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Codes Based On The k-Fibonacci Sequence Of The Heisenberg Group*

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Abstract

In this paper, we consider the Heisenberg group

$$H_{(t,l,m)} = \langle a, b, c | a^t = b^l = c^m = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle.$$

First, we study the $k-$Fibonacci sequence of $H_{(t,l,m)}$. Then we introduce a new coding scheme using the k-Fibonacci sequence of the Heisenberg group as an application of groups in coding theory. It is shown that the error-correcting capability of these codes is very high.

1 Introduction

The mathematical foundation of error-correcting codes [2, 15] is generally well-known but not always appreciated. Error-correcting codes were first investigated by Hamming [10]. He introduced many fundamental concepts including the Hamming distance and Hamming bound. In [20], Shannon established the fundamental concepts of information theory which are the basis for modern communication and cryptographic systems. This paper presents new results on error-correcting codes based on recent extensions to the Fibonacci sequence and its algebraic generalizations [3, 6, 7, 12, 13] within the broader field of information theory [21]. One generalization relevant to error-correcting codes and their relationship to number theory is the k-Fibonacci sequence [5, 9, 14, 16, 17, 19].

Definition 1 *For $k \geq 2$, the k-Fibonacci sequence, $\{F_n^k\}_{n=0}^{\infty}$ is*

$$F_n^k = F_{n-1}^k + F_{n-2}^k + \cdots + F_{n-k}^k,\ n \geq k,$$

where $F_0^k = 0$, $F_1^k = 0$, $\ldots$, $F_{k-2}^k = 0$, and $F_{k-1}^k = 1$. Let $K_k(m)$ denote the minimal period of the sequence $\{F_n^k \pmod m\}_{n=0}^{\infty}$. This is called the Wall number of the k-Fibonacci sequence modulo m [5].

Definition 2 *A k-Fibonacci sequence in a finite group $G = \langle X\rangle$ is a sequence of group elements x_1, x_2, $\ldots$, x_n, $\ldots$ for which given an initial (seed) set $X = \{a_1, \ldots, a_j\}$, the elements are*

$$x_n = \begin{cases} a_n, & n \leq j, \\ x_1x_2\ldots(x_{n-1}), & j < n \leq k, \\ x_{n-k}\ldots(x_{n-1}), & n > k. \end{cases}$$

The k-Fibonacci sequence of the group $G = \langle X\rangle$ and its period are denoted by $F_t(G; X)$ and $L_t(G; X)$, respectively [16].

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Lemma 1 (**[11]**) *For integers n and $m \geq 2$, if*

$$\begin{cases} F^k_n \equiv 0 & \pmod{m}, \\ F^k_{n+1} \equiv 0 & \pmod{m}, \\ \quad\vdots & \\ F^k_{n+k-2} \equiv 0 & \pmod{m}, \\ F^k_{n+k-1} \equiv 0 & \pmod{m}. \end{cases}$$

Then $K_k(m) \mid n$.

Definition 3 (**[18]**) *For all t, l, $m \in Z$, each element of the Heisenberg group, $H_{(t,l,m)}$, has the form*

$$\begin{bmatrix} 1 & t & m+tl \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$H_{(t,l,m)} = \langle a, b, c | a^t = b^l = c^m = 1, \ [a, b] = c, \ [a, c] = [b, c] = 1 \rangle.$$

Based on the above, we have the following lemma.

Lemma 2 *Every element of $H_{(t,l,m)}$ can be written uniquely in the form $a^i b^j c^k$ where $1 \leq i \leq t$, $1 \leq j \leq l$ and $1 \leq k \leq m$.*

In [22], a new coding scheme was introduced based on generating matrices of the Fibonacci p-numbers. The (m, t)-extension of the Fibonacci p-numbers and corresponding codes were given in [1]. In [8], codes were constructed using a new class of Fibonacci sequences. The k-Fibonacci sequence for $k \geq 3$ was defined and studied in [14] and it was used to obtain new codes. In [17], codes were constructed using the t-extension of the p-Fibonacci lower and upper triangular Pascal matrices. It was shown that the corresponding error correction capability is 100%. In this paper, we derive the k-Fibonacci sequence of the Heisenberg group. Then new codes are introduced using this sequence.

The rest of this paper is organized as follows. The 3-Fibonacci sequence of the Heisenberg group is examined in Section 2. This is generalized to the k-Fibonacci sequences in Section 3. In Section 4, a new coding scheme is defined using the Heisenberg group, and the error correction capability is shown to be high.

2 The 3-Fibonacci Sequence of $H_{(t,l,m)}$

In this section, we obtain the 3-Fibonacci sequence of $H_{(t,l,m)}$ and determine its period. To study the 3-Fibonacci sequence of $H_{(t,l,m)}$ with respect to $X = \{a, b, c\}$, we require the following sequences

$$\begin{aligned} F^3_n &:= F_n, \\ T(n) &= F_{n-3} + F_{n-2}, \\ g(1) &= g(2) = 0, g(3) = 1, \\ g(n) &= g(n-1) + g(n-2) + g(n-3) - (T(n-3)F_{n-4} + (T(n-3) + T(n-2))F_{n-3}), \ n \geq 4. \end{aligned}$$

We next provide a standard form for the 3-Fibonacci sequence x_4, x_5, ..., of $H_{(t,l,m)}$.

