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Best Proximity Point Results For Extended Proximal Cyclic α-Contraction Mappings in C^*-Algebra Valued Metric Space With Applications*

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Abstract

In this paper, we introduce a new class of mappings in C^*-algebra valued metric space which generalizes the class of cyclic α-contraction mappings. Some best proximity point as well as common best proximity point results are established considering such mappings. Practical applications of the derived results are demonstrated in solving matrix equation and nonlinear Volterra integral equation of convolution type.

1 Introduction

The concept of fixed point was introduced by Brouwer [8] in 1910, based on the foundational research of Poincare and Picard in the study of differential equations. Subsequently, Birkhoff and Kellogg [6], along with Schauder [31], Banach [4] and Kannan [16], further expanded and developed the theory of fixed points, with its applications in infinite-dimensional spaces and Banach spaces. In 1971, Reich [28] established a fixed point result that is more generalized than Banach's and Kannan's fixed point theorems. Extensive study in this area is going on by different prominent researchers. Nevertheless, the entirety of fixed point theory primarily focused on self-mappings. However, investigations for non-self-mappings paved the way for the emergence of the best proximity point theory. In 1969, a best approximation theorem was introduced by Fan [12], and since then several researchers ([1], [2], [3], [5], [13], [15], [23], [27], [29], [25], [30]) have studied the theory of best proximity point in different spaces like metric space, Banach space, partial metric space, partial b-metric space etc.

The notion of C^*-algebra valued metric space was introduced by Ma et al. [18] in 2014 with several results on fixed point of mappings and their applications. In 2016, Kamran et al. [15] showed that a C^*-algebra valued metric space is a C^*-algebra valued b-metric space but the converse does not hold. They obtained some results on fixed point in C^*-algebra valued metric space with an application. Later Xin et al. [33] derived some results on coincidence point and common fixed point in a complete C^*-algebra valued metric space involving some contractive conditions. In 2017, Mondal et al. [22] proved the existence and uniqueness of common fixed points for discontinuous self mappings with expansive conditions and deduced some best proximity point theorems in C^*-algebra valued metric space with an application to integral equation. Shen et al. [32] introduced the concept of C^*-algebra valued G-metric space and proved some interesting results involving fixed points for self mappings with contractive or expansive conditions and gave an application to a second order differential equation. Again in 2019, Chandok et al. [9] defined the notion of C^*-algebra valued partial metric space and did some investigation of fixed point with examples. Omran and Masmali [26] introduced α-admissible continuous mappings in C^*-algebra valued b-metric space using Lipschitz contraction and gave some non-trivial examples and applications. Recently, Bouftouh et al. [7] defined C^*-algebra valued

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asymmetric metric space and the notion of forward and backward C^*-algebra valued asymmetric contractions with establishment of fixed point results and applications. Also, Mani et al. [20] derived some fixed point results for generalized contraction in C^*-algebra valued partial b-metric space and gave application to the Fredholm integral equation.

Motivated by the promising outcomes of these works and recognizing the broad utility of the concept of best proximity point, in this paper, we develop a new class of mappings on a C^*-algebra valued metric space with the help of a specific property. We study best proximity point with respect to a C^*-algebra valued metric for such mappings. Moreover, considering the vast applicability of fixed point theory in different practical fields including medical diagnosis, we have shown the application of our derived results in solving matrix equation and Volterra integral equation of convolution type occurred in the SIR model for endemic infectious diseases.

2 Preliminaries

Throughout the paper, $\mathbb{A}$ denotes a C^*-algebra and $\mathbb{A}_h$ denotes the set of all Hermitian elements of $\mathbb{A}$. An element $\xi \in \mathbb{A}$ is called a positive element of $\mathbb{A}$ and denoted by $\theta \preccurlyeq \xi$ (θ being the zero element of $\mathbb{A}$) if $\xi \in \mathbb{A}_h$ and $\sigma(\xi) \subset [0, \infty)$, where $\sigma(\xi)$ is the spectrum of ξ.

A partial ordering on $\mathbb{A}$ is defined by $\xi \preccurlyeq \eta$ if and only if $\theta \preccurlyeq \eta - \xi$. The set $\{\xi \in \mathbb{A} : \theta \preccurlyeq \xi\}$ is denoted by $\mathbb{A}^+$ and we denote $|\xi|$ as $(\xi^*\xi)^{\frac{1}{2}}$ (refer to [7, 19, 20]). Let $\mathbb{A}'^+$ be the set $\{\xi \in \mathbb{A}^+ : \xi\eta = \eta\xi \text{ for all } \eta \in \mathbb{A}\}$. A mapping $\alpha : \mathbb{A}^+ \to \mathbb{A}^+$ is said to be strictly increasing with respect to " $\preccurlyeq$ " if and only if $\xi \preccurlyeq \eta$ implies $\alpha(\xi) \preccurlyeq \alpha(\eta)$.

Lemma 1 ([17]) *Let $\mathbb{A}$ be a C^*-algebra with unit I and let $a, b \in \mathbb{A}$.*

(i) If a is self-adjoint, then $a \preccurlyeq ||a||I$.

(ii) If $\theta \preccurlyeq a \preccurlyeq b$, then $||a|| \leq ||b||$.

Lemma 2 ([11, 24]) *Suppose that $\mathbb{A}$ is a unital C^*-algebra with unit element I.*

(i) For any $\xi \in \mathbb{A}^+$, $\xi \preccurlyeq I$ if and only if $||\xi|| \leq 1$.

(ii) If $a \in \mathbb{A}^+$ with $||a|| \leq \frac{1}{2}$, then $I - a$ is invertible and $||a(I - a)^{-1}|| < 1$.

(iii) Suppose that $a, b \in \mathbb{A}$ with $\theta \preccurlyeq a, b$ and $ab = ba$, then $\theta \preccurlyeq ab$.

(iv) Suppose that $\mathbb{A}' = \{a \in \mathbb{A} : ab = ba \text{ for all } b \in \mathbb{A}\}$. Let $a \in \mathbb{A}'$. If $b, c \in \mathbb{A}'$ with $\theta \preccurlyeq c \preccurlyeq b$ and $I - a$ is a positive element in $\mathbb{A}'$, then $(I - a)^{-1}c \preccurlyeq (I - a)^{-1}b$.

Replacing the set of non-negative real numbers in the definition of metric space by the set of positive elements of a C^*-algebra, Ma et al. [18] introduced the following definition of a C^*-algebra valued metric space considering the partial ordering " $\preccurlyeq$ " on $\mathbb{A}$.

Definition 1 ([18]) *Let X be a nonempty set and $\mathbb{A}$ be a C^*-algebra. Suppose that the mapping $d^* : X \times X \to \mathbb{A}^+$ satisfies:*

(i) $\theta \preccurlyeq d^*(\xi, \eta)$ for all $\xi, \eta \in X$ and $d^*(\xi, \eta) = \theta$ if and only if $\xi = \eta$;

(ii) $d^*(\xi, \eta) = d^*(\eta, \xi)$ for all $\xi, \eta \in X$; and

(iii) $d^*(\xi, \eta) \preccurlyeq d^*(\xi, \zeta) + d^*(\zeta, \eta)$ for all $\xi, \eta, \zeta \in X$.

Then d is called a C^*-algebra valued metric on X and $(X, \mathbb{A}, d^*)$ is called a C^*-algebra valued metric space.

