that the operator $\mathcal{V}$ has a unique fixed point, which is indeed a unique solution to the problem (1)–(2). This completes the proof. ■

3 Examples

Consider the following system of coupled sequential fractional differential equations equipped with closed boundary conditions:

$$
\begin{cases}
({}^cD^{1.57}+9/5\ {}^cD^{0.57})x(t)=f(t,x(t),y(t)), \qquad t\in J:=[0,2],\\
({}^cD^{1.4}+2/3\ {}^cD^{0.4})y(t)=g(t,x(t),y(t)),\\
x(2)=y(0)+10/9y'(0),\quad 2x'(2)=-3/8y(0)+8/9y'(0),\\
y(2)=-2/5x(0)-14/3x'(0),\quad 2y'(2)=7/2x(0)-2x'(0).
\end{cases}
\tag{22}
$$

Here, $T=2$, $q=1.57$, $p=1.4$, $k_1=9/5$, $k_2=2/3$, $\alpha_1=1$, $\alpha_2=-2/5$, $\beta_1=5/9$, $\beta_2=-7/3$, $\delta_1=4/9$, $\delta_2=-1$, $\gamma_1=-3/8$, $\gamma_2=7/2$ and f,g will be defined later. With the given data, it is found that $S_1=1.669994615$, $S_2=1.553972248$, $S_3=3.089558371$ and $S_4=5.711755201$ (S_1, S_2, S_3, and S_4 are given in (15).

(i) (Illustration of Theorem 1). Let us consider

$$
\begin{aligned}
f(t,x(t),y(t)) &= \tfrac{1}{(2t+4)^4}\tfrac{|x|x^2}{1+x^2} + \tfrac{7}{40}\sin y + \tfrac{1}{t^2+144},\\
g(t,x(t),y(t)) &= \tfrac{e^{-t}}{60}x\tan^{-1}x + \tfrac{y^2}{75(1+|y|)} + \tfrac{1}{\sqrt{t^2+400}}.
\end{aligned} \tag{23}
$$

Observe that $a_0 = 1/144$, $a_1 = 1/256$, $a_2 = 7/40$, $b_0 = 1/20$, $b_1 = \pi/120$, $b_2 = 1/75$ as

$$|f(t,x(t),y(t))| \le \frac{1}{144} + \frac{1}{256}|x| + \frac{7}{40}|y|, \quad |g(t,x(t),y(t))| \le \frac{1}{20} + \frac{\pi}{120}|x| + \frac{1}{75}|y|.$$

In addition,

$$a_1(S_1+S_4) + b_1(S_2+S_3) = 0.1254735973 < 1, \quad a_2(S_1+S_4) + b_2(S_2+S_3) = 0.1883333333 < 1.$$

Thus, by the conclusion of Theorem 1, the problem (22) with f and g given by (23) has at least one solution on $[0,2]$.

(ii) (Illustration of Theorem 2). In this case, we take

$$
\begin{aligned}
f(t,x(t),y(t)) &= \tfrac{1}{(t+4)^3}\tfrac{|x|}{1+|x|} + \tfrac{7}{60}\sin y + \tfrac{\cos t}{\sqrt{t^2+256}},\\
g(t,x(t),y(t)) &= \tfrac{e^{-t}}{50}\tan^{-1}x + \tfrac{1}{(t+5)^2}\cos y + \tfrac{t+1}{t^2+75},
\end{aligned} \tag{24}
$$

and note that

$$
\begin{aligned}
|f(t,x_2,y_2) - f(t,x_1,y_1)| &\le \frac{1}{64}|x_2-x_1| + \frac{7}{60}|y_2-y_1|,\\
|g(t,x_2,y_2) - g(t,x_1,y_1)| &\le \frac{1}{50}|x_2-x_1| + \frac{1}{25}|y_2-y_1|.
\end{aligned}
$$

From the above inequalities, it follows that $\kappa_1 = \frac{1}{64}$, $\kappa_2 = \frac{7}{60}$, $\overline{\kappa}_1 = \frac{1}{50}$, $\overline{\kappa}_2 = \frac{1}{25}$. Further, we have

$$[(\kappa_1+\kappa_2)(S_1+S_4) + (\overline{\kappa}_1+\overline{\kappa}_2)(S_2+S_3)] = 0.1922916667 < 1.$$

Thus, the hypothesis of Theorem 2 is satisfied and consequently, the problem (22) with f and g given by (24) has a unique solution on $[0,2]$.

3.1 Conclusions

A new boundary value problem involving coupled nonlinear sequential fractional differential equations supplemented with closed boundary conditions has been studied in this paper. We apply Leray-Schauder's alternative to prove an existence result for the given problem, while Banach's contraction mapping principle is used to establish the uniqueness of solutions to the problem at hand. Several results follow from the obtained ones as special cases by fixing the parameters appearing in the boundary conditions. Some of these results are listed below.

(i) By setting $\alpha_1 = 1 = \alpha_2$, $\beta_1 = 0 = \beta_2$, $\gamma_1 = 0 = \gamma_2$ and $\delta_1 \ne 1 \ne \delta_2$, we obtain the results for a nonlinear sequential fractional-order coupled system (1) with semi-periodic coupled boundary conditions of the form: $x(0) = y(T)$, $x'(T) = \delta_1 y'(0)$, $y(0) = x(T)$, and $y'(T) = \delta_2 x'(0)$;

(ii) Letting $\beta_1 = 0 = \beta_2$, $\gamma_1 = 0 = \gamma_2$ and by choosing $\alpha_1 = 1 = \alpha_2$ and $\delta_1 = -1 = \delta_2$, our results specialize to the ones for semi-periodic coupled boundary conditions of the form: $x(0) = y(T)$, $x'(T) = -y'(0)$, $y(0) = x(T)$, and $y'(T) = -x'(0)$;

(iii) Our results correspond to the combination of coupled periodic and anti-periodic boundary conditions of the form: $x(0) = y(T)$, $x'(0) = y'(T)$, $y(0) = -x(T)$, $y'(0) = -x'(T)$ for $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = 0 = \beta_2$, $\gamma_1 = 0 = \gamma_2$ and $\delta_1 = -1$, $\delta_2 = 1$;

(iv) Our results become the ones associated with the boundary conditions: $x(0) = -y(T)$, $x'(0) = -y'(T)$, $y(0) = x(T)$, $y'(0) = x'(T)$ when $\alpha_1 = 1$, $\alpha_2 = -1$, $\beta_1 = 0 = \beta_2$, $\gamma_1 = 0 = \gamma_2$ and $\delta_1 = 1$, $\delta_2 = -1$.

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