Lemma 3 *Every element of $F_3(H_{(t,l,m)}; X)$ can be represented by $x_n = a^{F_{n-2}} b^{T(n)} c^{g(n)}$, $n \geq 4$.*

Proof. For $n=4$ and $n=5$, we have $x_4=a^1b^1c^1=a^{F_2}b^{T(4)}c^{g(4)}$ and $x_5=abc^2=a^{F_3}b^{T(5)}c^{g(5)}$. Then by induction on n we obtain

$$\begin{aligned}
x_n &= x_{n-3}x_{n-2}x_{n-1}\\
&= a^{F_{n-5}}b^{T(n-3)}c^{g(n-3)}a^{F_{n-4}}b^{T(n-2)}c^{g(n-2)}a^{F_{n-3}}b^{T(n-1)}c^{g(n-1)}\\
&= a^{F_{n-5}}b^{T(n-3)}a^{F_{n-4}}b^{T(n-2)}a^{F_{n-3}}b^{T(n-1)}c^{g(n-3)+g(n-2)+g(n-1)}\\
&= a^{F_{n-5}+F_{n-4}}b^{T(n-3)+T(n-2)}b^{T(n-2)}a^{F_{n-3}}b^{T(n-1)}c^{g(n-3)+g(n-2)+g(n-1)-(T(n-3))F_{n-4}}\\
&= a^{F_{n-5}+F_{n-4}+F_{n-3}}b^{T(n-3)+T(n-2)+T(n-1)}c^{g(n-3)+g(n-2)+g(n-1)-(T(n-3)F_{n-4}+(T(n-3)+T(n-2)F_{n-3})}\\
&= a^{F_{n-2}}b^{T(n)}c^{g(n)},
\end{aligned}$$

which completes the proof. ∎

Lemma 4 *The following statements (i) and (ii) hold:*

(i) For $n \geq 6$, $g(n)=F_{n-2}-\Sigma_{i=1}^{n-6}F_{n-(i+4)}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})$.

(ii) For i *an integer and* $s=2^i$, *let* $n=K_3(2^{i+1})$. *Then we have*

$$\begin{cases} g(n+1)\equiv 0 \pmod{s},\\ g(n+2)\equiv 0 \pmod{s},\\ g(n+3)\equiv 0 \pmod{s}.\end{cases}$$

Proof.

(i) For $n=6$

$$g(6)=F_4-\Sigma_{i=1}^{6-6}F_{6-(i+4)}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})=2.$$

Then by induction on n,

$$\begin{aligned}
g(n) &= g(n-1)+g(n-2)+g(n-3)-(T(n-3)F_{n-4}+(T(n-3)+T(n-2))F_{n-3})\\
&= F_{n-3}-\Sigma_{i=1}^{n-7}F_{n-(i+5)}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})+F_{n-4}\\
&\quad -\Sigma_{i=1}^{n-8}F_{n-(i+6)}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})\\
&\quad +F_{n-5}-\Sigma_{i=1}^{n-9}F_{n-(i+7)}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})\\
&\quad -(T(n-3)F_{n-4}+(T(n-3)+T(n-2))F_{n-3})\\
&= (F_{n-3}+F_{n-4}+F_{n-5})-(\Sigma_{i=1}^{n-7}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})\\
&\quad +\Sigma_{i=1}^{n-8}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})\\
&\quad +\Sigma_{i=1}^{n-9}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})+T(n-3)F_{n-4}+(T(n-3)+T(n-2))F_{n-3})\\
&= F_{n-2}-\Sigma_{i=1}^{n-6}F_{n-(i+4)}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3}).
\end{aligned}$$

(ii) By induction on i, we prove that $g(n+2)\equiv 0 \pmod{s}$. We have that

$g(K_3(2^2)+2)=g(10)=F_8-\Sigma_{i=1}^{8-4}F_{8-(i+4)}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})\equiv -390\equiv 0 \pmod{2}$.

If $g(K_3(2^{v+1})+2)\equiv 0 \pmod{2^v}$, $1\leq v\leq i-1$. Then

$$\begin{aligned}
g(K_3(2^{v+1})+2) &= F_{K_3(2^{v+1})}-\Sigma_{i=1}^{K_3(2^{v+1})-4}F_{n-(i+4)}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})\\
&\equiv \Sigma_{i=1}^{K_3(2^{v+1})-4}F_{n-(i+4)}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})\\
&\equiv \Sigma_{i=1}^{K_3(2^{v})}F_{n-(i+4)}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})\\
&\quad +\Sigma_{i=K_3(2^v)+1}^{K_3(2^{v+1})}F_{n-(i+4)}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})\\
&\equiv 2\Sigma_{i=1}^{K_3(2^{v})}F_{n-(i+4)}((F_i+F_{i+1})F_{i+2}+(F_{i+1}+F_{i+3})F_{i+3})\equiv 0 \pmod{2^v},
\end{aligned}$$

and the result follows.

