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Stepsize Control For The Runge-Kutta-Fehlberg Method Using The Discrete Stochastic Arithmetic To Solve Initial Value Problems*

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Abstract

Finite precision computations can affect the stability of algorithms and the accuracy of computed solutions on a computer. In this paper, we present a method for estimating the number of common significant digits between the exact and computed solutions of a one-dimensional initial value problem solved using the Runge-Kutta-Fehlberg method. Subsequently, we introduce an algorithm that exploits stochastic arithmetic and the CESTAC method to control the impact of round-off errors on the computed solution and stepsize. Finally, we present numerical examples to demonstrate the effectiveness of the proposed method.

1 Introduction

Consider the initial-value problems (IVPs)

$$\begin{cases} \frac{dx}{dt} = f(t,x), & t \in [a,b], \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where x is a real-valued function of t, and $t_0 \in [a,b]$. Several numerical methods can be employed to solve Eq. 1, including one-step schemes like the Runge-Kutta methods and linear multi-step methods such as Adams-Bashforth. In each method, an expression of the form Ch^p, known as truncation error, is disregarded. As the value of h decreases, the truncation error reduces; however the round-off error within the method may increase. The practical error, known as the global error, is a combination of these two error sources. All numerical method that use a stepsize h are influenced by both truncation and round-off errors. Therefore, the goal is to find the optimal stepsize h_{opt} which minimizes the global error.

La Porte and Vignes developed the CESTAC (Contrôle et Estimation Stochastique des Arrondis de Calculs) method, that enables one to estimate the number of exact significant digits of any computed result by a computer [24, 21, 22]. The CESTAC method relies on a probabilistic approach to round-off errors, employing a random rounding mode. The CADNA (Control of Accuracy and Debugging for Numerical Applications) was created to implement the CESTAC method for estimating the accuracy of the solution of problems obtained from numerical methods implemented on a computer [10, 23].

Chesneaux and Jézéquel in [4] introduced a strategy using the CESTAC method and CADNA software to estimate the optimal stepsize for numerical integration using the trapezoidal and Simpson's rules. Their approach has been applied to several problems in numerical analysis. For instance, Abbasbandy and Fariborzi Araghi implemented the same idea for the interpolation problem in [1] and for the closed Newton-Cotes integration rules in [2]. In [15], Salkuyeh et al. proposed an algorithm with stepsize control for solving a system of n one-dimensional initial value problems using single- or multi-step methods. In [18], Salkuyeh

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presented a method to control the stepsize for the finite difference method for solving ordinary differential equations. Jézéquel in [11] further explored dynamical control of converging sequences computation. Additionally, in [16], Salkuyeh and Toutounian used the CADNA library to determine the optimal iteration for the Power and inverse Power methods for computing the largest and smallest eigenvalues of a matrix. In [17], Salkuyeh proposed a procedure to control the stepsize for determining interpolating cubic spline functions. The authors of [6] applied the stochastic arithmetic to validate the Sinc-collocation method, utilizing either single or double exponential decay, to compute an approximate solution for Fredholm integral equations of the second kind. In [3], Barzegar et al. by the CESTAC method estimated the optimal shape parameter and optimal number of points in RBF-meshless methods to solve differential equations. Recently, Noeiaghdam et al. in solving Volterra integral equation of the second kind with discontinuous kernel by an iterative method have found the optimal error, optimal iteration of the method and optimal approximation using the CADNA library [13]. In [14], Noeiaghdam et al. applied the CESTAC method, implemented in the CADNA software, to validate the numerical results and solve second-kind integral equations using the homotopy perturbation method.

In this paper, we develop a strategy to estimate the optimal stepsize and the exact solution for solving initial value problems (IVPs) 1 using the Runge-Kutta-Fehlberg method, based on the CESTAC method and CADNA software.

This paper is organized as follows. In Section 2, we present some theoretical results. Section 3 provides a brief description of stochastic round-off error analysis, the CESTAC method, and the CADNA library. Using the CADNA library, we present an algorithm to control the stepsize in the Runge-Kutta-Fehlberg method in Section 4. Some numerical examples are given in Section 5. Finally, some concluding remarks are given in Section 6.