Example 1 (**[33]**) *Let $X = [0,1]$, $\mathbb{A} = \mathbb{M}_2(\mathbb{R})$ and the C^*-algebra of bounded linear operators on the Hilbert space $\mathbb{R}^2$ with norm $||M||_\infty = \max_{1\le i\le 2}\sum_{j=1}^{2}|a_{ij}|$, where $M = (a_{ij}) \in \mathbb{A}$ and $M^* = (\overline{a_{ji}})$. Define $d^* : X \times X \to \mathbb{A}^+$ by*

$$d^*(\xi,\eta) = \begin{bmatrix} |\xi-\eta| & 0 \\ 0 & 2|\xi-\eta| \end{bmatrix}$$

where $\xi,\eta \in \mathbb{R}$. Then, $(X,\mathbb{A},d^*)$ is a C^*-algebra valued metric space. For some other examples of C^*-algebra valued metric space we refer to [18] and [32].

Definition 2 (**[18]**) *Let $(X,\mathbb{A},d^*)$ be a C^*-algebra valued metric space and $\{\xi_n\} \subset X$. If for any $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $||d^*(\xi_n,\xi)|| \le \epsilon$, then $\{\xi_n\}$ is said to be convergent to ξ with respect to $\mathbb{A}$ and it is denoted by $\lim_{n\to\infty}\xi_n = \xi$.*

If for any $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $m,n > n_0$, $||d^*(\xi_n,\xi_m)|| \le \epsilon$, then $\{\xi_n\}$ is said to be a Cauchy sequence with respect to $\mathbb{A}$. If every Cauchy sequence $\{\xi_n\}$ with respect to $\mathbb{A}$ is convergent, then the C^*-algebra valued metric space is said to be complete.

A subset Q of a C^*-algebra valued metric space $(X,\mathbb{A},d^*)$ is said to be sequentially compact if every sequence in Q has a convergent subsequence with respect to $\mathbb{A}$.

In [22], Mondal et al. gave the following definition of best proximity point in C^*-algebra valued metric space using a norm of the set X.

Definition 3 (**[22]**) *Let P and Q be two nonempty subsets of a C^*-algebra valued metric space $(X,\mathbb{A},d^*)$. For a mapping $T : P \to Q$, a point $\xi \in P$ is said to be a best proximity point of T, if it satisfies $d^*(\xi,T\xi) = d^*(P,Q)$, where*

$$d^*(P,Q) = \inf\{d^*(\xi,\eta) \in \mathbb{A}^+ : \xi \in P, \eta \in Q \text{ and } d^*(\xi,\eta) = ||\xi-\eta||I\}.$$

In 2009, Al-Thagafi et al. [2] introduced the cyclic α-contraction mapping by generalizing the cyclic contraction mappings and derived some existence and convergence results for best proximity points for such mappings.

Definition 4 (**[2]**) *Let P and Q be two nonempty subsets of a metric space (X,d). For a strictly increasing function $\alpha : [0,\infty) \to [0,\infty)$, if the mapping $T : P\cup Q \to P\cup Q$ satisfies the following conditions:*

(i) $T(P) \subseteq Q$ and $T(Q) \subseteq P$; and

(ii) $d(T\xi,T\eta) \le d(\xi,\eta) - \alpha(d(\xi,\eta)) + \alpha(d(P,Q))$ for all $\xi \in P$ and $\eta \in Q$.

Then T is called cyclic α-contraction mapping.

Example 2 (**[2]**) *Let $X = \mathbb{R}$ with the usual metric $d(\xi,\eta) = |\xi-\eta|$ for $\xi,\eta \in \mathbb{R}$. Let $P = Q = [0,1]$ and $T : P\cup Q \to P\cup Q$ be defined by*

$$T\xi = \frac{\xi}{\xi+1} \quad \text{and} \quad \alpha(t) = \frac{t^2}{t+1} \quad \text{for } t \ge 0.$$

Then T is a cyclic α-contraction mapping.

In 1971, Reich [28] extended the Banach and Kannan fixed point theorems as follows:

Theorem 1 (**[28]**) *Let (X,d) be a metric space and $T : X \to X$ be a mapping such that*

$$d(T\xi,T\eta) \le ld(T\xi,\xi) + md(T\eta,\eta) + nd(\xi,\eta)$$

where l,m,n are non-negative and $l+m+n < 1$. Then the mapping T has a unique fixed point.

3 Main Results

First we define the following property for a pair of subsets in C^*-algebra valued metric space. With the help of this, we give a new definition of best proximity point.

Definition 5 ((P-d^*) property) *Let $(X, \mathbb{A}, d^*)$ be a C^*-algebra valued metric space and P, Q be two nonempty subsets of X. The pair (P, Q) is said to satisfy proximal property with respect to d^* ((P-d^*) property) if there exist $p \in P$, $q \in Q$ such that $d^*(p, q) \preccurlyeq d^*(\xi, \eta)$ for all $\xi \in P$, $\eta \in Q$. For convenience, we denote $d^*(p, q)$ by $d_0(P, Q)$.*

Example 3 *Consider $P = (-\infty, 1]$, $Q = [5, \infty)$ and $X = \mathbb{A} = \mathbb{R}$ with the usual norm $||\xi|| = |\xi|$ for all $\xi \in \mathbb{R}$, and C^*-algebra valued metric $d^*(\xi, \eta) = |\xi - \eta|$ for all $\xi, \eta \in \mathbb{R}$. Then for $p = 1$ and $q = 5$, $d^*(1, 5) = 4 \leq d^*(\xi, \eta)$ for all $\xi \in P$, $\eta \in Q$. Thus, (P, Q) satisfies (P-d^*) property and $d_0(P, Q) = 4$.*

Example 4 *Consider $X = l^\infty$ and $\mathbb{A} = \mathbb{R}^2$. Then $\mathbb{A}$ is a C^*-algebra with norm $||(\xi, \eta)|| = (\xi^2 + \eta^2)^{\frac{1}{2}}$ (refer to [21]). Let C^*-algebra valued metric on X be defined by*

$$d^*(\xi, \eta) = (\sup_{j \in \mathbb{N}} |\xi_j - \eta_j|, 0) \text{ for all } \xi = \{\xi_j\},\ \eta = \{\eta_j\} \in l^\infty.$$

Suppose that

$$P = \{e_{2n-1} : n \in \mathbb{N}\} \cup \{\frac{1}{5}e_1\} \quad \text{and} \quad Q = \{e_{2n} : n \in \mathbb{N}\} \cup \{\frac{1}{6}e_{2n} : n \in \mathbb{N}\} \cup \{e_0\},$$

where $e_n = \{0, 0, ..., 1, 0, ...\}$, 1 being in the n^{th} place $(n \in \mathbb{N})$, and $e_0 = \{0, 0, ...\}$. Then for $p = \frac{1}{5}e_1$ and $q = e_0$, $d^*(\frac{1}{5}e_1, e_0) = (\frac{1}{5}, 0) \leq d^*(\xi, \eta)$ for all $\xi \in P$ and $\eta \in Q$. Thus, (P, Q) satisfies (P-d^*) property.

Now we give the following definition of best proximity point with respect to a C^*-algebra valued metric d^* for subsets satisfying (P-d^*) property. It is worth mentioning that this definition does not have the requirement that X should be a normed space.

Definition 6 *Let (P, Q) be a pair of nonempty subsets satisfying (P-d^*) property in a C^*-algebra valued metric space $(X, \mathbb{A}, d^*)$, and $T : P \to Q$ be a mapping. A point $\xi \in P$ is said to be a best proximity point with respect to d^* if it satisfies $d^*(\xi, T\xi) = d_0(P, Q)$.*

It is clear that for $\mathbb{A} = \mathbb{R}$, $d_0(P, Q)$ reduces to $d(P, Q) = \inf\{d(\xi, \eta) : \xi \in P, \eta \in Q\}$, provided the infimum is attained for some $p \in P$, $q \in Q$. In that case, the above definition of best proximity point with respect to d^* reduces to best proximity point in the usual sense.

