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Common Fixed Points Of Multivalued Interpolative Contractions In Super Metric Space With An Application To Dynamical Process*

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Abstract

In this paper, we introduce a multivalued interpolative contraction mapping and establish common fixed point theorems in the framework of super metric space. Relevant examples are given to support this new result. As an application, we obtain the solution of a functional equation arising in dynamic programming.

1 Introduction

The extension of the Banach contraction principle [4] to various spaces, such as b-metric space [10], fuzzy metric space [31], partial metric space [26], modular metric space [9], cone metric space [12], and others has been explored by numerous researchers. In 2007, Huang and Jhang [12] proposed the concept of a cone metric space, serving as a generalization of the metric space. Various authors have subsequently derived fixed-point results for different types of contractions within these spaces ([1], [3], [14], [18], [25]). In 2015, Jleli and Samet [13] introduced a novel generalization of metric spaces, encompassing a broad class of topological spaces, including standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces. More recently, a new extension of the metric space termed as super metric space, was introduced by Karapinar and Khojasteh [23].

In 1969, Nadler [27] made a significant contribution to the field of topology by introducing the concept of multivalued mappings. This novel approach expanded the traditional understanding of mappings, which typically associated a single output with each input. Nadler's exploration of multivalued mappings allowed for a more nuanced and versatile representation of relationships between sets. The motivation behind Nadler's work stemmed from the recognition that many real-world processes and systems exhibit inherently multivalued behaviors. For instance, in dynamical systems, where a system's state evolves over time, a single initial condition might lead to multiple possible future states. Multivalued mappings provide a powerful tool for modeling such complex and diverse relationships. Multivalued mapping has applications in various branches of mathematics including functional analysis, topology and optimization.

As it is well-established, a mapping that satisfies the Banach contraction condition is necessarily continuous. This prompts a relevant question: Can a mapping that is discontinuous and has a similar contractive condition to the Banach contraction, still exhibit a fixed point? In 1968, Kannan [15] addressed this question affirmatively. According to Kannan, a mapping U is termed as Kannan contraction if there exists a $\alpha \in [0, 1/2)$ such that for any x, $y \in X$,

$$d(Ux, Uy) \leq \alpha\{d(x, Ux) + d(y, Uy)\}$$

where U is not a continuous map. He established that if X is a complete metric space and U is a Kannan contraction mapping on X. Then it possesses a unique fixed point. A Kannan mapping may have

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discontinuity in the given space but it always continuous at it's fixed points. In 1999, Pant [28] achieved the initial breakthrough in understanding discontinuity at fixed points. Recently, Pant et al. ([8], [29]) have contributed new insights to this problem.

In 2018, Karapinar [16] revisited Kannan-type contractions, incorporating the interpolation technique. An interpolative contraction mapping is characterized by a contraction factor k that allows the adjustment or "squeezing"of the distance between points at any desired rate within the range of 0 to 1. For a standard metric space (M, b), Karapinar extended the Kannan-type contraction using the interpolative approach, presenting the following generalization:

$$b(Ux, Uy) \leq \lambda([b(x, Ux)]^q [b(y, Uy)]^{1-q}),$$

for all $x, y \in M \backslash Fix(U)$ and $\lambda \in [0, 1)$.

Recently, numerous studies have been done in this direction (see [2], [11], ([17], [19], [22], [24] and [30]). In 2020, Karapinar et al. [20] introduced the concept of interpolative Boyd-Wong type contractions and Matkowski-type contractions for both standard metric spaces and partial metric spaces and established some fixed point theorems for these mappings.

Inspired by these findings, we introduce the concept of a multivalued interpolative contraction mapping in setting of a super metric space. Our result is an extension and a generalization of many results existing in the literature. We present illustrative examples to underscore the relevance of our results in comparison to existing literature. The conclusion of our paper is highlighted through the application in solving functional equations in dynamic programming.

2 Preliminary

In this section we revisit some definitions and results that will contribute to the development of our main findings. We begin with recalling the definition of super metric space.