2 Theoretical Description

The Fehlberg method of order 4 is a Runge-Kutta type (hereafter, it is denoted by RK) for solving Eq. 1 and uses the following formulas [7, 8, 5]

$$\begin{cases} K_1 = hf(t, x), \\ K_2 = hf(t + \frac{1}{4}h, x + \frac{1}{4}K_1), \\ K_3 = hf(t + \frac{3}{8}h, x + \frac{3}{32}K_1 + \frac{9}{32}K_2), \\ K_4 = hf(t + \frac{12}{13}h, x + \frac{1932}{2197}K_1 - \frac{7200}{2197}K_2 + \frac{7296}{2197}K_3), \\ K_5 = hf(t + h, x + \frac{439}{216}K_1 - 8K_2 + \frac{3680}{513}K_3 - \frac{845}{4104}K_4), \\ x(t+h) = x(t) + \frac{25}{216}K_1 + \frac{1408}{2565}K_3 + \frac{2197}{4104}K_4 - \frac{1}{5}K_5. \end{cases} \tag{2}$$

The Fehlberg scheme requires one more function evaluation than the classical Runge-Kutta method of order 4. However, with an additional function evaluation

$$K_6 = hf(t + \frac{1}{2}h, x - \frac{8}{27}K_1 + 2K_2 - \frac{3544}{2565}K_3 + \frac{1859}{4104}K_4 - \frac{11}{40}K_5),$$

we can get a fifth-order Runge-Kutta method, namely,

$$x(t+h) = x(t) + \frac{16}{135}K_1 + \frac{6656}{12825}K_3 + \frac{28561}{56430}K_4 - \frac{9}{50}K_5 + \frac{2}{55}K_6. \tag{3}$$

The aforementioned scheme will be referred to as the Runge-Kutta-Fehlberg (RKF) method. The difference between the values of $x(t+h)$ obtained from the RK and RKF methods provides an estimate of the local truncation error in the fourth-order procedure. Specifically, for a given $\epsilon > 0$,

$$|x^{RKF}(t+h) - x^{RK}(t+h)| < \epsilon,$$

can be considered as a stopping criterion, where $x^{RKF}(t+h)$ and $x^{RK}(t+h)$ are, respectively, the computed solution using Eqs. 3 and 2. Nevertheless, here, the question is how small the value of h should be taken to achieve the best approximate solution that the computer can calculate. A large value of h typically yields a poor approximation, while too small h lead to increased error propagation. Therefore, h should be chosen in such a way that the sum of the truncation error and the round-off error is minimized. In the sequel, we provide a method to calculate the best stepsize h.

For later use, let $F_N(a,b)$ denote the set of functions f for which all partial derivatives up to and including order N exist on the strip $S = \{(t,x) | a \le t \le b, x \in \mathbb{R}^n\}$, a, b finite, and are continuous and bounded there.

Theorem 1 ([19]) *Let $f(t,x) \in F_{M+1}(a,b)$ and let $\eta(t;h)$ be the approximate solution obtained by a one-step method of order p, $p \le M$, to the solution $x(t)$ of IVP 1. Then $\eta(t,h)$ has an asymptotic expansion of the form*

$$\eta(t;h) = x(t) + c_p(t)h^p + c_{p+1}(t)h^{p+1} + \cdots + c_M(t)h^M + C_{M+1}(t;h)h^{M+1},$$

which is valid for all $t \in [a,b]$ and all $h = (t-t_0)/n$, $n = 1, 2, \ldots$, where $c_k(t_0) = 0$, for $k = p, p+1, \ldots$. The functions $c_i(t)$ therein are differentiable and independent of h, and the remainder term $E_{M+1}(t;h)$ is bounded for fixed t and all $h = (t-t_0)/n$, $n = 1, 2, \ldots$.

According to Theorem 1 if $f(t,x) \in F_{M+1}(t,x)$. Then the approximate solutions $\eta^{RKF}(t;h)$ and $\eta^{RK}(t;h)$, computed by the RKF and RK methods have the following expansions

$$\eta^{RK}(t;h) = x(t) + c_4(t)h^4 + c_5(t)h^5 + \cdots + c_M(t)h^M + C_{M+1}(t;h)h^{M+1}, \tag{4}$$

$$\eta^{RKF}(t;h) = x(t) + d_5(t)h^5 + d_6(t)h^6 + \cdots + d_M(t)h^M + D_{M+1}(t;h)h^{M+1}. \tag{5}$$

Definition 1 ([4, 12]) *Let p and q be two real numbers. The number of significant digits that are common to p and q can be defined in $(-\infty, +\infty)$ by*

1. *for $p \ne q$,*
$$C_{p,q} = \log_{10}\left|\frac{p+q}{2(p-q)}\right|.$$
2. $\forall p \in \mathbb{R},\ C_{p,p} = +\infty.$

The next theorem gives a relationship between the number of common significant digits of $\eta^{RKF}(t;h)$ and $\eta^{RK}(t;h)$ with of that of $\eta^{RK}(t;h)$ and $x(t)$. Without loss of generality, we assume that $x(t) \ne 0$ and $c_4(t) \ne 0$.