■

Lemma 5 *If $L_3(H_{(t,l,m)}; X) = s$, then s is the least integer such that*

$$\begin{cases} F_{s-1} \equiv 1 & \pmod t, \\ F_s \equiv 0 & \pmod t, \\ F_{s+1} \equiv 0 & \pmod t, \\ T(s+1) \equiv 0 & \pmod l, \\ T(s+2) \equiv 1 & \pmod l, \\ T(s+3) \equiv 0 & \pmod l, \\ g(s+1) \equiv 0 & \pmod m, \\ g(s+2) \equiv 0 & \pmod m, \\ g(s+3) \equiv 1 & \pmod m. \end{cases}$$

Moreover, $K_3(t)$ divides $L_3(H_{(t,l,m)}; X)$.

Proof. By Lemma 3 we have $x_n = a^{F_{n-2}} b^{T(n)} c^{g(n)}$. Further, $x_{s+1} = a, x_{s+2} = b$, and $x_{s+3} = c$. Every element of $H_{(t,l,m)}$ can be written uniquely in the form $a^i b^j c^k$ where $1 \leq i \leq t$, $1 \leq j \leq l$, and $1 \leq k \leq m$, so we have

$$\begin{cases} F_{s-1} \equiv 1 & \pmod t, \\ F_s \equiv 0 & \pmod t, \\ F_{s+1} \equiv 0 & \pmod t, \\ T(s+1) \equiv 0 & \pmod l, \\ T(s+2) \equiv 1 & \pmod l, \\ T(s+3) \equiv 0 & \pmod l, \\ g(s+1) \equiv 0 & \pmod m, \\ g(s+2) \equiv 0 & \pmod m, \\ g(s+3) \equiv 1 & \pmod m. \end{cases}$$

Then Lemma 1 yields that $K_3(t) \mid s$. ■

Theorem 1 *For integer $u \geq 1$ and $t = l = m = 2^u$ we have $L_3(H_{(t,l,m)}; X) = K_3(2^{u+1})$.*

Proof. Let $s = 2^u$. Then

$$x_{K_3(s)+1} = a^{F_{K_3(s)-1}} b^{T(K_3(s)+1)} c^{g(K_3(s)+1)} = a,$$
$$x_{K_3(s)+2} = a^{F_{K_3(s)}} b^{T(K_3(s)+2)} c^{g(K_3(s)+2)} = b,$$
$$x_{K_3(s)+3} = a^{F_{K_3(s)+1}} b^{T(K_3(s)+3)} c^{g(K_3(s)+3)} = c.$$

From Lemmas 1 and 5

$$F_{K_3(s)-1} \equiv F_{-1} \equiv 1 \pmod s,\ F_{K_3(s)} \equiv F_0 \equiv 0 \pmod s,\ \text{and } F_{K_3(s)+1} \equiv F_1 \equiv 0 \pmod s,$$
$$T_{K_3(s)+1} \equiv 0 \pmod s,\ T_{K_3(s)+2} \equiv 1 \pmod s,\ \text{and } T_{K_3(s)+3} \equiv 0 \pmod s,$$
$$g_{K_3(2^{u+1})+1} \equiv 0 \pmod s,\ g_{K_3(2^{u+1})+2} \equiv 0 \pmod s,\ \text{and } g_{K_3(2^{u+1})+3} \equiv 1 \pmod s.$$

Therefore, $x_{n+1} = a$, $x_{n+2} = b, x_{n+3} = c$, i.e. $L_3(H_{(t,l,m)}; X) \mid K_3 2^{u+1}$. Let $i = L_3(H_{(t,l,m)}, X)$. Then

$$\begin{cases} F_{i-1} \equiv 1 & \pmod s, \\ F_i \equiv 0 & \pmod s, \\ F_{i+1} \equiv 0 & \pmod s, \\ T(i+1) \equiv 0 & \pmod s, \\ T(i+2) \equiv 1 & \pmod s, \\ T(i+3) \equiv 0 & \pmod s, \\ g(i+1) \equiv 0 & \pmod{2^{u+1}}, \\ g(i+2) \equiv 0 & \pmod{2^{u+1}}, \\ g(i+3) \equiv 1 & \pmod{2^{u+1}}. \end{cases}$$

Table 1: Period of $L_3(H_{(t,l,m)};X)$ for $1 \leq n \leq 10$ and $t=l=m=2^n$.

n	$K_3(2^{n+1})$	$L_3(H_{(t,l,m)};X)$	n	$K_3(2^{n+1})$	$L_3(H_{(t,l,m)};X)$
1	8	8	6	256	256
2	16	16	7	512	512
3	32	32	8	1024	1024
4	64	64	9	2048	2048
5	128	128	10	4096	4096

If $i = j \times K_3(m)$, then by Lemma 4 we have

$$\begin{cases} g(i+1) \equiv 0 \pmod{2^{u+1}}, \\ g(i+2) \equiv 0 \pmod{2^{u+1}}, \\ g(i+3) \equiv 1 \pmod{2^{u+1}}. \end{cases}$$

so $K_3(m^2)$ is a divisor of $L_3(H_{(t,l,m)};X)$ and therefore $L_3(H_{(t,l,m)};X) = L_3(2^{u+1})$. ■

Table 1 gives the period of $L_3(H_{(t,l,m)};X)$ for $1 \leq n \leq 10$ and $t=l=m=2^n$.