Definition 1 ([23]) *Let X be a nonempty set and $m : X \times X \to [0, \infty)$. Then m is said to be super metric if the following conditions hold:*

(m_1) *for all x, $y \in X$, if $m(x, y) = 0$. Then $x = y$;*

(m_2) *$m(x, y) = m(y, x)$ for all $x, y \in X$;*

(m_3) *there exists $s \geq 1$ such that for every $y \in X$, there exist two distinct sequences $\{x_n\}, \{y_n\} \subset X$, with $m(x_n, y_n) \to 0$ when $n \to \infty$ and*

$$\limsup_{n\to\infty} m(y_n, y) \leq s \limsup_{n\to\infty} m(x_n, y).$$

Here, (X, m, s) is called a super metric space.

Example 1 *Let $X = [0, +\infty)$, $s = 2$ and $m : X \times X \to [0, \infty)$ defined as*

$$m(x, y) = |x - y|, \quad \forall x \in [0, \infty)\backslash\{2\},$$

$$m(2, y) = m(y, 2) = (2 - y)^2, \quad \text{otherwise.}$$

It is easy to verify that the conditions (m_1) and (m_2) are fulfilled. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $[0, \infty)\backslash\{2\}$ such that $m(x_n, y_n) \to 0$ as $n \to \infty$. Then, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c$ and for $x \in [0, \infty)\backslash\{2\}$, we have

$$\limsup_{n\to\infty} m(y_n, y) = \limsup_{n\to\infty} |x_n - y| = |c - y| \leq s|c - y| = \limsup_{n\to\infty} m(x_n, y).$$

If $y = 2$, by opting for identical sequences $\{x_n\}$, $\{y_n\} \in X$ it follows that (m_3) holds. Hence, (X, m, s) forms a super metric space.

Definition 2 ([23]) *On a super metric space (X, m, s), a sequence $\{x_n\}$ is said to be*

(i) convergent and converges to x in X iff, $\lim_{n\to\infty} m(x_n, x) = 0$; and

(ii) a Cauchy sequence in X iff,

$$\lim_{n\to\infty} \sup\{m(x_n, x_p) : p > n \text{ and } p, n \in \mathbb{N}\} = 0.$$

Proposition 1 ([21]) *The limit of a convergent sequence is unique in a super metric space.*

Definition 3 ([23]) *A super metric space is complete iff, every Cauchy sequence converges to a point in the super metric space.*

Definition 4 ([27]) *Let $T : X \to CB^m(X)$ is a multivalued mapping. Then, $x \in X$ is said to be a fixed point of T if $x \in Tx$. Also a point $x \in X$ is called a common fixed point of two multivalued mappings T, $S : X \to CB^m(X)$ if $x \in Tx \bigcap Sx$.*

Let $CB^m(X)$ be a collection of all non empty closed bounded subsets of X. For $A \in CB^m(X)$, we define

$$m(a, A) = \inf\{m(a, x) : x \in A\}.$$

For B, $C \in CB^m(X)$,

$$\delta_m(B, C) = \sup\{m(b, C) : b \in B\}$$

and

$$H_m(B, C) = \max\{\delta_m(B, C), \delta_m(C, B)\}.$$

Lemma 1 ([24]) *Let (X, m, s) be a super metric space and A, $B \in CB^m(X)$. Then*

(i) $\delta_m(x, B) \leq m(x, y)$ for any $y \in B$ and $x \in X$;

(ii) $\delta_m(x, B) \leq H_m(A, B)$ for any $x \in A$.

Lemma 2 ([24]) *Consider A, $B \in CB^m(X)$ and $x \in A$. Then for any $\epsilon > 0$, there exists $y \in B$ such that*

$$m(x, y) \leq H_m(A, B) + \epsilon.$$

Lemma 3 *Let $\{K_n\}$ be a sequence in $CB^m(X)$ and for some $K \in CB^m(X)$, $\lim_{n\to\infty} H_m(K_n, K) = 0$. If $\{u_n\} \in \{K_n\}$ and for some $u \in X$, $\lim_{n\to\infty} m(u_n, u) = 0$, then $u \in K$.*

3 Main Result

Definition 5 *Let (X, m, s) is a super metric space. A mapping $T : X \to CB^m(X)$ is called a multivalued interpolative contraction if there exists $\alpha \in [0, 1)$ and $\lambda_1, \lambda_2, \lambda_3 \in (0, 1)$ such that $\lambda_1 + \lambda_2 + \lambda_3 < 1$ and*

$$H_m(Tx, Ty) \leq \alpha[[m(x, y)]^{\lambda_1}[m(x, Tx)]^{\lambda_2}[m(y, Ty)]^{\lambda_3}[\frac{1}{3}(m(x, Ty) + m(y, Tx))]^{1-\lambda_1-\lambda_2-\lambda_3}] \qquad (1)$$

for all x, $y \in X$ with x, $y \notin Fix(T)$.