Theorem 2 *Let $f(t,x) \in F_7(a,b)$. Then*

$$C_{\eta^{RKF}(t;h),\eta^{RK}(t;h)} = C_{\eta^{RK}(t;h),x(t)} + \frac{1}{\ln 10}\frac{d_5(t)}{c_4(t)}h + \mathcal{O}(h^2), \tag{6}$$

where $x(t)$ is the exact solution of 1.
Proof. *For the sake of brevity, we use η^{RKF} and η^{RK} as shorthand for $\eta^{RKF}(t;h)$ and $\eta^{RK}(t;h)$, respectively. At first, we see that*

$$\begin{aligned} C_{\eta^{RKF},\eta^{RK}} - C_{\eta^{RK},x(t)} &= \log_{10}\left|\frac{\eta^{RKF}+\eta^{RK}}{2\left(\eta^{RKF}-\eta^{RK}\right)}\right| - \log_{10}\left|\frac{\eta^{RK}+x(t)}{2\left(\eta^{RK}-x(t)\right)}\right| \\ &= \log_{10}\left|\frac{\eta^{RKF}+\eta^{RK}}{\eta^{RK}+x(t)}\right| + \log_{10}\left|\frac{\eta^{RK}-x(t)}{\eta^{RKF}-\eta^{RK}}\right|. \end{aligned} \tag{7}$$

Using Eqs. 4 and 5 with $M = 6$ we have

$$\eta^{RKF} + \eta^{RK} = 2x(t) + c_4(t)h^4 + (c_5(t) + d_5(t))h^5 + \mathcal{O}(h^6), \tag{8}$$

$$\eta^{RK} + x(t) = 2x(t) + c_4(t)h^4 + c_5(t)h^5 + \mathcal{O}(h^6), \tag{9}$$

$$\eta^{RKF} - \eta^{RK} = -c_4(t)h^4 + (d_5(t) - c_5(t))h^5 + \mathcal{O}(h^6), \tag{10}$$

$$\eta^{RK} - x(t) = c_4(t)h^4 + c_5(t)h^5 + \mathcal{O}(h^6). \tag{11}$$

Using Eqs. 8, 9 and the Taylor expansion of $\ln(1+z)$ *for* $|z| < 1$*, we get*

$$\begin{aligned} \log_{10}\left|\frac{\eta^{RKF} + \eta^{RK}}{\eta^{RK} + x(t)}\right| &= \log_{10}\left|\frac{2x(t) + c_4(t)h^4 + (c_5(t) + d_5(t))h^5 + \mathcal{O}(h^6)}{2x(t) + c_4(t)h^4 + c_5(t)h^5 + \mathcal{O}(h^6)}\right| \\ &= \frac{1}{2\ln 10}\frac{d_5(t)}{x(t)}h^5 + \mathcal{O}(h^6). \end{aligned} \tag{12}$$

Similarly, from Eqs. 10 and 11 we see that

$$\log_{10}\left|\frac{\eta^{RK} - x(t)}{\eta^{RKF} - \eta^{RK}}\right| = \log_{10}\left|\frac{c_4(t) + c_5(t)h + \mathcal{O}(h^2)}{c_4(t) - (d_5(t) - c_5(t))h + \mathcal{O}(h^2)}\right| = \frac{1}{\ln 10}\frac{d_5(t)}{c_4(t)}h + \mathcal{O}(h^2). \tag{13}$$

Now, substituting Eqs. 12 and 13 in Eq. 7 gives the desired result. ■

This theorem shows that, for sufficiently small value of h, the number of common significant digits of $\eta^{RKF}(t;h)$ and $\eta^{RK}(t;h)$ are the same as that of $\eta^{RK}(t;h)$ and $x(t)$, up to one digit.

3 The CESTAC Method and The CADNA Library

The basic idea of the CESTAC method involves executing the same code N times, each time with a different round-off error propagation. The goal is to estimate the common part of the results obtained from these runs and consider this common part as representative of the exact result. In practice, these various round-off error propagations are achieved by using the random rounding mode.