3 The k-Fibonacci Sequence of $H_{(t,l,m)}$

In this section, we determine the k-Fibonacci sequence of $H_{(k,l,m)}$ for $k>3$ and investigate its period. Before proving the main results, we present some useful lemmas. For $k=4$, we have

$$h_n(4) = F^4_{n-2} + F^4_{n-1},$$

$$W_1(4) = W_2(4) = 0, W_3(4) = W_4(4) = 1,$$

$$\begin{aligned} W_n(4) &= W_{n-4}(4) + W_{n-3}(4) + W_{n-2}(4) + W_{n-1}(4) \\ &\quad -(F^4_{n-4}h_{n-3}(4) + (F^4_{n-4} + F^4_{n-3})h_{n-2}(4)) + (F^4_{n-4} + F^4_{n-3} + F^4_{n-2})h_{n-1}(4))) \end{aligned}$$

for $n \geq 5$.

Lemma 6 *Every element of $F_3(H_{(t,l,m)};X)$ can be represented by $x_n = a^{h_n(4)} b^{F^4_n} c^{W_n(4)}$, $n \geq 5$.*

Proof. For $n=4$ and $n=5$, we have $x_4 = a^1b^1c^1 = a^{h_4(4)}b^{F^4_4}c^{W_4(4)}$ and $x_5 = a^2b^2c^1 = a^{h_5(4)}b^{F^4_5}c^{W_5(4)}$. Then by induction on n

$$\begin{aligned} x_n &= x_{n-4}x_{n-3}x_{n-2}x_{n-1} \\ &= a^{h_{n-4}(4)}b^{F^4_{n-4}}c^{W_{n-4}(4)}a^{h_{n-3}(4)}b^{F^4_{n-3}}c^{W_{n-3}(4)}a^{h_{n-2}(4)}b^{F^4_{n-2}}c^{W_{n-2}(4)}a^{h_{n-1}(4)}b^{F^4_{n-1}}c^{W_{n-1}(4)} \\ &= a^{h_{n-4}(4)}b^{F^4_{n-4}}a^{h_{n-3}(4)}b^{F^4_{n-3}}a^{h_{n-2}(4)}b^{F^4_{n-2}}a^{h_{n-1}(4)}b^{F^4_{n-1}}c^{W_{n-4}+W_{n-3}+W_{n-2}+W_{n-1}} \\ &= a^{h_{n-4}(4)+h_{n-3}(4)}b^{F^4_{n-4}+F^4_{n-3}}a^{h_{n-2}(4)}b^{F^4_{n-2}}a^{h_{n-1}(4)}b^{F^4_{n-1}}c^{W_{n-4}+W_{n-3}+W_{n-2}+W_{n-1}-F^4_{n-4}h_{n-3}(4)} \\ &\vdots \\ &= a^{h_n(4)}b^{F^4_n}c^{W_{n-4}(4)+W_{n-3}(4)+W_{n-2}(4)+W_{n-1}(4)-(F^4_{n-4}h_{n-3}(4)+(F^4_{n-4}+F^4_{n-3})h_{n-2}(4))+(F^4_{n-4}+F^4_{n-3}+F^4_{n-2})h_{n-1}(4)))} \\ &= a^{h_n(4)}b^{F^4_n}c^{W_n(4)}, \end{aligned}$$

and the result follows. ■

Table 2 gives the period of $L_4(H_{(t,l,m)};X)$ for $2<p<29$, p prime, and $t=l=m=p$.

Table 2: Period of $L_4(H_{(t,l,m)};X)$ for $2<p<29$, p prime, and $t=l=m=p$.

p	$K_4(p)$	$L_4(H_{(t,l,m)};X)$	p	$K_4(p)$	$L_4(H_{(t,l,m)};X)$
3	26	78	13	84	1092
5	312	1560	17	4912	43504
7	342	2394	19	6858	130302
11	120	1320	23	12166	279818

We now generalize the 4-Fibonacci sequence to the k-Fibonacci sequence of $H_{(t,l,m)}$. First, we have that

$$h_n(k)=F^k_{n-2}+F^k_{n-1}+\cdots+F^k_{n+k-5},\ n\geq k+2,$$

$$h_1(k)=h_1(k-1),h_2(k)=h_2(k-1),\ \cdots,\ h_k(k)=h_k(k-1),$$

$$h_{k+1}(k)=h_{k+1}(k-1)+F^{k-1}_{k-2},$$

$$W_1(k)=W_1(k-1),W_2(k)=W_2(k-1),\ \cdots,\ W_{k+1}(k)=W_{k+1}(k-1),$$

$$W_{k+2}(k)=W_{k+2}(k-1)-(h_4(k-1)+h_5(k-1)+h_6(k-1)+(F^{k-1}_4+F^{k-1}_5)F^{k-1}_{k-2}),$$

$$\begin{aligned} W_n(k) &= W_{n-1}(k)+W_{n-2}(k)+\cdots+W_{n-k}(k)\\ &\quad -(h_{n-(k-1)}F^k_{n-4}+(F^k_{n-4}+F^k_{n-3})h_{n-(k-2)}+\cdots+(F^k_{n-4}+F^k_{n-3}+\cdots+F^k_{n+k-5})h_{n-1}(k)).\end{aligned}$$