Example 5 *Taking $(X, \mathbb{A}, d^*)$ and P, Q as in the Example 3, we take the mapping $T : P \to Q$ as*

$$T\xi = \begin{cases} 11, & \xi \in (-\infty, 1), \\ 5, & \xi = 1. \end{cases}$$

Then $d^*(T1, 1) = d^*(5, 1) = d_0(P, Q)$. Hence 1 is a best proximity point of T with respect to d^*.

Example 6 *For the Example 4, we take $T : P \to Q$ as*

$$T\xi = \begin{cases} \frac{1}{6}e_{2n}, & \xi = e_{2n-1},\ n \in \mathbb{N}, \\ e_0, & \xi = \frac{1}{5}e_1. \end{cases}$$

Then $d^*(\frac{1}{5}e_1, T(\frac{1}{5}e_1)) = d^*(\frac{1}{5}e_1, e_0) = (\frac{1}{5}, 0) = d_0(P, Q)$. So, $\frac{1}{5}e_1$ is a best proximity point of T with respect to d^*.

Next we define extended proximal cyclic α-contraction mapping for subsets with $(P\text{-}d^*)$ property.

Definition 7 *Let (P,Q) be a pair of nonempty subsets satisfying $(P\text{-}d^*)$ property in a C^*-algebra valued metric space $(X,\mathbb{A},d^*)$ and $\alpha:\mathbb{A}^+\to\mathbb{A}^+$ be a strictly increasing mapping with respect to the partial ordering "$\preccurlyeq$" on $\mathbb{A}$. A mapping $T:P\cup Q\to P\cup Q$ is said to be extended proximal cyclic α-contraction mapping if the following conditions are satisfied:*

(EC1) $T(P)\subseteq Q$ *and* $T(Q)\subseteq P$;

(EC2) $\alpha(d^*(\xi,\eta))-\alpha(d_0(P,Q))\preccurlyeq d^*(\xi,\eta)$ *for all* $\xi\in P$ *and* $\eta\in Q$; *and*

(EC3) For some $a,b,c\in\mathbb{A}'^+$ *with* $||a||^2+||b||^2+||c||^2\leq\frac{1}{2}$,

$$d^*(T\xi,T\eta)\preccurlyeq\frac{1}{2}(d^*(\xi,\eta)-\alpha(d^*(\xi,\eta))+\alpha(d_0(P,Q)))+a^*d^*(T\xi,\xi)a \\ +b^*d^*(T\eta,\eta)b+c^*d^*(\xi,\eta)c \quad \text{for all } \xi\in P,\eta\in Q. \tag{1}$$

Remark 1 *Considering metric space in place of C^*-algebra valued metric space, it is seen that every cyclic α-contraction mapping is an extended proximal cyclic α-contraction mapping for $a=b=c=\theta$. However, the converse is not true in general, which can be shown by the following example.*

Example 7 *Let $X=\mathbb{A}=\mathbb{R}$ with the usual norm $||\xi||=|\xi|$ for all $\xi\in\mathbb{R}$ and the C^*-algebra valued metric $d^*(\xi,\eta)=|\xi-\eta|$ for all $\xi,\eta\in\mathbb{R}$. Let $P=Q=[0,1]$. Clearly (P,Q) satisfies $(P\text{-}d^*)$ property with $p=0=q$ and $d_0(P,Q)=0$. Suppose that $T:P\cup Q\to P\cup Q$ is defined by*

$$T(\xi)=\begin{cases}\frac{1}{16}, & \xi=1,\\ \frac{1}{8}, & \xi\in[0,1).\end{cases}$$

Let $\alpha:[0,\infty)\to[0,\infty)$ be the identity mapping, and $a=c=\frac{1}{4}$, $b=\frac{1}{8}$. Clearly (EC1) and (EC2) are satisfied. For (EC3),

$$\frac{1}{2}(d^*(\xi,\eta)-\alpha(d^*(\xi,\eta))+\alpha(d_0(P,Q)))=0.$$

Case 1: *If $\xi=1$ and $\eta\in[0,1)$, then*

$$d^*(T\xi,T\eta)=|T\xi-T\eta|=|\frac{1}{16}-\frac{1}{8}|=\frac{1}{16}$$

and

$$\begin{aligned}a^*d^*(T\xi,\xi)a+b^*d^*(T\eta,\eta)b+c^*d^*(\xi,\eta)c&=\frac{1}{4}|T\xi-\xi|\frac{1}{4}+\frac{1}{8}|T\eta-\eta|\frac{1}{8}+\frac{1}{4}|\xi-\eta|\frac{1}{4}\\&=\frac{1}{16}|\frac{1}{16}-1|+\frac{1}{64}|\frac{1}{8}-\eta|+\frac{1}{16}|1-\eta|\\&=\frac{1}{16}.\frac{15}{16}+\frac{1}{64}|\frac{1}{8}-\eta|+\frac{1}{16}|1-\eta|\\&\geq\frac{1}{16},\ \text{for all } \eta\in[0,1).\end{aligned}$$

Case 2: *If $\xi=\eta=1$, clearly the condition (EC3) is satisfied.*
Case 3: *If $\xi,\eta\in[0,1)$,*

$$d^*(T\xi,T\eta)=0$$

and

$$a^*d^*(T\xi,\xi)a+b^*d^*(T\eta,\eta)b+c^*d^*(\xi,\eta)c=\frac{1}{4}|T\xi-\xi|\frac{1}{4}+\frac{1}{8}|T\eta-\eta|\frac{1}{8}+\frac{1}{4}|\xi-\eta|\frac{1}{4}$$

$$= \frac{1}{16}|\frac{1}{8} - \xi| + \frac{1}{64}|\frac{1}{8} - \eta| + \frac{1}{16}|\xi - \eta|$$
$$\geq 0, \text{ for all } \xi, \eta \in [0, 1).$$

Hence, (EC3) is satisfied. Therefore, T is an extended proximal cyclic α-contraction mapping. But T is not a cyclic α-contraction, since

$$\begin{aligned} d^*(T\xi, T\eta) &\leq d^*(\xi, \eta) - \alpha(d^*(\xi, \eta)) + \alpha(d_0(P, Q)) \text{ for all } \xi \in P, \eta \in Q \\ &= d_0(P, Q) = 0 \text{ for all } \xi \in P, \eta \in Q, \end{aligned}$$

which is impossible for $\xi = 1$ and $\eta < 1$.

Remark 2 *It may be noted here that in metric space, a Reich type mapping (refer to [28]) satisfying (EC1) and (EC2) is an extended proximal cyclic α-contraction mapping. But the converse is not true. For this, we present the following example.*

Example 8 *Let $(X, \mathbb{A}, d^*)$ be as defined in Example 4. Let $P = (-\infty, 0]$ and $Q = [5, \infty)$. Then (P, Q) satisfies (P-d^*) property with $p = 0$, $q = 5$ and $d_0(P, Q) = 5$. Suppose that $T : P \cup Q \to P \cup Q$ is defined by*

$$T(\xi) = \begin{cases} 5 & \xi \in P, \\ 0, & \xi \in Q. \end{cases}$$

Let $\alpha : [0, \infty) \to [0, \infty)$ be the identity mapping, and $a = b = c = \frac{1}{\sqrt{6}}$. For $\xi \in P$ and $\eta \in Q$,

$$d^*(T\xi, T\eta) = |5 - 0| = 5$$

and

$$\begin{aligned} &\frac{1}{2}(d^*(\xi, \eta) - \alpha(d^*(\xi, \eta)) + \alpha(d_0(P, Q))) + a^* d^*(T\xi, \xi)a + b^* d^*(T\eta, \eta)b + c^* d^*(\xi, \eta)c \\ &= \frac{5}{2} + \frac{1}{6}(|5 - \xi| + |0 - \eta| + |\xi - \eta| \\ &\geq 5 \text{ for all } \xi \in P, \eta \in Q. \end{aligned}$$

Hence, T is an extended proximal cyclic α-contraction mapping. But T is not a Reich type mapping since

$$a^* d(T\xi, \xi)a + b^* d^*(T\eta, \eta)b + c^* d^*(\xi, \eta)c < 5 = d^*(T\xi, T\eta) \text{ for all } \xi \in P, \eta \in Q.$$

The following result shows the existence of best proximity point for the above class of mappings. Here $\mathbb{A}$ denotes a unital C^*-algebra with unit element I.