Let F represent the set of all values that are representable in the computer. Therefore, every real value r is represented as $R \in F$ in the computer. In binary floating-point arithmetic with a mantissa of P bits, the rounding error arising from the assignment operator is given by

$$R = r - \epsilon 2^{E-P}\alpha,$$

where ϵ represents the sign of r, $2^{-P}\alpha$ denotes the lost part of the mantissa due to round-off error, and E is the binary exponent of the result. For single precision on a personal computer, the value of P is 24, and for double precision, it is 53. If α is considered a random variable uniformly distributed on $[-1, 1]$. Then the computed result R becomes a random variable, and its precision is determined by its mean value and standard deviation.

If a code is executed N times with a computer using random rounding, resulting in N values R_k, where $k = 1, \ldots, N$. Under certain assumptions, it can be demonstrated that these N results follow a quasi-Gaussian distribution centered around the exact result r. Consequently, in practical applications, the mean value $\overline{R}$ of the R_k can be considered as the computed result. By applying Student's test, a confidence interval for $\overline{R}$ with a probability of $(1-\beta)$ can be determined. This allows for the estimation of the number of significant digits of $\overline{R}$ using the formula (see [20, 24])

$$C_{\overline{R}} = \log_{10}\frac{\sqrt{N}|\overline{R}|}{\tau_\beta \sigma}, \tag{14}$$

where

$$\overline{R} = \frac{1}{N}\sum_{i=1}^{N} R_i, \quad \sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(R_i - \overline{R})^2.$$

Here, τ_β represents the value of the Student distribution for $N-1$ degrees of freedom at a probability level of $1-\beta$. In practical scenarios, setting $N=3$ and $\beta = 0.05$ yields $\tau_\beta = 4.4303$.

Definition 2 **([22])** *Each result provided by the CESTAC method is an informatical zero denoted by @.0 iff one of the two conditions holds:*

1) $\forall i,\ i = 1, \ldots, N,\ R_i = 0.$

2) $C_{\overline{R}} \leq 0$, *(*$C_{\overline{R}}$ *obtained by Eq. (14)).*

When $C_{\overline{R}} \leq 0$. *Then* $\overline{R}$ *is an unsignificant value.*

Definition 3 **([22])** *Let* X *and* Y *be* N*-samples provided by the CESTAC method, discrete stochastic equality denoted by* $s=$ *is defined as:* $Xs = Y$ *if* $X - Y =$ @.0.

As was mentioned in Section 1, the CADNA software applies the CESTAC method to estimate the accuracy of any computed result whose digits are affected by round-off errors. Indeed, CADNA implements all the concepts and definitions of the stochastic arithmetic and enables a real numerical debugging by detecting numerical instabilities which usually involve numerical noise. When a stochastic variable is printed, only its significant digits are displayed to point out its accuracy. If a number has no significant digit (i.e., a computed zero), the symbol @.0 is displayed. For more details about CADNA the reader is referred to https://www-pequan.lip6.fr/cadna

4 Stepsize Estimation for RKF

From Theorem 2 we see that as h approaches to zero, the number of common significant digits between $\eta^{RKF}(t;h)$ and $\eta^{RK}(t;h)$ is equal to the number of common significant digits between $\eta^{RK}(t;h)$ and $x(t)$, up to one digit. Now, we are in the situation where our algorithm for estimating the optimal stepsize can be estimated using the CADNA library. To compute the solution at $t \in [a,b]$, we choose a coarse stepsize h and the solution $x(t)$ is computed using the RKF method. If

$$e(t;h) := |\eta^{RKF}(t;h) - \eta^{RK}(t;h)| = 0.@.$$

Then h is considered as an estimation of the optimal stepsize and $\eta^{RK}(t;h)$ is chosen the best solution which the computer can compute. Otherwise, the stepsize is halved and the above precess is repeated. Indeed, in the mth repetition the stepsize is set $h = (t-a)/2^m$.

