Applied Mathematics E-Notes, 25(2025), 329-338 ©
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/ ISSN 1607-2510

New Aspects For The Oscillation Of q-Fractional Nonlinear Neutral Difference Equations*

Nagamanickam Nagajothi[†], Vadivel Sadhasivam[‡], Marappan Sathish Kumar[§]

Received 25 March 2024

Abstract

In this work, we study a class of quantum fractional nonlinear difference equations with neutral terms. The results are developed in the sense of the q-analogue of the R-L fractional difference operator. New sufficient criteria for oscillation of the considered q-fractional equations with R-L type fractional derivatives are established using the integral average and generalized Riccati technique. Essential preliminaries for q-fractional calculus are stated prior to giving the main results. To the best of the authors' knowledge, nothing is known about the oscillatory behaviour of the considered equation, and so this article begins the study. An example is provided to illustrate the importance of the main results.

1 Introduction

Fractional differential equations (FDEs) have recently received great attention. Several researchers have studied FDEs due to their effectiveness in various fields of engineering and science (see references [1, 4, 5, 13, 16, 18, 22, 27, 30, 31, 32, 35] and many references therein). In the last years, the theory of oscillation of FDEs was also investigated [14]. Nonlinear neutral differential/difference equations arise in modeling various phenomena, such as biology, population growth, chemical reactions, economics, and neural networks [6, 15, 24, 25, 36].

The concept of q-difference equations has acquired great interest in the past few years. For details on q-calculus, the interested reader is referred to the seminal texts [20, 26, 33]. The q-analog concept has been initiated in [23], the q-analog of the beta and gamma functions was discussed in [11], and their integral representations was investigated in [17]. For models from Mathematical biology where oscillation actions may be formulated by means of cross-diffusion terms. In dynamical models, oscillation effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g [28]. Oscillation theory is essential in biology for explaining how predating and competition cause synchrony in animal and plant populations. Numerous researchers have conducted systematic studies on the oscillation and non-oscillation of solutions to integer-order differential and difference equations in response to such applications; we direct the reader to the excellent monograph [6] and the papers [7, 8, 19, 29]. Al-Salam and Waleed [9] introduced the concept of q-analog of the Riemann-Liouville (R-L) fractional integral operator, as well as the q-analogue of the Cauchy's formula. Agarwal [3] addressed the q-analog of both the fractional derivative and integral for R-L operators. Rajković et al. [34] discussed fractional derivatives and integrals in q-calculus. Atici and Eloe [12] studied the fractional q-calculus on time scales. Some sufficient criteria for the oscillation of solutions of R-L and Caputo q-fractional difference equations were studied in [2] based on applying Young's inequality.

Despite the research mentioned in the above discussion, there are no works available in the literature addressing oscillation criteria for fractional q-difference equations based on the integral averaging and Riccati techniques. Therefore, we construct the new problem of the oscillation of quantum nonlinear neutral

*Mathematics Subject Classifications: 39A70, 39A13, 26A33

[†]Department of Mathematics, Nandha Engineering College (Autonomous), Erode-638 052, Tamil Nadu, India

[‡]Post Graduate and Research Department of Mathematics, Thiruvalluvar Government Arts College (Affli. to Periyar University) Rasipuram - 637 401, Namakkal, Tamil Nadu, India

[§]Department of Mathematics, Paavai Engineering College (Autonomous), Namakkal, 637 018, Tamil Nadu, India

difference equations of the form:

$$D_q\left(a_1(r)\varphi_2\left(D_q^\alpha z(r)\right)\right)+a_2(r)\varphi_2\left(D_q^\alpha z(r)\right)+a_3(r)\varphi_1\left(\int_0^r r^{-\alpha-1}\left(\frac{qs}{r};q\right)_{-\alpha-1}y(s)d_qs\right)=0, \quad (1)$$

for $r>0$. Where $z(r)=y(r)+\Lambda(r)y(r-\sigma)$, the constant $\sigma>0$, $\alpha\in(0,1)$ is a constant, $q\in(0,1)$ and $q\in\mathbb{R}$. The operator D_q denotes the quantum difference operator and D_q^α denotes the R-L quantum fractional derivative of order α of y with respect to r.