Theorem 2 *Let (P, Q) be a pair of nonempty subsets satisfying (P-d^*) property in a C^*-algebra valued metric space $(X, \mathbb{A}, d^*)$. For a strictly increasing function $\alpha : \mathbb{A}^+ \to \mathbb{A}^+$ with respect to the partial order "$\preccurlyeq$" on $\mathbb{A}$, let $T : P \cup Q \to P \cup Q$ be an extended proximal cyclic α-contraction mapping. For $\xi_0 \in P$, let $\{\xi_n\}$ be the Picard's sequence such that $\{\xi_n\}$ has a convergent subsequence in $P \cup Q$. Then there exists $\xi \in P \cup Q$ such that $d^*(\xi, T\xi) = d_0(P, Q)$.*

Proof. We have, by (EC3),

$$\begin{aligned} d^*(\xi_{n+2}, \xi_{n+1}) &= d^*(T\xi_{n+1}, T\xi_n) \\ &\preccurlyeq \frac{1}{2}(d^*(\xi_{n+1}, \xi_n) - \alpha(d^*(\xi_{n+1}, \xi_n)) + \alpha(d_0(P, Q))) \\ &\quad + a^* d^*(T\xi_{n+1}, \xi_{n+1})a + b^* d^*(T\xi_n, \xi_n)b + c^* d^*(\xi_{n+1}, \xi_n)c. \end{aligned}$$

Since α is increasing and $d_0(P,Q) \preccurlyeq d^*(\xi_{n+1},\xi_n)$, the above expression becomes

$$\begin{aligned} d^*(\xi_{n+2},\xi_{n+1}) &\preccurlyeq \frac{1}{2}d^*(\xi_{n+1},\xi_n) + ||a||^2 I d^*(\xi_{n+2},\xi_{n+1}) + ||b||^2 I d^*(\xi_{n+1},\xi_n) \\ &\quad + ||c||^2 I d^*(\xi_{n+1},\xi_n), \end{aligned}$$

i.e.,

$$(I - ||a||^2 I) d^*(\xi_{n+2},\xi_{n+1}) \preccurlyeq (\frac{1}{2}I + ||b||^2 I + ||c||^2 I) d^*(\xi_{n+1},\xi_n),$$

i.e.,

$$d^*(\xi_{n+2},\xi_{n+1}) \preccurlyeq d^*(\xi_{n+1},\xi_n) \text{ since } ||a^2|| + ||b^2|| + ||c^2|| \leq \frac{1}{2}.$$

So $\{d^*(\xi_{n+2},\xi_{n+1})\}$ is monotonically decreasing and bounded below and so, there exists a number $\beta_0 \geq 0$ such that $\lim_{n\to\infty} d^*(\xi_{n+1},\xi_n) = \beta_0$. Again,

$$\begin{aligned} d^*(\xi_{n+2},\xi_{n+1}) &\preccurlyeq \frac{1}{2}(d^*(\xi_{n+1},\xi_n) - \alpha(d^*(\xi_{n+1},\xi_n)) + \alpha(d_0(P,Q))) \\ &\quad + a^* d^*(T\xi_{n+1},\xi_{n+1})a + b^* d^*(T\xi_n,\xi_n)b + c^* d^*(\xi_{n+1},\xi_n)c. \end{aligned}$$

This implies

$$\begin{aligned} \frac{1}{2}\alpha(d^*(\xi_{n+1},\xi_n)) &\preccurlyeq \frac{1}{2}(d^*(\xi_{n+1},\xi_n) + \alpha(d_0(P,Q))) + ||a||^2 I d^*(\xi_{n+2},\xi_{n+1}) \\ &\quad + ||b||^2 I d^*(\xi_{n+1},\xi_n) + ||c||^2 I d^*(\xi_{n+1},\xi_n) - d^*(\xi_{n+2},\xi_{n+1}) \\ &\preccurlyeq \frac{1}{2}\alpha(d_0(P,Q)) + (\frac{1}{2}I + ||b||^2 I + ||c||^2 I) d^*(\xi_{n+1},\xi_n) + (||a||^2 I - I) d^*(\xi_{n+2},\xi_{n+1}) \\ &\preccurlyeq \frac{1}{2}\alpha(d_0(P,Q)) + (\frac{1}{2}I + ||b||^2 I + ||c||^2 I) d^*(\xi_{n+1},\xi_n) - (\frac{1}{2}I + ||b||^2 I + ||c||^2 I) d^*(\xi_{n+1},\xi_n) \\ &= \frac{1}{2}\alpha(d_0(P,Q)). \end{aligned}$$

Taking limit as $n \to \infty$, from the above expression, we get

$$\lim_{n\to\infty} \alpha(d^*(\xi_{n+1},\xi_n)) \preccurlyeq \alpha(d_0(P,Q).$$

Again, $d_0(P,Q) \preccurlyeq d^*(\xi_{n+1},\xi_n) \implies \alpha(d_0(P,Q)) \preccurlyeq \alpha(d^*(\xi_{n+1},\xi_n))$. Hence,

$$\lim_{n\to\infty} \alpha(d^*(\xi_{n+1},\xi_n)) = \alpha(d_0(P,Q)).$$

Since $\lim_{n\to\infty} d^*(\xi_{n+1},\xi_n) = \beta_0$, we see that$d_0(P,Q) \leq \beta_0 \leq d^*(\xi_{n+1},\xi_n)$. So

$$\alpha(d_0(P,Q)) \leq \alpha(\beta_0) \leq \alpha(d^*(\xi_{n+1},\xi_n)).$$

Thus

$$\lim_{n\to\infty} \alpha(\beta_0) = \alpha(d_0(P,Q)) \implies \beta_0 = d_0(P,Q). \tag{2}$$

Let $\{\xi_{n_j}\}$ be a convergent subsequence of $\{\xi_n\}$ in $P \cup Q$ such that $\lim_{j\to\infty} \xi_{n_j} = \xi$.

$$\begin{aligned} d_0(P,Q) &\preccurlyeq d^*(\xi_{n_j-1},\xi) \\ &\preccurlyeq d^*(\xi_{n_j-1},\xi_{n_j}) + d^*(\xi_{n_j},\xi). \end{aligned}$$