Theorem 1 *Let (X, m, s) is a complete super metric space and T is a multivalued interpolative contraction mapping. Then T has a fixed point.*

Proof. Let $x_0 \in X$, choose $x_1 \in Tx_0$. From Lemma 2 we can select $x_2 \in Tx_1$, such that $m(x_2, x_1) \leq H_m(Tx_1, Tx_0) + \epsilon$. Similarly, we may consider $x_3 \in Tx_2$ such that $m(x_3, x_2) \leq H_m(Tx_2, Tx_1) + \epsilon$ and so on. Proceeding in the same manner, we generate a sequence $\{x_n\}$ satisfying $x_{n+1} \in Tx_n$ such that

$$m(x_{n+1}, x_n) \leq H_m(Tx_n, Tx_{n-1}) + \epsilon.$$

Suppose that $x_n \notin Tx_n$, $\forall n \in \mathbb{N}$. Otherwise, T has a fixed point in X. Thus, $m(x_n, Tx_n) > 0 \, \forall \, n \in \mathbb{N}$. Taking $x = x_{n-1}$ and $y = x_n$ in (1), we have

$$\begin{aligned} m(x_n, Tx_n) &\leq H_m(Tx_{n-1}, Tx_n) \\ &\leq \alpha[[m(x_{n-1}, x_n)]^{\lambda_1}[m(x_{n-1}, Tx_{n-1})]^{\lambda_2}[m(x_n, Tx_n)]^{\lambda_3} \\ &\quad [\frac{1}{3}(m(x_{n-1}, Tx_n) + m(x_n, Tx_{n-1}))]^{1-\lambda_1-\lambda_2-\lambda_3}] \\ &= \alpha[[m(x_{n-1}, x_n)]^{\lambda_1}[m(x_{n-1}, x_n)]^{\lambda_2}[m(x_n, x_{n+1})]^{\lambda_3} \\ &\quad [\frac{1}{3}(m(x_{n-1}, x_{n+1}) + m(x_n, x_n))]^{1-\lambda_1-\lambda_2-\lambda_3}] \\ &= \alpha[[m(x_{n-1}, x_n)]^{\lambda_1+\lambda_2}[m(x_n, x_{n+1})]^{\lambda_3}[\frac{1}{3}(m(x_{n-1}, x_{n+1}) + m(x_n, x_n))]^{1-\lambda_1-\lambda_2-\lambda_3}]. \end{aligned}$$

If $m(x_n, x_{n+1}) > m(x_{n-1}, x_n)$,

$$m(x_n, x_{n+1}) = m(x_n, Tx_n) \leq \alpha[m(x_n, x_{n+1})]^{1-\lambda_1-\lambda_2} \leq \alpha m(x_n, x_{n+1}).$$

This is a contradiction as $\alpha \in [0, 1)$. If

$$m(x_n, x_{n+1}) \leq m(x_{n-1}, x_n),$$

then

$$m(x_n, x_{n+1}) \leq m(x_n, Tx_n) \leq \alpha[m(x_{n-1}, x_n)]^{1-\lambda_2-\lambda_3} \leq \alpha[m(x_{n-1}, x_n)]$$

for all $n \in \mathbb{N}$. Hence, $\{m(x_n, Tx_n)\}$ is non-increasing and non negative sequence and

$$m(x_n, x_{n+1}) \leq \alpha[m(x_{n-1}, x_n)] \leq \alpha^2[m(x_{n-2}, x_{n-1})] ... \leq \alpha^n[m(x_0, x_1)]. \tag{2}$$