Lemma 7 *Every element of $F_k(H_{(t,l,m)};X)$ can be represented by $x_n(k)=a^{h_n(k)}b^{F^k_{n+k-4}}c^{W_n(k)}$, $k\geq 4$, $n\geq 4$.*

Proof. We use induction on both k and n. By Lemma 6, we have $x_n(4)=a^{h_n(4)}b^{F^4_n}c^{W_n(4)}$ and if $x_n(s)=a^{h_n(s)}b^{F^s_{n+s-4}}c^{W_n(s)}$, $5\leq s\leq k-1$, it is sufficient to show that

$$x_n(s)=a^{h_n(k)}b^{F^k_{n+k-4}}c^{W_n(k)}.$$

For this, we use induction on n. If $4\leq v\leq k-1$, from the definitions of F_n,h_n, and W_n we have $F^k_v=F^{k-1}_v, h_v(k)=h_v(k-1)$, and $W_v(k)=W_v(k-1)$, respectively, so $x_v(k)=x_v(k-1)$. Then by induction on k

$$x_v(k)=a^{h_v(k)}b^{F^k_{v+k-4}}c^{W_v(k)}.$$

Now suppose that the hypothesis holds for all $v\leq n-1$. From the definition of $x_n(k)$, if $t_n:=t_n(k)$, $F_n:=F^k_n$, and $x_n:=x_n(k)$. Then

$$\begin{aligned} x_n &= x_{n-k}x_{n-k+1}x_{n-k+2}\cdots x_{n-2}(x_{n-1})\\ &= a^{h_{n-k}}b^{F_{n-4}}c^{W_{n-k}}a^{h_{n-(k-1)}}b^{F_{n-3}}c^{W_{n-(k-1)}}\times\cdots\times a^{h_{n-1}}b^{F_{n+k-5}}c^{W_{n-1}}\\ &= a^{h_{n-k}}b^{F_{n-4}}a^{h_{n-(k-1)}}b^{F_{n-3}}c^{W_{n-k}+W_{n-(k-1)}}\times\cdots\times a^{h_{n-1}}b^{F_{n+k-5}}c^{W_{n-1}}\\ &= a^{h_{n-k}+h_{n-(k-1)}}b^{F_{n-4}+F_{n-3}}c^{W_{n-k}+W_{n-(k-1)}-F_{n-4}h_{n-(k-1)}}\times\cdots\times a^{h_{n-1}}b^{F_{n+k-5}}c^{W_{n-1}}\\ &= a^{h_{n-k}+h_{n-(k-1)}+\cdots+h_{n-1}}b^{F_{n-4}+F_{n-3}+\cdots+F_{n+k-5}}\\ &\quad +c^{W_{n-1}+W_{n-2}+\cdots+W_{n-k}-(h_{n-(k-1)}F_{n-4}+(F_{n-4}+F_{n-3})h_{n-(k-2)}+\cdots+(F_{n-4}+F_{n-3}+\cdots+F_{n+k-5})h_{n-1})}\\ &= a^{h_n}b^{F_{n+k-4}}c^{W_n},\end{aligned}$$

and the proof is complete. ■

Theorem 2 *For integer $t\geq 1$ and prime p we have*

$$K_k(l)|L_k(H_{(t,l,m)};X).$$

Proof. The proof is similar to that of Lemma 5 and so is omitted. ■

We end this section with the following open question. Prove or disprove for $t = l = m = p$ and p a prime that

$$L_k(H_{(t,l,m)}; X) = K_k(p^2).$$

4 Error Correcting Codes From the 3-Fibonacci Sequence of $H_{(t,l,m)}$

In this section, a new class of error-correcting codes is obtained using the 3-Fibonacci sequence of $H_{(k,l,m)}$, and the error detection and correction capability of these codes is examined. For this, from Lemmas 3 and 4 we have that

$$x_n = a^{F_{n-2}} b^{T(n)} c^{g(n)},$$

where

$$g(n) = F_{n-2} - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}).$$

Using Lemma 2, define

$$U_n := \begin{bmatrix} 1 & F_{n-2} & F_{n-2} \times (F_{n-2} + F_{n-3}) + (F_{n-2} - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})) \\ 0 & 1 & F_{n-2} + F_{n-3} \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$U_n^T := \begin{bmatrix} 1 & 0 & 0 \\ F_{n-2} & 1 & 0 \\ a & F_{n-2} + F_{n-3} & 1 \end{bmatrix},$$

where

$$a := F_{n-2} \times (F_{n-2} + F_{n-3}) + (F_{n-2} - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})).$$

For a message $M_{3\times 3}$, the transformation

$$E = U_n^T \times M \times U_n,$$

is called 3-Fibonacci Heisenberg group encoding and the transformation

$$M = (U_n^T)^{-1} \times E \times (U_n)^{-1},$$

is called 3-Fibonacci Heisenberg group decoding. The matrix E is called the code matrix and all elements of M are positive, i.e.

$$M := \begin{bmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{bmatrix},$$

where m_i, $1 \leq i \leq 9$, is a nonnegative integer. The following example explains this method.