Algorithm 1 ***Stepsize control in the RKF method***

1. *Choose an initial stepsize* h
2. *Compute* $\eta^{RKF}(t;h)$ *and* $\eta^{RK}(t;h)$
3. *If* $|\eta^{RKF}(t;h) - \eta^{RK}(t;h)| = 0.@$. *Then Stop*
4. *Set* $h := h/2$, *and go to Step 2*

Table 1: Results for Example 1.

m	$\eta^{RK}(1;h)$	$\eta^{RKF}(1;h)$	$\mid \eta^{RKF}(1;h)-\eta^{RK}(1;h) \mid$	$\mid \eta^{RK}(1;h)-x(1) \mid$
0	0.920370730213091E+008	-0.400953432733725E+009	0.492990505755034E+009	0.920370726534296E+008
1	0.142214886239385E+014	-0.264908329475341E+014	0.407123215714726E+014	0.142214886239381E+014
2	0.996208024173633E+019	-0.622543420409556E+019	0.161875144458319E+020	0.996208024173633E+019
3	0.1521258970723E+016	-0.3440572426867E+014	0.1555664694992E+016	0.1521258970723E+016
4	-0.57849299453042E+019	-0.14478070211645E+019	0.43371229241397E+019	0.57849299453042E+019
5	0.367840380283578E+000	0.367866037437244E+000	0.25657153666E-004	0.39060887863E-004
6	0.367878291987841E+000	0.367879013592116E+000	0.721604274E-006	0.114918360E-005
7	0.367879413226051E+000	0.367879430628179E+000	0.27945390E-007	0.17402128E-007
8	0.367879440416371E+000	0.367879440886100E+000	0.4697283E-009	0.755070E-009
9	0.367879441149589E+000	0.367879441163187E+000	0.13597E-010	0.21852E-010
10	0.367879441170785E+000	0.367879441171194E+000	0.408E-012	0.656E-012
11	0.36787944117142E+000	0.36787944117143E+000	0.12E-013	0.1E-013
12	0.36787944117144E+000	0.36787944117144E+000	@.0	@.0

5 Numerical Experiments

We present three examples to demonstrate the proposed procedure. We initially set $h = t - a$ and apply Algorithm 1 to compute the solution at the end of the given interval.

Example 1 *Consider the following IVP*

$$\begin{cases} x' = -100x + 99e^t, & in\ [0,1], \\ x(0) = 0. \end{cases}$$

The exact solution of this problem is $x(t) = e^{-t} - e^{-100t}$. Algorithm 1 is used to compute the solution of the problem at $t = 1$ and the numerical results are presented in Table 1. As we see, the optimal value of stepsize is estimated as $h = 1/2^{12}$ and the approximate solution is

$$x(1) \approx \eta^{RK}(1; \frac{1}{2^{12}}) = 0.36787944117144E + 000.$$

The exact solution of the problem is

$$x(1) = e^{-1} - e^{-100} = 0.367879441171442E + 000.$$

By comparing the exact and approximate solutions we see that all the significant digits of the approximate solution are correct. In Table 2, we present the computed solutions of the problem using the estimated optimal stepsize at some points in the interval $[0, 1]$. As we observe, up to one digit, the significant digits of $\eta^{RK}(t; h)$ are the same with those of $x(t)$.

Table 2: Results for intemediate values for Example 1.

t	$\eta^{RK}(t;h)$	$x(t)$	$\mid \eta^{RK}(t;h)-x(t) \mid$
0.100097656250000E+000	0.90470410059243E+000	0.904704100592434E+000	@.0
0.199951171875000E+000	0.81877072907030E+000	0.818770729070311E+000	@.0
0.300048828125000E+000	0.74078204880005E+000	0.740782048800052E+000	@.0
0.399902343750000E+000	0.67038551017407E+000	0.670385510174073E+000	@.0
0.500000000000000E+000	0.60653065971263E+000	0.606530659712633E+000	@.0
0.600097656250000E+000	0.54875804382454E+000	0.548758043824541E+000	@.0
0.700195312500000E+000	0.49648832394525E+000	0.496488323945259E+000	@.0
0.800048828125000E+000	0.44930702476202E+000	0.449307024762028E+000	@.0
0.900146484375000E+000	0.40651010799990E+000	0.406510107999904E+000	@.0
0.100000000000000E+001	0.36787944117144E+000	0.367879441171442E+000	@.0

Table 3: Results for Example 2.