Using (2), we get

$$\lim_{j\to\infty} d^*(\xi_{n_j-1},\xi) = d_0(P,Q). \tag{3}$$

Again, $d_0(P,Q) \preccurlyeq d^*(\xi, T\xi)$ and

$$\begin{aligned} d^*(\xi,T\xi) &\preccurlyeq d^*(\xi,\xi_{n_j}) + d^*(\xi_{n_j},T\xi) \\ &\preccurlyeq d^*(\xi,\xi_{n_j}) + d^*(T\xi_{n_j-1},T\xi) \\ &\preccurlyeq d^*(\xi,\xi_{n_j}) + \frac{1}{2}(d^*(\xi_{n_j-1},\xi) - \alpha(d^*(\xi_{n_j-1},\xi)) + \alpha(d_0(P,Q))) \\ &+ a^*d^*(T\xi_{n_j-1},\xi_{n_j-1})a + b^*d^*(T\xi,\xi)b + c^*d^*(\xi_{n_j-1},\xi)c \\ &\preccurlyeq d^*(\xi,\xi_{n_j}) + \frac{1}{2}d^*(\xi_{n_j-1},\xi) + ||a||^2 Id^*(\xi_{n_j},\xi_{n_j-1}) \\ &+ ||b||^2 Id^*(T\xi,\xi) + ||c||^2 Id^*(\xi_{n_j-1},\xi), \end{aligned}$$

i.e.,

$$(I - ||b||^2 I)d^*(\xi,T\xi) \preccurlyeq d^*(\xi,\xi_{n_j}) + \frac{1}{2}d^*(\xi_{n_j-1},\xi) + ||a||^2 Id^*(\xi_{n_j},\xi_{n_j-1}) + ||c||^2 Id^*(\xi_{n_j-1},\xi).$$

Taking limit as $j \to \infty$ in the above equation and using (2) and (3), we get,

$$(I - ||b||^2 I)d^*(\xi,T\xi) \preccurlyeq (\frac{1}{2}I + ||a||^2 I + ||c||^2 I)d_0(P,Q).$$

Thus, $d^*(\xi,T\xi) \preccurlyeq d_0(P,Q)$, since $||a||^2 + ||b||^2 + ||c||^2 \leq \frac{1}{2}$. Hence, $d^*(\xi,T\xi) = d_0(P,Q)$. ■

Corollary 1 *Let (P,Q) be a pair of nonempty closed subsets satisfying (P-d^*) property in a unital C^*-algebra valued metric space $(X,\mathbb{A},d^*)$. For a strictly increasing function $\alpha : \mathbb{A}^+ \to \mathbb{A}^+$ with respect to the partial order " $\preccurlyeq$ " on $\mathbb{A}$, and a self mapping T on $P \cup Q$, let $\{T_n : P \cup Q \to P \cup Q : n \in \mathbb{N}\}$ be a sequence of extended proximal cyclic α-contraction mappings such that $\{T_n x\}$ converges to Tx for each x in $P \cup Q$. For $\xi_0 \in P$, $\{\xi_n\}$ be the Picard's sequence having a convergent subsequence in $P \cup Q$. Then T has a best proximity point in $P \cup Q$ with respect to d^*.*

The following example exhibits Theorem 2.

Example 9 *Let $X = \mathbb{R}^2$ and $\mathbb{A} = \mathbb{R}$ with the norm $||\xi|| = |\xi|$ for all $\xi \in \mathbb{R}$. The C^*-algebra valued metric on X is defined by,*

$$d^*(\xi,\eta) = \max\{|\xi_1 - \eta_1|, |\xi_2 - \eta_2|\} \quad \textit{for all } \xi = (\xi_1,\xi_2) \textit{ and } \eta = (\eta_1,\eta_2) \in X.$$

Let $P = \{(\frac{1}{6},1),(\frac{1}{7},1)\}$ and $Q = \{(-1,0),(-2,0)\}$. Clearly (P,Q) satisfies (P-d^*) property with $p = (\frac{1}{7},1)$, $q = (-1,0)$ and $d_0(P,Q) = \frac{8}{7}$. Let $T : P \cup Q \to P \cup Q$ be defined by

$$T(\xi) = \begin{cases} (-1,0), & \xi \in P, \\ (\frac{1}{7},1), & \xi \in Q. \end{cases}$$

We take α as the identity mapping on $[0,\infty)$, and $a = b = c = \frac{1}{\sqrt{6}}$. Then T is an extended proximal cyclic α-contraction mapping. For $\xi_0 = (\frac{1}{6},1)$,

$$\begin{aligned} \xi_1 &= T(\xi_0) = T((\frac{1}{6},1)) = (-1,0), \\ \xi_2 &= T(\xi_1) = T((-1,0)) = (\frac{1}{7},1), \\ \xi_3 &= T(\xi_2) = T((\frac{1}{7},1)) = (-1,0), \\ &\vdots \end{aligned}$$

$$\xi_j = \begin{cases} (\frac{1}{7}, 1), & j \text{ is even,} \\ (-1, 0), & j \text{ is odd.} \end{cases}$$

When $j \to \infty$, $\xi_{2j} \to (\frac{1}{7}, 1)$. *Thus,* T *satisfies the conditions of Theorem 2. Now,*

$$d^*((\frac{1}{7}, 1), T(\frac{1}{7}, 1)) = \frac{8}{7} = d_0(P, Q).$$

Thus, $(\frac{1}{7}, 1)$ *is a best proximity point of* T *with respect to* d^*.

Considering $P = Q = X$ and T as a self mapping on X, we have the following fixed point theorem. (Clearly, here $p = q = \theta$ since $\theta \preccurlyeq d^*(\xi, \eta)$ for all $\xi, \eta \in X$. Also, $d_0(P, Q) = \theta$.)

Theorem 3 *Let* $(X, \mathbb{A}, d^*)$ *be a complete unital* C^**-algebra valued metric space and* $\alpha : \mathbb{A}^+ \to \mathbb{A}^+$ *be a strictly increasing mapping. If* T *is an extended proximal cyclic* α*-contraction self mapping on* X*, then* T *has a unique fixed point.*

The notion of orbital continuity was introduced by Ćirić [10] in 1971. If $T : X \to X$, where (X, d) is a metric space, then the set $O(\xi, T) = \{T^j\xi : j = 0, 1, 2, ...\}$ is called the orbit of T at the point ξ and T is orbitally continuous if for any sequence $\{\xi_n\}$ in X, $\lim_{n\to\infty} \xi_n = z$ implies $\lim_{n\to\infty} T\xi_n = Tz$. Every continuous self mapping is orbitally continuous but the converse is not true. Similar concept holds in case of C^*-algebra valued metric space also. The following result deals with orbitally continuous extended proximal cyclic α-contraction mapping T on a partially ordered set X.

Theorem 4 *Let* (P, Q) *be a pair of nonempty subsets satisfying (*P*-*d^**) property in a unital* C^**-algebra valued metric space* $(X, \mathbb{A}, d^*)$ *where* P *is closed with respect to* d^* *and* X *is partially ordered. For a strictly increasing function* $\alpha : \mathbb{A}^+ \to \mathbb{A}^+$ *with respect to the partial order "* $\preccurlyeq$ *" on* $\mathbb{A}$*, let* $T : P \cup Q \to P \cup Q$ *be an extended proximal cyclic* α*-contraction mapping. Assume that there exists* $\xi_0 \in P$ *such that* $\xi_0 \preccurlyeq T^2\xi_0 \preccurlyeq T\xi_0$*. If* T *is orbitally continuous and every bounded monotone sequence in* X *is convergent, then there exists* $\xi \in P$ *such that* $d^*(\xi, T\xi) = d_0(P, Q)$*.*

Proof. For $\xi_0 \in P$, we consider the Picard's sequence $\{\xi_n\}$. Since T is an extended proximal cyclic α-contraction mapping, similar to Theorem 2, we can easily show that

$$\lim_{n\to\infty} d^*(\xi_{n+2}, \xi_{n+1}) = \lim_{n\to\infty} d^*(T\xi_{n+1}, T\xi_n) = d_0(P, Q).$$

By the assumption $\xi_0 \preccurlyeq T^2\xi_0 \preccurlyeq T\xi_0$ we get,

$$\xi_0 \preccurlyeq \xi_2 \preccurlyeq \xi_4 \preccurlyeq ... \preccurlyeq \xi_{2n} \preccurlyeq ... \preccurlyeq \xi_1 \text{ for all } n \in \mathbb{N}.$$