First, we will assume that the following basic assumptions hold:

(A_1) $a_1(r)\in C_q^1([r_0,\infty),\mathbb{R}_+)$ and $a_2(r),a_3(r)>0$ are continuous functions on $[r_0,\infty)$ for $r_0>0$;

(A_2) $\Lambda\in C_q^{1+\alpha}([r_0,\infty),\mathbb{R}_+)$, $0\leq\Lambda<1$;

(A_3) $\varphi_1,\varphi_2:\mathbb{R}\to\mathbb{R}$ are continuous function with $x\varphi_1(u)>0, x\varphi_2(u)>0$, and there exist constants $l_1>0$, $l_2>0$ such that $\frac{\varphi_1(u)}{u}\geq l_1$, $\frac{(u)}{\varphi_2(u)}\geq l_2$ for all $u\neq 0$;

(A_4) $\varphi_2^{-1}\in C(\mathbb{R},\mathbb{R})$ is a continuous functions with $u\varphi_2^{-1}(u)>0$ for $u\neq 0$, and there exists some nonnegative constant ν such that $\varphi_2^{-1}(uv)\geq\nu\varphi_2^{-1}(u)\varphi_2^{-1}(v)$ for $uv\neq 0$.

By a solution of problem (1), we mean a nontrivial function $y(r)\in C_q^{1+\alpha}[r_y,\infty)$, $r_y\geq r_0$ such that

$$K_q(r)=\int_0^r r^{-\alpha-1}\left(\frac{qs}{r};q\right)_{-\alpha-1}y(s)d_qs\in C_q^1\left([r_y,\infty),\mathbb{R}\right),$$

which has the property $a_1(r)\varphi_2 D_q^\alpha z(r)\in C_q^1[r_y,\infty)$ and satisfies the equation (1) for $r\geq r_y$, where $C_q^{1+\alpha}$ is the space of quantum difference functions of fractional order $(1+\alpha)$.

A solution $y(r)$ of (1) is considered oscillatory if it is neither eventually negative nor eventually positive. If one of the previous cases happened, we call it non-oscillatory. Moreover, equation (1) is said to be oscillatory if each of its solutions is oscillatory.

This rest of the current paper has the following construction. The preliminaries and definitions are given in the Section 2 along with the notations that are needed in the sequel. In the third section, we provide an intensive discussion about the oscillation of the problem mentioned in (1). Last but not least, an example to illustrate the main results is presented in the fourth section.

2 Preliminaries

One can start by recalling some existing fundamental ideas for fractional quantum calculus, which are essential to what follows.

Definition 1 ([10]) *The q-analogue of the R-L fractional integral of a positive order α, and $0<q<1$ of a mapping $x:\mathbb{R}^+\to\mathbb{R}$ on the positive real line can be represented as:*

$$I_q^\alpha x(r)=\frac{r^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^r\left(\frac{qs}{r};q\right)_{\alpha-1}x(s)d_qs, \quad \textit{for } r>0,$$

given that the right-hand side is point wise stated on the positive real line, such that Γ_q is the q-gamma mapping.

Definition 2 ([10]) *The q-analogue of the R-L fractional derivative of a positive order α, and $0<q<1$ of a mapping $x:\mathbb{R}^+\to\mathbb{R}$ on the positive real line that can be represented as the following*

$$D_q^\alpha x(r):=(D_q^{\lceil\alpha\rceil}I_q^{\lceil\alpha\rceil-\alpha}x)(r),\ \alpha>0,$$

where $\lceil\alpha\rceil$ is the ceiling function of α.

Definition 3 ([10]) *The q-analogue of the R-L fractional derivative of a positive order α of a mapping $x : \mathbb{R}_+ \to \mathbb{R}_+$ on the positive real line can be represented as*

$$(D_q^\alpha x)(r) := \frac{d_q^{\lceil \alpha \rceil}}{dt_q^{\lceil \alpha \rceil}} \left(I_q^{\lceil \alpha \rceil - \alpha} x \right)(r),$$

$$(D_q^\alpha x)(r) = \frac{r^{\lceil \alpha \rceil - \alpha - 1}}{\Gamma_q \lceil \alpha \rceil - \alpha} \frac{d_q^{\lceil \alpha \rceil}}{dt_q^{\lceil \alpha \rceil}} \int_0^r \left(\frac{qs}{r}; q \right)_{\lceil \alpha \rceil - \alpha - 1} x(s) d_q s, \tag{2}$$

for $r > 0$, given that the right side is a pointwise mapping stated on the positive real line.