m	$\eta^{RK}(2;h)$	$\eta^{RKF}(2;h)$	$\mid \eta^{RKF}(2;h) - \eta^{RK}(2;h) \mid$	$\mid \eta^{RK}(2;h) - x(2) \mid$
0	0.704674469242015E+001	0.12777366955309E+001	0.576900799688921E+001	0.468897303912866E+001
1	0.35034880566331E+001	0.30253342931123E+001	0.47815376352078E+000	0.11457164033416E+001
2	0.236209477875104E+001	0.236024802796166E+001	0.184675078937E-002	0.4323125459560E-002
3	0.235782549709573E+001	0.235780024371002E+001	0.25253385714E-004	0.53843804251E-004
4	0.235777275154523E+001	0.235777222608914E+001	0.525456086E-006	0.109825374E-005
5	0.235777168116066E+001	0.235777166783582E+001	0.1332483E-007	0.2786918E-007
6	0.235777165407673E+001	0.235777165370238E+001	0.37434E-009	0.78524E-009
7	0.23577716533147E+001	0.23577716533036E+001	0.1108E-010	0.233E-010
8	0.23577716532921E+001	0.23577716532918E+001	0.33E-012	0.70E-012
9	0.23577716532915E+001	0.23577716532914E+001	0.1E-013	@.0
10	0.23577716532921E+001	0.2357771653291E+001	@.0	@.0

Example 2 *We consider the following IVP*

$$\begin{cases} x' = -x^2 + 2x + 1, & in\ [0,2], \\ x(0) = 0. \end{cases}$$

The exact solution of this problem is [9]

$$x(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}\right).$$

By computing the exact solution at $t = 2$ we get $x(2) = 0.235777165329148E{+}001$. Numerical results obtained by Algorithm 1 are given in Table 3. This table shows that the optimal value of h is estimated as $h = 2/2^{10}$ and the approximate solution corresponding to this h is

$$\eta^{RK}(2; \frac{2}{2^{10}}) = 0.2357771653291E + 001.$$

As we observe all the significant digits of exact and approximate solutions are correct.

Example 3 *Consider the following IVP*

$$\begin{cases} x'(t) = tx + te^{t^2}, & in\ [0,2], \\ x(0) = 1. \end{cases}$$

The exact solution of this problem is $x(t) = e^{t^2}$. The value of this function at $t = 2$ is

$$x(2) = 0.545981500331442E + 002.$$

Numerical results are given in Table 4. As we observe the optimal value of h is estimated as $h = \frac{2}{2^{11}}$. On the other hand, the approximate solution of the problem computed by Algorithm 1 is given by

$$\eta^{RK}(2; \frac{2}{2^{11}}) = 0.5459815003314E + 002.$$

By comparing the exact and approximate solution, we find that all the significant digits of computed solution are correct.

6 Conclusion

We have presented a procedure to estimate the optimal stepsize in the Range-Kutta-Fehlberg method for solving IVPs. This method not only estimates the optimal stepsize but also evaluates the best solution

Table 4: Results for Example 3.

m	$\eta^{RK}(2;h)$	$\eta^{RKF}(2;h)$	$\mid \eta^{RKF}(2;h) - \eta^{RK}(2;h) \mid$	$\mid \eta^{RK}(2;h) - x(2) \mid$
0	0.497678282611901E+002	0.502677027522395E+002	0.4998744910493E+000	0.48303217719540E+001
1	0.541643909567675E+002	0.542723026907965E+002	0.1079117340289E+000	0.4337590763766E+000
2	0.545893393925204E+002	0.545833673924956E+002	0.59720000248E-002	0.881064062378E-002
3	0.545994686260350E+002	0.545977233040428E+002	0.17453219922E-002	0.13185928908E-002
4	0.545982683103826E+002	0.545981407326407E+002	0.1275777419E-003	0.118277238E-003
5	0.545981558349879E+002	0.545981498553491E+002	0.59796388E-005	0.58018437E-005
6	0.545981502580430E+002	0.54598150029862E+002	0.2281806E-006	0.224898E-006
7	0.54598150040956E+002	0.54598150033081E+002	0.78752E-008	0.7812E-008
8	0.54598150033401E+002	0.54598150033142E+002	0.2585E-009	0.25E-009
9	0.5459815003315E+002	0.5459815003314E+002	0.82E-011	0.8E-011
10	0.5459815003314E+002	0.5459815003314E+002	0.2E-012	@.0
11	0.5459815003314E+002	0.5459815003314E+002	@.0	@.0

that a computer can achieve. The proposed procedure employs the CADNA library which is based on the CESTAC method. To illustrate the effectiveness of this approach, numerical experiments were reported for three examples. Additionally, with some straightforward simplifications, this method can be extended to solve n one-dimensional initial value problems.

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