Since P is closed and every bounded monotone sequence in X is convergent, so, for the sequence $\{\xi_{2n}\}$, there exists $\xi \in P$ such that $\lim_{n\to\infty} \xi_{2n} = \xi$. Again T is orbitally continuous. So,

$$\begin{aligned} d_0(P, Q) &\preccurlyeq d^*(\xi_{2n}, T\xi) = d^*(T\xi_{2n-1}, T\xi) \\ &\preccurlyeq d^*(T\xi_{2n-1}, T\xi_{2n}) + d^*(T\xi_{2n}, T\xi) \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Taking limit as $n \to \infty$, from the above equation we get,

$$d_0(P, Q) \preccurlyeq d^*(\xi, T\xi) \preccurlyeq d_0(P, Q).$$

Hence, $d^*(\xi, T\xi) = d_0(P, Q)$. ∎

4 Common Best Proximity Point with Respect to d^*

In this section, we establish the existence of common best proximity point for a pair of extended proximal α-contraction mappings. For this we give the following definition.

Definition 8 *Let (P, Q) be a pair of nonempty subsets satisfying (P-d^*) property in a C^*-algebra valued metric space $(X, \mathbb{A}, d^*)$ and $\alpha : \mathbb{A}^+ \to \mathbb{A}^+$ be a strictly increasing mapping with respect to the partial ordering " $\preccurlyeq$ " on $\mathbb{A}$. For self mappings S and T on $P \cup Q$, T is said to be extended proximal cyclic α-contraction mapping with respect to S if the following conditions are satisfied:*

(ECS1) $T(P) \subseteq Q$ *and* $T(Q) \subseteq P$*;*

(ECS2) $\alpha(d^*(S\xi, S\eta)) - \alpha(d_0(P, Q)) \preccurlyeq d^*(S\xi, S\eta)$ *for all* $\xi \in P$ *and* $\eta \in Q$*; and*

(ECS3) For some $a, b, c \in \mathbb{A}'^+$ *with* $||b|| \leq ||a||$ *and* $||a||^2 + ||b||^2 + ||c||^2 \leq \frac{1}{2}$,

$$\begin{aligned} d^*(T\xi, T\eta) \quad &\preccurlyeq \quad \frac{1}{2}(d^*(S\xi, S\eta) - \alpha(d^*(S\xi, S\eta)) + \alpha(d_0(P, Q))) + a^* d^*(S\xi, T\xi)a \\ &\quad + b^* d^*(S\eta, T\eta)b + c^* d^*(S\xi, S\eta)c \text{ for all } \xi \in P, \eta \in Q. \end{aligned}$$

Theorem 5 *Let $(X, \mathbb{A}, d^*)$ be a unital complete C^*-algebra valued metric space and (P, Q) be a pair of nonempty subsets of X satisfying (P-d^*) property with Q sequentially compact with respect to d^*. For a strictly increasing continuous mapping $\alpha : A^+ \to A^+$ with respect to the partial ordering " $\preccurlyeq$ " and a self mapping S on $P \cup Q$, let T be an extended proximal cyclic α-contraction mapping on $P \cup Q$ with respect to S. If the following conditions are satisfied:*

(i) $T(P) \subseteq S(P) \subseteq Q$ *and* $T(Q) \subseteq S(Q) \subseteq P$*;*

(ii) S is continuous and S, T commute; and

Then there exists a common best proximity point of S and T with respect to d^*.

Proof. Let $\xi_0 \in P$. From condition (i), there exists $\xi_1 \in P$ such that

$$T(\xi_0) = S(\xi_1).$$

Again, since $T(\xi_1) \in S(P)$, there exists $\xi_2 \in P$ such that

$$T(\xi_1) = S(\xi_2).$$

In this way, we get a sequence $\{\xi_n\}$ in P such that

$$T(\xi_n) = S(\xi_{n+1}), n \in \mathbb{N} \cup \{0\}.$$

Since Q is sequentially compact, there exists a convergent subsequence $\{T\xi_{n_k}\}$ of the sequence $\{T\xi_n\}$ in Q. Clearly, $\{T\xi_{n_k}\}$ is a Cauchy sequence in $(X, \mathbb{A}, d^*)$. Since $(X, \mathbb{A}, d^*)$ is complete, $\{T\xi_{n_k}\}$ converges to some $\eta \in Q$. So, $T\xi_{n_k} \to \eta$ and also $S\xi_{n_k} \to \eta$ as $k \to \infty$. Since S is continuous, $ST\xi_{n_k} \to S\eta$ and $SS\xi_{n_k} \to S\eta$ as $k \to \infty$.

Again, S and T commute, so $\lim_{k\to\infty} ST\xi_{n_k} = \lim_{k\to\infty} TS\xi_{n_k}$. Now, from (ECS3) we have

$$\begin{aligned} d^*(T\xi_{n_k}, TS\xi_{n_k}) \quad &\preccurlyeq \quad \frac{1}{2}(d^*(S\xi_{n_k}, SS\xi_{n_k}) - \alpha(d^*(S\xi_{n_k}, SS\xi_{n_k})) + \alpha(d_0(P, Q))) + a^* d^*(T\xi_{n_k}, S\xi_{n_k})a \\ &\quad + b^* d^*(TS\xi_{n_k}, SS\xi_{n_k})b + c^* d^*(S\xi_{n_k}, SS\xi_{n_k})c, \end{aligned}$$

i.e.,

$$\frac{1}{2}\alpha(d^*(S\xi_{n_k}, SS\xi_{n_k})) \quad \preccurlyeq \quad \frac{1}{2}(d^*(S\xi_{n_k}, SS\xi_{n_k}) + \alpha(d_0(P, Q))) + ||a||^2 I d^*(T\xi_{n_k}, S\xi_{n_k})$$

$$+||b||^2Id^*(TS\xi_{n_k}, SS\xi_{n_k}) + ||c||^2Id^*(S\xi_{n_k}, SS\xi_{n_k}) - d^*(T\xi_{n_k}, TS\xi_{n_k}).$$

Taking limit as $k \to \infty$, the above expression becomes

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{2}\alpha(d^*(S\xi_{n_k}, SS\xi_{n_k})) &\preccurlyeq \frac{1}{2}(d^*(\eta, S\eta) + \alpha(d_0(P,Q))) + ||a||^2Id^*(\eta,\eta)\\ &+ ||b||^2Id^*(S\eta, S\eta) + ||c||^2Id^*(\eta, S\eta) - d^*(\eta, S\eta)\\ &= \frac{1}{2}\alpha(d_0(P,Q)) + (\frac{1}{2}I - I + ||c||^2I)d^*(\eta, S\eta)\\ &= \frac{1}{2}\alpha(d_0(P,Q)) - (\frac{1}{2}I - ||c||^2I)d^*(\eta, S\eta)\\ &\preccurlyeq \frac{1}{2}\alpha(d_0(P,Q)),\end{aligned}$$

i.e.,

$$\lim_{k\to\infty}\alpha(d^*(S\xi_{n_k}, SS\xi_{n_k})) \preccurlyeq \alpha(d_0(P,Q)).$$