Since $\alpha \in [0, 1)$, and taking limit in (2) as $n \to \infty$, we have

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0.$$

Now, for any positive integer k,

$$\begin{aligned} m(x_n, x_{n+k}) &\leq m(x_n, x_{n+1}) + m(x_{n+1}, x_{n+2}) + ... + m(x_{n+k-1}, x_{n+k}) \\ &\leq \alpha^n m(x_0, x_1) + \alpha^{n+1} m(x_0, x_1) + ... + \alpha^{n+k-1} m(x_0, x_1) \\ &= (\alpha^n + \alpha^{n+1} + ... + \alpha^{n+k-1}) m(x_0, x_1) \\ &\leq \frac{\alpha^n}{1-\alpha} m(x_0, x_1). \end{aligned} \tag{3}$$

Letting $n \to \infty$ in (3), we obtain $m(x_n, x_{n+k}) \to 0$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since (X, m, s) is complete, we see that $\lim_{n\to\infty} x_n = x$ for some $x \in X$.

Next, we show that x is a fixed point of T. Now,

$$\begin{aligned} m(x_{n+1}, Tx) &\leq H_m(Tx_n, Tx) \\ &\leq \alpha[[m(x_n, x)]^{\lambda_1}[m(x_n, Tx_n)]^{\lambda_2}[m(x, Tx)]^{\lambda_3}[\frac{1}{3}(m(x_n, Tx) + m(x, Tx_n))]^{1-\lambda_1-\lambda_2-\lambda_3}] \\ &= \alpha[[m(x_n, x)]^{\lambda_1}[m(x_n, x_{n+1})]^{\lambda_2}[m(x, Tx)]^{\lambda_3}[\frac{1}{3}(m(x_n, Tx) + m(x, x_{n+1}))]^{1-\lambda_1-\lambda_2-\lambda_3}]. \end{aligned}$$

Letting $n \to \infty$, we obtain $\lim_{n\to\infty} m(x_{n+1}, Tx) = \lim_{n\to\infty} m(x, Tx) = 0$. Therefore $x \in Tx$ or T has a fixed point in X. ∎

Example 2 *Consider $X = [0, \infty)$ and $m(x, y) = |x - y|^2$ for all $x, y \in X$. Construct $T : X \to CB^m(X)$ such that*

$$T(x) = \begin{cases} \{0\}, & x \in [0, 1), \\ \{x, x+1\}, & x \geq 1. \end{cases}$$

For $s = 1$, clearly (X, m, s) is a complete super metric space. Let $x = 0$. Then $T(0) = \{0\}$, therefore 0 is a fixed point of T. Let $x \geq 1$. Then $x \in T(x)$, therefore every $x \geq 1$ is a fixed point of T. Now let $x, y \notin Fix(X)$. Then $x, y \in (0, 1)$. Now, $H_m(Tx, Ty) = H_m(\{0\}, \{0\}) = 0$. Thus, T is a multivalued interpolative contraction mapping for any $\alpha \in [0, 1)$ and $\lambda_1, \lambda_2, \lambda_3 \in (0, 1)$ with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Hence, all the conditions of Theorem 1 satisfied. Thus T has infinitely many fixed points.

If we take T as a single valued mapping in Theorem 1. Then we get the following corollary as generalization of interpolative Hardy-Rogers type contraction obtained in [19] on super metric space.

Corollary 1 *Let (X, m, s) be a complete super metric space and $T : X \to X$ satisfying for all x, $y \in X$. Then*

$$H_m(Tx, Ty) \leq \alpha \max \left\{ m(x, y), m(x, Tx), m(y, Ty), \frac{1}{3}(m(x, Ty) + m(y, Tx)) \right\}$$

for all x, $y \in X$ and some $\alpha \in [0, 1)$. Then T has a fixed point in X.