Example 1 *Consider the message given by*

$$M := \begin{bmatrix} 5 & 7 & 2 \\ 4 & 3 & 6 \\ 10 & 2 & 1 \end{bmatrix}.$$

If $n = 8$ *we have*

$$U_8 = \begin{bmatrix} 1 & F_6 & F_6 \times (F_6 + F_5) + (F_6 - \Sigma_{i=1}^{2} F_{8-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})) \\ 0 & 1 & F_6 + F_5 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 7 & 53 \\ 0 & 1 & 11 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using 3-Fibonacci Heisenberg group encoding gives $E = U_n^T \times M \times U_n$ so that

$$E = U_6^T \times M \times U_6 = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 53 & 11 & 1 \end{bmatrix} \begin{bmatrix} 5 & 7 & 2 \\ 4 & 3 & 6 \\ 10 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 & 53 \\ 0 & 1 & 11 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 42 & 344 \\ 11 & 129 & 1175 \\ 319 & 2639 & 21546 \end{bmatrix}.$$

Then 3-Fibonacci Heisenberg group decoding results in

$$M = (U_6^T)^{-1} \times M \times (U_6)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 24 & -11 & 1 \end{bmatrix} \begin{bmatrix} 5 & 42 & 344 \\ 11 & 129 & 1175 \\ 319 & 2639 & 21546 \end{bmatrix} \begin{bmatrix} 1 & -7 & 24 \\ 0 & 1 & -11 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 2 \\ 4 & 3 & 6 \\ 10 & 2 & 1 \end{bmatrix}.$$

Lemma 8 *For $E = U_n^T \times M \times U_n$, we have $\det(E) = \det(M)$.*

Proof. Since $\det(U_n) = 1$, the result follows. ■

The generalization of 3-Fibonacci Heisenberg group coding to $k-$Fibonacci Heisenberg group coding for $k \geq 4$ is as follows. k-Fibonacci Heisenberg group encoding is

$$E = U_{(n,k)}^T \times M \times U_{(n,k)},$$

and k-Fibonacci Heisenberg group decoding is

$$M = (U_{(n,k)}^T)^{-1} \times E \times (U_{(n,k)})^{-1},$$

where

$$U_{(n,k)} := \begin{bmatrix} 1 & h_n(k) & W_n(k) + h_n(k) F_{n+k-4}^k \\ 0 & 1 & F_{n+k-4}^k \\ 0 & 0 & 1 \end{bmatrix},$$

and $U_{(n,k)}^T$ denotes transpose.

For 3-Fibonacci Heisenberg group coding, we have the following relationship for the elements of the code matrix E

$$\begin{aligned} &(U_n^T)^{-1} \times E \times (U_n)^{-1} \\ = &\begin{bmatrix} 1 & 0 & 0 \\ F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}) & 1 & 0 \\ -F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})) & -(F_{n-2} + F_{n-3}) & 1 \end{bmatrix} \\ &\times \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix} \times \begin{bmatrix} 1 & a & -F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})) \\ 0 & 1 & -(F_{n-2} + F_{n-3}) \\ 0 & 0 & 1 \end{bmatrix} \\ = &\begin{bmatrix} e_1 & b & c \\ d & f & g \\ h & o & u \end{bmatrix}, \end{aligned}$$

where

$$a := F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}),$$

$$b := (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))e_1 + e_2,$$

$$c := (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_1 - (F_{n-2} + F_{n-3})e_2 + e_3,$$

$$d := (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))e_1 + e_4,$$

$$\begin{aligned} f \quad : \quad &= (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})e_1 + e_4) \\ &\times (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})) \\ &+ (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))e_2 + e_5, \end{aligned}$$

$$\begin{aligned} g \quad : \quad &= (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})e_1 + e_4) \\ &\times (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))) - (F_{n-2} + F_{n-3}) \times (F_{n-2} \\ &\times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})e_2 + e_5) \\ &+ (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))e_3 + e_6, \end{aligned}$$

$$h := -F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_1 - (F_{n-2} + F_{n-3})e_4 + e_7,$$

$$\begin{aligned} o \quad : \quad &= (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_1 - (F_{n-2} + F_{n-3})e_4 + e_7) \\ &\times (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})) \\ &+ (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_2 - (F_{n-2} + F_{n-3})e_5 + e_8, \end{aligned}$$

$$\begin{aligned} u \quad : \quad &= (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_1 - (F_{n-2} + F_{n-3})e_4 + e_7) \\ &\times (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))) \\ &- (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_2 \\ &- (F_{n-2} + F_{n-3})e_5 + e_8) \times (F_{n-2} + F_{n-3}) \\ &+ (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_3 \\ &- (F_{n-2} + F_{n-3})e_6 + e_9), \end{aligned}$$