Again, $\alpha(d_0(P,Q)) \preccurlyeq \alpha(d^*(S\xi_{n_k}, SS\xi_{n_k}))$. Hence, $\lim_{k\to\infty} d^*(S\xi_{n_k}, SS\xi_{n_k}) = d^*(\eta, S\eta) = d_0(P,Q)$. Again using (ECS3),

$$\begin{aligned}d^*(T\xi_{n_k}, T\eta) \quad &\preccurlyeq \quad \frac{1}{2}(d^*(S\xi_{n_k}, S\eta) - \alpha(d^*(S\xi_{n_k}, S\eta)) + \alpha(d_0(P,Q)))\\ &+||a||^2Id^*(S\xi_{n_k}, T\xi_{n_k}) + ||b||^2Id^*(S\eta, T\eta) + ||c||^2Id^*(S\xi_{n_k}, S\eta).\end{aligned}$$

Taking limit as $k \to \infty$ in the above expression, we get,

$$\begin{aligned}d^*(\eta, T\eta) \quad &\preccurlyeq \quad \frac{1}{2}(d^*(\eta, S\eta) - \alpha(d^*(\eta, S\eta)) + \alpha(d_0(P,Q))) + ||a||^2Id^*(\eta,\eta)\\ &+||b||^2Id^*(S\eta, T\eta) + ||c||^2Id^*(\eta, S\eta)\\ &= \quad \frac{1}{2}d_0(P,Q) + ||b||^2I(d^*(S\eta,\eta) + d^*(\eta, T\eta)) + ||c||^2Id_0(P,Q).\end{aligned}$$

Since $||b|| \leq ||a||$, the above equation becomes

$$d^*(\eta, T\eta) \preccurlyeq \frac{1}{2}d_0(P,Q) + ||a||^2Id^*(S\eta,\eta) + ||b||^2Id^*(\eta, T\eta) + ||c||^2Id_0(P,Q),$$

i.e.,

$$(I - ||b||^2I)d^*(\eta, T\eta) \preccurlyeq (\frac{1}{2}I + ||a||^2I + ||c||^2I)d_0(P,Q),$$

i.e.,

$$d^*(\eta, T\eta) \preccurlyeq d_0(P,Q), \text{ since } ||a||^2 + ||b||^2 + ||c||^2 \leq \frac{1}{2}.$$

Therefore, $d^*(\eta, T\eta) = d_0(P,Q) = d^*(\eta, S\eta)$. Thus, η is a common best proximity point of S and T with respect to d^*. ■

Remark 3 *It can be seen from the following example that the best proximity point in the above theorem is not necessarily unique.*

Example 10 *Let $X = \mathbb{R}^2$ and $\mathbb{A} = \mathbb{R}$ with the norm $||\xi|| = |\xi|$ for all $\xi \in \mathbb{R}$. The C^*-algebra valued metric on X is defined by*

$$d^*(\xi,\eta) = \max\{|\xi_1 - \eta_1|, |\xi_2 - \eta_2|\} \textit{ for all } \xi = (\xi_1, \xi_2) \textit{ and } \eta = (\eta_1, \eta_2) \in X.$$

Let $P = \{(0, \xi) : 0 \leq \xi \leq 1\}$ *and* $Q = \{(1, \xi) : 0 \leq \xi \leq 1\}$. *Clearly* (P, Q) *satisfies (P-*d^**) property with* $p = (0, 0)$ *and* $q = (1, 0)$. *The mappings* $T, S : P \cup Q \to P \cup Q$ *be defined by*

$$T((\xi,\eta)) = \begin{cases} (1, \frac{\xi}{2}), & (\xi,\eta) \in P, \\ (0, \frac{\eta}{2}), & (\xi,\eta) \in Q, \end{cases}$$

and

$$S((\xi,\eta)) = \begin{cases} (1, \xi), & (\xi,\eta) \in P, \\ (0, \eta), & (\xi,\eta) \in Q. \end{cases}$$

Let α *be the identity mapping on* $[0, \infty)$ *and* $a = b = 0$ *and* $c = \frac{1}{\sqrt{2}}$. *Then* T *is an extended proximal cyclic* α*-contraction mapping with respect to* S. *Clearly,* T *satisfies the conditions of Theorem 5, and* $(0, 0)$ *and* $(0, 1)$ *are the common best proximity points of* S *and* T *with respect to* d^*.

5 Application to Matrix Equations

This section deals with the application of our derived result for solving linear matrix equations of the type

$$X - \Omega_1^* X \Omega_1 - \Omega_2^* X \Omega_2 - .... - \Omega_k^* X \Omega_k = \Theta \tag{4}$$

or

$$X + \Omega_1^* X \Omega_1 + \Omega_2^* X \Omega_2 + ... + \Omega_k^* X \Omega_k = \Theta, \tag{5}$$

where $\Omega_1, \Omega_2, ...\Omega_k$ are arbitrary $n \times n$ matrices and Θ is an $n \times n$ positive definite matrix. Using our derived result, we show the existence of a Hermitian matrix solution to the above matrix equations.

Theorem 6 *Suppose that* $\beta(H)$ *is the set of all bounded linear operators on a Hilbert space* H. *Let* $\Omega_1, \Omega_2, ..., \Omega_k \in \beta(H)$ *be such that* $\sum_{j=1}^k ||\Omega_j|| < \frac{1}{2}$. *Then the operator equation* $X - \sum_{j=1}^k \Omega_j^* X \Omega_j = \Theta$ *has a unique solution in* $\beta(H)$*, where* $\Theta \in \beta(H)^+$.

Proof. Let $X = \mathbb{A} = \beta(H)$ with the metric d^* on $\beta(H)$ be defined by $d^*(U, V) = ||U - V||\Psi$ for $U, V \in \beta(H)$ and a positive operator $\Psi \in \beta(H)$. Let the mapping $T : \beta(H) \to \beta(H)$ be defined by

$$T(U) = \sum_{j=1}^{k} \Omega_j^* U \Omega_j + \Theta, \ U \in \beta(H)^+.$$

Then $T \in \beta(H)^+$. Now, for $U, V \in \beta(H)$, we have

$$\begin{aligned} d^*(T(U), T(V)) &= ||T(U) - T(V)||\Psi \\ &= ||\sum_{j=1}^{k} \Omega_j^*(U - V)\Omega_j||\Psi \\ &\preccurlyeq \sum_{j=1}^{k} ||\Omega_j||^2 ||U - V||\Psi \\ &\preccurlyeq \frac{1}{4} d^*(U, V) \\ &= (\frac{1}{2}I)^* d^*(U, V)(\frac{1}{2}I) \\ &\preccurlyeq \frac{1}{2}(d^*(U, V) - \alpha(d^*(U, V)) + \alpha(d_0(P, Q))) + a^* d^*(TU, U)a \end{aligned}$$

$$+b^* d^*(TV,V)b + c^* d^*(U,V)c,$$

where α is the identity mapping on $\beta(H)^+$, $P = Q = X$, $d_0(P,Q) = \theta$ (the zero operator), and $a = b = c = \frac{1}{4}I$, so that $a, b, c \in \beta(H)'^+$ with $||a||^2 + ||b||^2 + ||c||^2 \leq \frac{1}{2}$. Using the Theorem 3, there exists a unique fixed point U in $\beta(H)$. Since T is a positive operator, the solution U is a Hermitian operator. ■

Using the above theorem, we can say that the matrix equation given by (4) has a unique solution which is Hermitian. Similar result can be obtained for the matrix equation given by (5). Next we consider the following type of matrix equation:

$$X - A^{-1}B + A^{-1}X^tC = \Theta,$$

where A, B, C are arbitrary $n \times n$ matrices with A invertible, Θ is an $n \times n$ positive definite matrix, and X^t denotes the transpose of the $n \times n$ matrix X. The following result shows that this matrix equation also has a unique solution X under some specific conditions.