Lemma 1 *Suppose that y is a solution of (1) as well as*

$$K_q(r) := r^{-\alpha - 1} \int_0^r \left(\frac{qs}{r}; q \right)_{-\alpha - 1} y(s) d_q s, \quad \text{for } \alpha \in (0, 1) \text{ and } r > 0. \tag{3}$$

Then

$$D_q((K_q(r))) = \Gamma_q(1 - \alpha)(D_q^\alpha y)(r). \tag{4}$$

Proof. Using the previously mentioned definition of q-fractional integral as well as (2), we obtain:

$$\begin{aligned} D_q(K_q(r)) &= D_q \left[r^{\alpha - 1} \int_r^\infty \left(\frac{qs}{r}; q \right)_{\alpha - 1} y(s) d_q s \right] \\ &= \Gamma_q(1 - \alpha) \frac{d_q}{dt_q} \left[\frac{r^{\alpha - 1}}{\Gamma_q(1 - \alpha)} \int_0^r \left(\frac{qs}{r}; q \right)_{-\alpha - 1} y(s) d_q s \right] \\ &= \Gamma_q(1 - \alpha) \frac{d_q^{\lceil \alpha \rceil}}{dt_q^{\lceil \alpha \rceil}} \left[\frac{r^{\lceil \alpha \rceil - \alpha - 1}}{\Gamma_q(\lceil \alpha \rceil - \alpha)} \int_0^r \left(\frac{qs}{r}; q \right)_{\lceil \alpha \rceil - \alpha - 1} y(s) d_q s \right] \\ &= \Gamma_q(1 - \alpha) \left[\frac{d_q^{\lceil \alpha \rceil}}{dt_q^{\lceil \alpha \rceil}} \frac{r^{\lceil \alpha \rceil - \alpha - 1}}{\Gamma_q(\lceil \alpha \rceil - \alpha)} \int_0^r \left(\frac{qs}{r}; q \right)_{\lceil \alpha \rceil - \alpha - 1} y(s) d_q s \right], \end{aligned}$$

$$D_q K_q(r) = \Gamma_q(1 - \alpha) D_q^\alpha y(r).$$

■

Lemma 2 ([7]) *Let $y(r) > 0$ be a solution of (1) on $T \geq 0$. Then the function $z(r) = y(r) + \Lambda(r) y(r - \sigma)$ satisfies one of the following cases:*
Case I: $z(r) > 0, D_q^\alpha z(r) > 0,\ D_q^\alpha \left(a_1(r) \varphi_2 \left(D_q^\alpha z(r) \right) \right) \leq 0$;
Case II: $z(r) > 0, D_q^\alpha z(r) < 0,\ D_q^\alpha \left(a_1(r) \varphi_2 \left(D_q^\alpha z(r) \right) \right) \leq 0$; for all $r \geq T$.

Lemma 3 ([21]) *If $X > 0$, $Y > 0$. Then*

$$mXY^{m-1} - X^m \leq (m - 1) Y^m.$$

3 Main Results

We provide in this section several sufficient conditions for the oscillation of (1) under the hypothesis

$$\int_{r_0}^\infty \frac{1}{a_1(r)} d_q r = \infty.$$

Theorem 1 *Assume that* (A_1)*–*(A_4) *hold and that* $D_q(\varphi_1(k_q(r))) \geq \mu$ *for some* $\mu > 0$ *and for all* $k_q(r) > 0$. *Furthermore, assume that there is a non negative and nonzero mapping* $\delta \in C_q^\alpha((0,\infty),\mathbb{R}_+)$ *such that*

$$\int_{r_0}^{\infty} \varphi_2^{-1}\left(\frac{1}{a_1(s)e_q^{R_q(s)}}\right) d_q s = \infty, \tag{5}$$

$$\limsup_{r\to\infty} \int_{r_0}^{r} \left[\delta(qs)e_q^{R_q(s)}a_3(s) - \frac{(D_q(\delta(s))^2 e_q^{R_q(s)} a_1(s)}{4\mu l_2 \delta(qs)\Gamma_q(1-\alpha)}\right] d_q s = \infty \tag{6}$$

where

$$R_q(s) = \int_{r_0}^{r} \frac{a_2(s)}{a_1(s)} d_q s. \tag{7}$$

Then every solution of (1) is oscillatory.