Theorem 2 *Let (X, m, s) be a complete super metric space and T, $S : X \to CB^m(X)$ be two multivalued mappings satisfying the following condition*

$$H_m(Tx, Sy) \leq \alpha[[m(x, y)]^{\lambda_1}[m(x, Tx)]^{\lambda_2}[m(y, Sy)]^{\lambda_3}[\frac{1}{3}(m(x, Sy) + m(y, Tx)]^{1-\lambda_1-\lambda_2-\lambda_3}]$$

for all x, $y \in X$ and some $\alpha \in [0, 1)$, $\lambda_1, \lambda_2, \lambda_3 \in (0, 1)$ such that $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Then T and S have a common fixed point. Additionally, if either T or S is single-valued. Then the common fixed point is unique.

Proof. Let $x_0 \in X$ with $x_0 \notin Tx_0$ and $m(x_0, Tx_0) > 0$. Choose $x_1 \in Tx_0$. By Lemma 2, there exists $x_2 \in Sx_1$ such that

$$m(x_2, x_1) \leq H_m(Sx_1, Tx_0).$$

Similarly, we may choose $x_3 \in Tx_2$ such that

$$m(x_3, x_2) \leq H_m(Tx_2, Sx_1).$$

Proceeding in similar manner, we construct a sequence $\{x_n\}$ such that $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ for all $n \in \mathbb{N}$ satisfying

$$m(x_{2n+2}, x_{2n+1}) \leq H_m(Sx_{2n+1}, Tx_{2n})$$

and

$$m(x_{2n+1}, x_{2n}) \leq H_m(Tx_{2n}, Sx_{2n-1}).$$

Now,

$$\begin{aligned} m(x_{2n+1}, Sx_{2n+1}) &\leq H_m(Tx_{2n}, Sx_{2n+1}) \\ &\leq \alpha[[m(x_{2n}, x_{2n+1})]^{\lambda_1}[m(x_{2n}, Tx_{2n})]^{\lambda_2}[m(x_{2n+1}, Sx_{2n+1})]^{\lambda_3} \\ &\quad [\frac{1}{3}(m(x_{2n}, Sx_{2n+1}) + m(x_{2n+1}, Tx_{2n})]^{1-\lambda_1-\lambda_2-\lambda_3}] \\ &\leq \alpha[[m(x_{2n}, x_{2n+1})]^{\lambda_1}[m(x_{2n}, x_{2n+1})]^{\lambda_2}[m(x_{2n+1}, x_{2n+2})]^{\lambda_3} \\ &\quad [\frac{1}{3}(m(x_{2n}, x_{2n+2}) + sm(x_{2n+1}, x_{2n+1})]^{1-\lambda_1-\lambda_2-\lambda_3}] \\ &= \alpha[[m(x_{2n}, x_{2n+1})]^{\lambda_1+\lambda_2}[m(x_{2n+1}, x_{2n+2})]^{1-\lambda_1-\lambda_2}] \\ &\leq \alpha[m(x_{2n+1}, x_{2n+2})]^{1-\lambda_1-\lambda_2} \end{aligned}$$

which is a contradiction. Therefore,

$$\begin{aligned} m(x_{2n+1}, x_{2n+2}) &\leq m(x_{2n}, x_{2n+1}) \\ &\leq \alpha[m(x_{2n}, x_{2n+1})]^{1-\lambda_3}[m(x_{2n+1}, x_{2n+2})]^{\lambda_3} \\ &\leq \alpha[m(x_{2n}, x_{2n+1})]. \end{aligned} \tag{4}$$

Similarly,

$$m(x_{2n}, x_{2n+1}) \leq \alpha[m(x_{2n-1}, x_{2n})]. \tag{5}$$

From (4) and (5), we have, $m(x_n, x_{n+1}) \leq \alpha[m(x_{n-1}, x_n)]$ we conclude that

$$m(x_{2n+1}, x_{2n+2}) \leq \alpha[m(x_{2n}, x_{2n+1})] \leq \alpha^2[m(x_{2n-1}, x_{2n})] ... \leq \alpha^n[m(x_0, x_1)].$$