Theorem 7 *Let $\beta(H)$ be the set of all bounded linear operators on H, where H is a Hilbert space. Consider the operator equation:*

$$X - A^{-1}B + A^{-1}X^tC = \Theta \tag{6}$$

where $A, B, C \in \beta(H)$ with A invertible, $\Theta \in \beta(H)^+$, and X^t denotes the transpose of X. If $\max(||A^{-1}||, ||C||) \leq \gamma$ *for some $\gamma \in (0, \frac{1}{4})$, then the operator equation (6) has a unique solution in $\beta(H)$ where $\Theta \in \beta(H)^+$.*

Proof. We consider X, $\mathbb{A}$ and d^* as in Theorem 6. Suppose the mapping

$$T : \beta(H) \to \beta(H)$$

be defined by

$$T(U) = A^{-1}B - A^{-1}U^tC + \Theta, \ U \in \beta(H).$$

Now, for $U, V \in \beta(H)$,

$$\begin{aligned}
d^*(T(U), T(V)) &= ||T(U) - T(V)||\Psi \\
&= ||A^{-1}B - A^{-1}U^tC + \Theta - A^{-1}B + A^{-1}V^tC - \Theta|||\Psi \\
&= ||A^{-1}U^tC - A^{-1}V^tC||\Psi \\
&\preccurlyeq ||A^{-1}||.||C||d^*(U,V) \\
&\preccurlyeq \gamma^2 d^*(U,V) \\
&= (\gamma I)^* d^*(U,V)(\gamma I) \\
&\preccurlyeq \frac{1}{2}(d^*(U,V) - \alpha(d^*(U,V)) + \alpha(d_0(P,Q))) + a^* d^*(TU,U)a \\
&\quad +b^* d^*(TV,V)b + c^* d^*(U,V)c,
\end{aligned}$$

where α is the identity mapping on $\beta(H)^+$, $P = Q = X$, $d_0(P,Q) = \theta$ (the zero operator), and $a = \frac{1}{4}I$, $b = \frac{1}{8}I$ and $c = \gamma I$, so that $a, b, c \in \beta(H)'^+$ with $||a||^2 + ||b||^2 + ||c||^2 \leq \frac{1}{2}$. Using the Theorem 3, there exists a unique fixed point U in $\beta(H)$. ■

6 Application to Volterra Integral Equation

Endemic infectious diseases that grant lasting immunity upon infection are characterized through a framework of nonlinear Volterra integral equations of convolution type. These models, with fixed parameters, encompass essential population dynamics like births, deaths, vaccination effects, and a distributed infectious period. The population under consideration is partitioned into distinct classes viz., the susceptible class (S), comprises individuals susceptible to infection; the exposed class (E), contains individuals who have been exposed to the pathogen but are not yet capable of transmitting it; the infective class (I), includes those who are actively spreading the infection and the removed class (R), encompasses individuals who have acquired lasting immunity due to either immunization or previous infection (refer to [14]).

In certain models, the (E) class is omitted when the exposure period is short or negligible. Here, we consider the integral equation for $I(t)$ involved in the SIR model and show that under certain conditions, this equation has a solution. This integral equation is represented by the following (refer to [14]):

$$I(t) = I_0(t)e^{-\mu t} + \int_0^t \beta S(x)I(x)P(t-x)e^{-\mu(t-x)}.$$

For convenience, we write it as

$$I(t) = \rho(t) + \int_0^t f(x, I(x))\phi(t,x)dx \tag{7}$$

where $f(x, I(x)) = \beta S(x)I(x)$ in which the constant contact rate β is the average number of contacts sufficient for transmission of an infective per unit time, $\rho(t) = I_0(t)e^{-\mu t}$ and $\phi(t,x) = P(t-x)e^{-\mu(t-x)}$. Here μ denotes the death rate, $I_0(t)e^{-\mu t}$ is the fraction of the population that was initially infectious and is still alive and infectious at time t and $P(t)$ is the probability that an infected at time $t_0 = 0$ remains infectious at time t. We consider the time $t \in [0,T]$, where T is sufficiently large.

Theorem 8 *Consider the nonlinear Volterra integral equation of convolution type (7) with a continuous function f from $[0,T]\times[0,1]$ to $[0,\infty)$ satisfying the conditions*

(i) $|f(x,\xi(x)) - f(x,\eta(x))| \leq |\xi(x) - \eta(x)|$ *for all* $x \in [0,T]$.

(ii) $\int_0^t |\phi(t,x)|dx \leq \gamma^2$, $t \in [0,T]$ *and for some* $\gamma \in (0, \frac{1}{4})$.

Then the integral equation (7) has a solution.

Proof. Let $X = C([0,T],[0,1])$, $\mathbb{A} = \mathbb{R}$ and $d^*(\xi,\eta) = \sup_{x\in[0,T]} |\xi(x) - \eta(x)|$ for all $\xi, \eta \in X$. Define $T : X \to X$ by

$$T\xi(t) = \rho(t) + \int_0^t f(x,\xi(x))\phi(t,x)dx$$

where $x \in [0,t]$, $t \in [0,T]$. Then,

$$\begin{aligned}
d^*(T\xi(t), T\eta(t)) &= \sup_{t\in[0,T]} |T\xi(t) - T\eta(t)| \\
&= \sup_{t\in[0,T]} \left| \int_0^t f(x,\xi(x))\phi(t,x)dx - \int_0^t f(x,\eta(x))\phi(t,x)dx \right| \\
&\leq \sup_{t\in[0,T]} \int_0^t |f(x,\xi(x)) - f(x,\eta(x))|.|\phi(t,x)|dx \\
&\leq \sup_{t\in[0,T]} \int_0^t |\xi(x) - \eta(x)|.|\phi(t,x)|dx \\
&\leq \sup_{t\in[0,T]} \int_0^t (\sup_{x\in[0,t]} |\xi(x) - \eta(x)|)|\phi(t,x)|dx \\
&\leq (\sup_{x\in[0,T]} |\xi(x) - \eta(x)|) \sup_{t\in[0,T]} \int_0^t |\phi(t,x)|dx \\
&\leq \gamma^2 d^*(\xi,\eta) \\
&= (\gamma I)^* d^*(\xi,\eta)(\gamma I) \\
&\preccurlyeq \frac{1}{2}(d^*(\xi,\eta) - \alpha(d^*(\xi,\eta)) + \alpha(d_0(P,Q))) + a^*d^*(T\xi,\xi)a + b^*d^*(T\eta,\eta)b + c^*d^*(\xi,\eta)c,
\end{aligned}$$

where α is the identity mapping, $P = Q = X$, $a = \frac{1}{4}I$, $b = \frac{1}{8}I$ and $c = \gamma I$ so that $||a||^2 + ||b||^2 + ||c||^2 \leq \frac{1}{2}$. Hence by Theorem 3, the integral equation (7) has a unique solution. ∎

7 Conclusion

In this paper, we have obtained some best proximity point and common best proximity point results in C^*-algebra valued metric space. The results are applied for matrix equations and an integral equation of convolution type which represents the SIR model for endemic infectious diseases. Further investigation can be done considering coupled best proximity point with application to some other type of matrix equation or biological models. We have mentioned in the introduction section that many researchers have generalized the C^*-algebra valued metric space and related fixed point theory in different aspects. The novelty of our work lies in the fact that we have generalized the cyclic α-contraction mapping in C^*-algebra valued metric space with different interesting consequences. Since in a C^*-algebra valued metric space distances are measured in terms of elements of C^*-algebra, the study of best proximity point in this area gives interesting formulation of proximity conditions. Also development of best proximity point results in C^*-algebra valued metric space can formulate tools for different applications in quantum mechanics and operator theory. In this context, the results of this paper may open up many scopes of further research in the domain of C^*-algebra valued metric space from the perspectives of application to matrix equations and mathematical modelling.

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