Since $M = (U_n^T)^{-1} \times E \times (U_n)^{-1}$ and $m_i > 0$, $1 \le i \le 9$, we have

$$m_1 = e_1 \ge 0,$$

$$m_2 = b := (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))e_1 + e_2 \ge 0,$$

$$m_3 = c := (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_1 - (F_{n-2} + F_{n-3})e_2 + e_3 \ge 0,$$

$$m_4 = d := (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))e_1 + e_4 \ge 0,$$

$$\begin{aligned} m_5 \quad = \quad & f := (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})e_1 + e_4) \\ &\times (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})) \\ &+ (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))e_2 + e_5 \ge 0, \end{aligned}$$

$$\begin{aligned} m_6 \quad = \quad & g := (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})e_1 + e_4) \\ &\times (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))) - (F_{n-2} + F_{n-3}) \times (F_{n-2} \\ &\times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})e_2 + e_5) \\ &+ (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))e_3 + e_6 \ge 0, \end{aligned}$$

$$m_7 = h := -F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_1 - (F_{n-2} + F_{n-3})e_4 + e_7 \ge 0,$$

$$\begin{aligned} m_8 \quad = \quad & o := (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_1 - (F_{n-2} + F_{n-3})e_4 + e_7) \\ &\times (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})) \\ &+ (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_2 - (F_{n-2} + F_{n-3})e_5 + e_8 \ge 0, \end{aligned}$$

$$
\begin{aligned}
m_9 =: u = &(-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_1 - (F_{n-2} + F_{n-3})e_4 + e_7) \\
&\times (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))) \\
&- (-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_2 \\
&- (F_{n-2} + F_{n-3})e_5 + e_8) \times (F_{n-2} + F_{n-3}) \\
&+(-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})))e_3 - (F_{n-2} + F_{n-3})e_6 + e_9) \geq 0.
\end{aligned}
$$

As

$$m_2 = (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))e_1 + e_2 \geq 0,$$

we have

$$(F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3}))e_1 \geq -e_2,$$

which gives

$$\frac{e_1}{e_2} \leq \frac{1}{-F_{n-2} \times (F_{n-2} + F_{n-3}) + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})}. \tag{1}$$

Similarly, we have

$$\frac{e_1}{e_3} \leq \frac{1}{i}, \tag{2}$$

$$\frac{e_1}{e_4} \leq \frac{1}{-F_{n-2} \times (F_{n-2} + F_{n-3}) + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})}, \tag{3}$$

$$\frac{e_2}{e_5} \leq \frac{1}{-F_{n-2} \times (F_{n-2} + F_{n-3}) + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})}, \tag{4}$$

$$\frac{e_3}{e_6} \leq \frac{1}{-F_{n-2} \times (F_{n-2} + F_{n-3}) + \Sigma_{i=1}^{n-6}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})}, \tag{5}$$

$$\frac{e_1}{e_7} \leq \frac{1}{i}, \tag{6}$$

$$\frac{e_2}{e_8} \leq \frac{1}{i}, \tag{7}$$

$$\frac{e_3}{e_9} \leq \frac{1}{i}, \tag{8}$$

where

$$
\begin{aligned}
i \quad = \quad &-F_{n-2} + \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})) \\
&+(F_{n-2} + F_{n-3}) \times (F_{n-2} \times (F_{n-2} + F_{n-3}) - \Sigma_{i=1}^{n-6} F_{n-(i+4)}((F_i + F_{i+1})F_{i+2} + (F_{i+1} + F_{i+3})F_{i+3})).
\end{aligned}
$$

Therefore, the determinant of the message M, $\det(M)$, is connected to the determinant of the code matrix E, so $\det(M)$ affects the entries of E. First, assume that only one error exists in the received matrix E. There are nine cases to consider

$$
(1) \begin{bmatrix} a & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix}, \qquad
(2) \begin{bmatrix} e_1 & b & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix}, \qquad
(3) \begin{bmatrix} e_1 & e_2 & c \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix},
$$

$$
(4) \begin{bmatrix} e_1 & e_2 & e_3 \\ d & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix}, \qquad
(5) \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix}, \qquad
(6) \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & f \\ e_7 & e_8 & e_9 \end{bmatrix},
$$

$$
(7) \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ g & e_8 & e_9 \end{bmatrix}, \qquad
(8) \begin{bmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & h & e_9 \end{bmatrix}, \qquad
(9) \begin{bmatrix} e_1 & e_2 & e_1 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & i \end{bmatrix},
$$

where a, b, $\cdots$, i are the possible erroneous entries. From $\det(E) = \det(M)$ we have

$$\begin{aligned}
&(1)\ \ a(e_5e_9 - e_6e_8) - e_2(e_4e_9 - e_6e_7) + e_3(e_4e_8 - e_5e_7) = \det(M),\\
&(2)\ \ e_1(e_5e_9 - e_6e_8) - b(e_4e_9 - e_6e_7) + e_3(e_4e_8 - e_5e_7) = \det(M),\\
&\qquad\qquad\vdots\\
&(9)\ \ e_1(e_5i - e_6e_8) - e_2(e_4i - e_6e_7) + e_3(e_4e_8 - e_5e_7) = \det(M).
\end{aligned}$$

Then we can consider double errors in E, for example

$$\begin{bmatrix} a & e_2 & b \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{bmatrix},$$

so there are $\binom{9}{2} = 36$ possibilities.