Since $\alpha < 1$, and taking $n \to \infty$, we have

$$m(x_{2n+1}, x_{2n+2}) \to 0. \tag{6}$$

For any positive integer k,

$$\begin{aligned} m(x_n, x_{n+k}) &\leq m(x_n, x_{n+1}) + m(x_{n+1}, x_{n+2}) + ... + m(x_{n+k-1}, x_{n+k}) \\ &\leq \alpha^n m(x_0, x_1) + \alpha^{n+1} m(x_0, x_1) + ... + \alpha^{n+k-1} m(x_0, x_1) \\ &= (\alpha^n + \alpha^{n+1} + ... + \alpha^{n+k-1}) m(x_0, x_1) \\ &\leq \frac{\alpha^n}{1-\alpha} m(x_0, x_1). \end{aligned} \tag{7}$$

Letting $n \to \infty$ in (7), we get $m(x_n, x_{n+k}) \to 0$. Therefore, $\{x_n\}$ is a Cauchy sequence. Since (X, m, s) is complete so $\lim_{n\to\infty} x_n = x$ for some $x \in X$. Next, we claim that x is a common fixed point of T and S. Now,

$$\begin{aligned} m(x_{2n+2}, Tx) &\leq H_m(Sx_{2n+1}, Tx) \\ &= H_m(Tx, Sx_{2n+1}) \\ &\leq \alpha[[m(x, x_{2n+1})]^{\lambda_1}[m(x, Tx)]^{\lambda_2}[m(x_{2n+1}, Sx_{2n+1})]^{\lambda_3} \\ &\quad [\frac{1}{3}(m(x, Sx_{2n+1}) + m(x_{2n+1}, Tx)]^{1-\lambda_1-\lambda_2-\lambda_3}] \\ &= \alpha[[m(x, x_{2n+1})]^{\lambda_1}[m(x, Tx)]^{\lambda_2}[m(x_{2n+1}, x_{2n+2})]^{\lambda_3} \\ &\quad [\frac{1}{3}(m(x, x_{2n+2}) + m(x_{2n+1}, Tx)]^{1-\lambda_1-\lambda_2-\lambda_3}]. \end{aligned} \tag{8}$$

Taking limit $n \to \infty$ in (8) and using (6), we obtain $m(x, Sx) = 0$, that is $x \in Sx$. Hence x is a common fixed point.

Additionally, if T or S is single valued, we show that the common fixed point is unique. Suppose that $u \in X$ is another common fixed point of T and S. Then by (5), we have

$$\begin{aligned} m(u, x) &\leq H_m(\{u\}, Sx) = H_m(Tu, Sx) \\ &\leq \alpha[[m(u, x)]^{\lambda_1}[m(u, Tu)]^{\lambda_2}[m(x, Sx)]^{\lambda_3}[\frac{1}{3}(m(u, Sx) + m(x, Tu)]^{1-\lambda_1-\lambda_2-\lambda_3}] \\ &\leq \alpha[[m(u, x)]^{\lambda_1}[m(u, u)]^{\lambda_2}[m(x, x)]^{\lambda_3}[\frac{1}{3}(m(u, x) + m(x, u)]^{1-\lambda_1-\lambda_2-\lambda_3}] \\ &\leq \alpha[[m(u, x)]^{1-\lambda_2-\lambda_3}] \end{aligned}$$

which follows $m(u, x) = 0$. Therefore $u = x$. ■

Example 3 *Assume* $X = [0,1]$ *with super metric* $m(x,y) = |x-y|^2$ *for all* x, $y \in X$. *Construct* T, $S : X \to CB^m(X)$, *such that*

$$Tx = \{\frac{x}{2} : x \in [0,1]\}$$

and

$$Sx = \{\frac{x}{4} : x \in [0,1]\}.$$

Assume that $x \neq 0$, $y \neq 0$ *and* $x < y$. *Then,*

$$m(x,Tx) = \left|x - \frac{x}{2}\right|^2, \quad m(y,Sy) = \left|y - \frac{y}{4}\right|^2, \quad m(x,Sy) = \left|x - \frac{y}{4}\right|^2,$$

$$m(y,Tx) = \left|y - \frac{x}{2}\right|^2 \quad \textit{and} \quad H_m(Tx,Sy) = \left|\frac{x}{2} - \frac{y}{4}\right|^2.$$

Now, we can check that

$$\left|\frac{x}{2} - \frac{y}{4}\right|^2 \leq \alpha \left[|x-y|^{2\lambda_1} \left|x - \frac{x}{2}\right|^{2\lambda_2} \left|y - \frac{y}{4}\right|^{2\lambda_3} \left[\frac{1}{3}\left(\left|x - \frac{y}{4}\right|^2 + \left|y - \frac{x}{2}\right|^2\right)\right]^{1-\lambda_1-\lambda_2-\lambda_3} \right]$$

is satisfied for $\alpha \in (0,1)$, $\lambda_1, \lambda_2, \lambda_3 \in (0,1)$ *such that* $\lambda_1 + \lambda_2 + \lambda_3 < 1$ *and for all* x, $y \in X$ *such that* $x \notin Tx$ *and* $y \notin Sy$. *Therefore all the conditions given in Theorem 2 are met and* $0 \in X$ *is a common fixed point of* T *and* S.

Example 4 *Let* $X = \{0,1,2\}$ *be a super metric space endowed with super metric* $m : X \times X \to \mathbb{R}^+$ *defined by*

$$m(0,0) = \frac{1}{2}, \quad m(1,1) = m(2,2) = \frac{2}{3}, \quad m(0,1) = m(1,0) = \frac{1}{3},$$

$$m(0,2) = m(2,0) = \frac{1}{9}, \quad m(1,2) = m(2,1) = \frac{1}{4}.$$

Also define T, $S : X \to CB^m(X)$ *by*

$$Tx = \begin{cases} \{0,2\}, & x = \{1,2\}, \\ \{2\}, & x = 0. \end{cases}$$

and

$$Sx = \begin{cases} \{0,2\}, & x = \{1,2\}, \\ \{1\}, & x = 0. \end{cases}$$

Observe that Tx *and* Sx *is closed and bounded for every* $x \in X$ *under the given super metric* m *also* (X,m,s) *is complete super metric space. Clearly,* Tx *and* Sx *satisfies (5) and* $x = 2$ *is a common fixed point of* T *and* S.

If we take $\lambda_2 = \lambda$ and $\lambda_3 = 1 - \lambda_2$ in Theorem 2, we get the following corollary as generalization of multivalued interpolative Kannan-type contraction obtained in Theorem 1 of [24] on super metric space.

Corollary 2 *Let* (X,m,s) *be a complete super metric space.* T, $S : X \to CB^m(X)$ *be two multivalued mappings satisfying for all* x, $y \in X$

$$H_m(Tx,Sy) \leq \alpha[[m(x,Tx)]^{\lambda}[m(y,Sy)]^{1-\lambda}]$$

where $\lambda \in (0,1)$ *and* $\alpha \in [0,1)$. *Then* T *and* S *have a common fixed point. Additionally, if* T *or* S *is single valued mapping. Then the common fixed point is unique.*

4 Application

Dynamic programming is commonly used in various domains such as optimization problems, sequence alignment, shortest path problems, computer programming and many others. Specifically, resolving the dynamic programming problem associated with a multistage process is equivalent to solving the functional equation

$$q(x) = \sup_{y\in D}\{f(x,y) + L(x,y,q(t(x,y)))\} - a, \quad x \in W$$

which can be reframed as

$$q(x) = \sup_{y\in D}\{l(x,y) + L(x,y,q(t(x,y)))\} - a, \quad x \in W \tag{9}$$

where $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space, and U and V are Banach spaces $t : W \times D \to W$ also $f,\ g : W \times D \to \mathbb{R}$, $L : W \times D \times \mathbb{R} \to \mathbb{R}$, $a > 0$. In this section, we investigate the existence and uniqueness of a bounded solution to the functional equation (9). For additional insights into the problem's context, readers can consult references ([5], [6], [7]) for a more comprehensive explanation. Let $B(W)$ denote the set of all bounded real-valued functions on W. And for an arbitrary $h \in B(W)$, define $||h|| = \sup|h(x)|$. Clearly, $(B(W), ||.||)$ endowed with the metric d defined by

$$d(h,k) = \sup_{x\in W}|h(x) - k(x)|$$

for all $h, k \in B(W)$ is a Banach space. Certainly, the convergence in the space $B(W)$ with respect to $||.||$ is uniform. Consequently, if we examine a Cauchy sequence $\{h_n\}$ in $B(W)$, the sequence $\{h_n\}$ uniformly converges to a bounded function, denoted as h^*. Thus $h^* \in B(W)$. Now, for every $h, k \in B(W)$, $x \in W$ and $a > 0$, we consider the super metric m given by

$$m(h,k) = d(h,k) + a$$

and $T : B(W) \to B(W)$ given by

$$T(h)(x) = \sup_{y\in D}\{l(x,y) + L(x,y,h(t(x,y)))\} - a. \tag{10}$$

The mapping T is well defined if the function l and L are bounded.