The number of possible error patterns is

$$\binom{9}{0} + \binom{9}{1} + \binom{9}{2} + \cdots + \binom{9}{9} = 2^9 = 512.$$

Using $\det(E) = \det(M)$ and the relations (1)–(8), we can correct zero, single, double, ..., and eight errors, but not nine errors. Thus, the probability of correct decoding is $\dfrac{510}{511} = 0.998$ (99.8%) if the error patterns are equiprobable [22].

The k-Fibonacci Heisenberg group coding has a high error correction capability compared to classical (algebraic) coding. The reason is that matrix theory is used rather than algebraic techniques that add parity symbols to information symbols. As in [22], we compare 3-Fibonacci Heisenberg group codes with the binary Hamming codes. For $r > 1$, the Hamming code parameters are $n = 2^r - 1$ and $k = 2^r - r - 1$. There are 2^n error patterns of which $2^r = 2^{n-k}$ can be corrected. Thus, the probability of correct decoding is

$$\frac{2^{n-k}}{2^n} = \frac{1}{2^k},$$

so the percentage decreases as r increases. As an example, let $r = 4$, so $n = 15$ and $k = 11$ [22]. This code can correct $2^{15-11} = 2^4 = 16$ error patterns so the probability of correct decoding is .0625. Conversely, the error correction capability of the 3-Fibonacci Heisenberg group codes is .998 which is much higher.

5 Conclusion

In this paper, we studied the k-Fibonacci sequence in the Heisenberg group and considered its application in error correction coding. A new class of codes was introduced and the coding/decoding examined. The error correction capability of this coding method was shown to be very high compared to the Hamming codes.

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References

[1] M. Basu and B. Prasad, Coding theory on the (m, t)-extension of the Fibonacci p-numbers, Discrete Math. Algorithms Appl., 3(2011), 259–267.

[2] E. R. Berlekamp, Algebraic Coding Theory, World Scientific, Singapore, 2015.

[3] L. Betthauser, Ö. Deveci and J. Hiller, A very general binomial matrix, Notes Number Theory Discrete Math., 27(2021), 125–133.

[4] A. Cronin and T. J. Laffey, On a conjecture of Deveci and Karaduman, Linear Algebra Appl., 471(2015), 569–574.

[5] Ö. Deveci, The k-nacci sequences and the generalized order k-Pell sequences in the semi-direct product of finite cyclic groups, Thai J. Sci., 40(2013), 89–98.

[6] Ö. Deveci and E. Karaduman, The Pell sequences in finite groups, Util. Math., 96(2015), 263–279.

[7] Ö. Deveci and A. Shannon, The quaternion-Pell sequence, Comm. Algebra 49(2018), 5403–5409.

[8] M. Esmaeili, M. Esmaeili and T. A. Gulliver, A new class of Fibonacci sequence based error correcting codes, Cryptogr. Commun., 9(2017), 379–396.

[9] S. Falcon, The k-Fibonacci matrix and the Pascal matrix, Cen. Eur. J. Math., 9(2011), 1403–1410.

[10] R. W. Hamming, Error detecting and error correcting codes, Bell Syst. Tech. J., 29(1950) 147–160.

[11] M. Hashemi and M. Pirzadeh, t-nacci sequences in some special groups of order m^3, Math. Rep., 20(2018), 389–399.

[12] M. Hashemi and E. Mehraban, The generalized order k-Pell sequences in some special groups of nilpotency class 2, Comm. Algebra, 50(2022), 1768–1784.

[13] M. Hashemi and E. Mehraban, Factorization of the t-extension of the p-Fibonacci matrix and Pascal matrix, J. Math. Model., 10(2022), 161–172.

[14] M. Hashemi and E. Mehraban, Some new codes on the k-Fibonacci sequence, Math. Probl. Eng., 2021(2021), 7660902.

[15] F. J. MacWilliams and N. J. A. Sloane, The Theory of Error-Correcting Codes, North-Holland, Amsterdam, The Netherlands, 1977.

[16] E. Mehraban and M. Hashemi, Fibonacci length and the generalized order k-Pell sequences of the 2-generator p-groups of nilpotency class 2, J. Algebra Appl., 22(2023), 24 pp.

[17] E. Mehraban and M. Hashemi, An application of the t-extension of the p-Fibonacci Pascal matrix in coding theory, Adv. Math. Phys., 2022(2022), 4619136.

[18] D. V. Osipov, The discrete Heisenberg group and its automorphism group, Math. Notes, 98(2015), 185–188.

[19] D. Tasci, N. Tuglu, and M. Asci, On Fibo-Pascal matrix involving k-Fibonacci and k-Pell matrices, Arab. J. Sci. Eng., 36(2011), 1031–1037.

[20] C. E. Shannon, A mathematical theory of communication, Bell Systems Tech. J., 27(1948), 379–423.

[21] N. J. A. Sloane and A. D. Wyner, Claude Elwood Shannon: Collected Papers, IEEE Press, Piscataway, NJ, USA, 1992.

[22] A. P. Stakhov, Fibonacci matrices, a generalization of the Cassini formula and new coding theory, Chaos Solitons Fractals, 30(2006), 56–66.