Theorem 3 *Assume that*

$$|L(x,y,h(x)) - L(x,y,k(x))| \le \alpha M(h,k)$$

where

$$M(h,k) = [m(h,k)]^{\lambda_1}[m(h,T(h))]^{\lambda_2}[m(k,T(k))]^{\lambda_3}[\frac{1}{3}(m(h,T(k)) + m(k,T(h))]^{1-\lambda_1-\lambda_2-\lambda_3}$$

and $x \in W$, $y \in D$, $\alpha \in [0,1)$, $\lambda_1, \lambda_2, \lambda_3 \in (0,1)$ *such that* $\lambda_1 + \lambda_2 + \lambda_3 < 1$. *Let* $T : B(W) \to B(W)$ *defined in (10), and functions* $L : W \times D \times \mathbb{R} \to \mathbb{R}$ *and* $l : W \times D \to \mathbb{R}$ *are bounded. Then the functional equation (9) has a unique bounded solution.*

Proof. $(B(W), m, s)$ is complete super metric space. Let β be an arbitrary positive number, $x \in W$ and $h_1, h_2 \in B(W)$. Then there exists $y_1, y_2 \in D$ such that

$$T(h_1(x)) < l(x,y_1) + L(x,y_1,h_1,(t(x,y_1))) - a + \beta, \tag{11}$$

$$T(h_2(x)) < l(x,y_2) + L(x,y_2,h_2,(t(x,y_2))) - a + \beta, \tag{12}$$

$$T(h_1(x)) \ge l(x,y_2) + L(x,y_2,h_1,(t(x,y_2))), \tag{13}$$

$$T(h_2(x)) \geq l(x, y_1) + L(x, y_1, h_2, (t(x, y_1))). \tag{14}$$

Now, from (11) and (14), it follows

$$\begin{aligned} T(h_1(x)) - T(h_2(x)) &< L(x, y_1, h_1, (t(x, y_1))) - L(x, y_1, h_2, (t(x, y_1))) - a + \beta \\ &\leq |L(x, y_1, h_1, (t(x, y_1))) - L(x, y_1, h_2, (t(x, y_1)))| - a + \beta \\ &\leq \alpha M(h_1, h_2) - a + \beta. \end{aligned}$$

Then we get

$$T(h_1(x)) - T(h_2(x)) < \alpha M(h_1, h_2) - a + \beta. \tag{15}$$

Similarly, from (12) and (13), we get

$$T(h_2(x)) - T(h_1(x)) < \alpha M(h_1, h_2) - a + \beta. \tag{16}$$

Therefore from (15) and (16), we have

$$|T(h_1(x)) - T(h_2(x))| < \alpha M(h_1, h_2) - a + \beta,$$

i.e. $s(T(h_1), T(h_2)) < \alpha M(h_1, h_2) + \beta$. ■

Since the above inequality does not depend on $x \in W$ and $\beta > 0$ is taken arbitrary. Then we conclude that

$$s(T(h_1), T(h_2)) \leq \alpha M(h_1, h_2).$$

And from Theorem 1 the mapping T has a unique fixed point, i.e., the functional equation (9) has a unique bounded solution.

5 Conclusion

In this paper, we have introduced the notion of multivalued interpolative contractions in a super metric space. Our results extend and generalize certain results in [19], [24] and some other results from existing literature. To validate our contributions, we have provided relevant examples. Furthermore, we have presented an application utilizing our primary result to address functional equations in dynamic programming. The outcomes presented in this paper offer valuable insights for future research, paving the way for exploring the existence of common fixed points for these contractions in various metric space settings and their applications across diverse fields.

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