Proof. *Suppose that the solution* $y(r)$ *is a non oscillatory solution of the problem (1). We may also suppose that* $y(r) > 0$ *is a solution of (1), and also assume that* $y(qr) > 0$, $y(r-\sigma) > 0$. *Then, there exists* $r_1 \geq r_0$ *such that* $y(r) > 0$ *and* $K_q(r) > 0$ *for* $r \geq r_1$. *Using (1) and (7), we have*

$$\begin{aligned} D_q\left[e_q^{R_q(r)}a_1(r)\left(\varphi_2(D_q^\alpha z(r))\right)\right] &= e_q^{R_q(r)}D_q\left[a_1(r)\left(\varphi_2(D_q^\alpha z(r))\right)\right] + e_q^{R_q(r)}a_2(r)\varphi_2(D_q^\alpha z(r)) \\ &= e_q^{R_q(r)}\left[D_q\left(a_1(r)\varphi_2\left(D_q^\alpha\left(z(r)\right)\right)\right) + a_2(r)\varphi_2\left(D_q^\alpha\left(z(r)\right)\right)\right] \\ &\leq e_q^{R_q(r)}\left[-a_3(r)\varphi_1(K_q(r))\right] < 0. \end{aligned} \tag{8}$$

Thus $D_q^\alpha z(r) \geq 0$ *or* $D_q^\alpha z(r) < 0$, *for some* $r_1 \geq r_0$. *We now claim that* $D_q^\alpha z(r) \geq 0$. *Suppose now* $D_q^\alpha z(r) < 0$ *and there exists* $r_2 \in [r_1,\infty)$ *such that* $D_q^\alpha z(r_2) < 0$. *Since* $e_q^{R_q(r)}a_1(r)\left(\varphi_2(D_q^\alpha z(r))\right)$ *is strictly decreasing on the interval* $[r_1,\infty)$. *Clearly,*

$$e_q^{R_q(r)}a_1(r)\left(\varphi_2(D_q^\alpha z(r))\right) < e_q^{R_q(r_2)}a_1(r_2)\left(\varphi_2(D_q^\alpha z(r_2))\right) := -c_1,$$

where c_1 *is a positive constant for* $r \in [r_2,\infty)$. *Therefore, from Lemma 1, we obtain*

$$\frac{D_q(K_q(r))}{\Gamma_q(1-\alpha)} = (D_q^\alpha z)(r) < \varphi_2^{-1}\left(\frac{-c_1}{a_1(r)e_q^{R_q(r)}}\right) \leq -\nu_1\varphi_2^{-1}\left(\frac{1}{a_1(r)e_q^{R_q(r)}}\right), \quad \textit{for } r \in [r_2,\infty),$$

where $\nu_1 = \nu\varphi_2^{-1}(c_1)$. *Therefore,*

$$\varphi_2^{-1}\left(\frac{1}{a_1(r)e_q^{R_q(r)}}\right) \leq \frac{-D_q\left(K_q(r)\right)}{\nu_1\Gamma_q(1-\alpha)}, \quad \textit{for } r \in [r_2,\infty).$$

q-integrating from r_2 *to* r,

$$\int_{r_2}^{r} \varphi_2^{-1}\left(\frac{1}{a_1(s)e_q^{R_q(s)}}\right) d_q s \leq -\frac{K_q(r) - K_q(r_2)}{\nu_1\Gamma_q(1-\alpha)} < \frac{K_q(r_2)}{\nu_1\Gamma_q(1-\alpha)}, \quad \textit{for } r \in [r_2,\infty).$$

Letting $r \to \infty$,

$$\int_{r_2}^{\infty} \varphi_2^{-1}\left(\frac{1}{a_1(s)e_q^{R_q(s)}}\right) d_q s \leq \frac{K_q(r_2)}{\nu_1\Gamma_q(1-\alpha)} < \infty,$$

which contradicts (5). Define the function $w_q(r)$ *by*

$$w_q(r) = \delta(r)\frac{a_1(r)e_q^{R_q(r)}\left(\varphi_2(D_q^\alpha z(r))\right)}{\varphi_1(K_q(r))}, \quad \textit{for } r \in [r_1,\infty). \tag{9}$$

Since $w_q(r) > 0$ *for* $r \in [r_1, \infty)$, *using (8),* (A_3), *and Lemma 1,*

$$w_q(r) = \delta(r)\left(\frac{e_q^{R_q(r)} a_1(r)\varphi_2(D_q^\alpha z(r))}{\phi_1(K_q(r))}\right).$$

Then $w_q(r) > 0$, *we get*

$$D_q w_q(r) = \delta(qr) D_q\left[\frac{e_q^{R_q(r)} a_1(r)\varphi_2(D_q^\alpha z(r))}{\phi_1(K_q(r))}\right] + \left[\frac{e_q^{R_q(r)} a_1(r)\varphi_2(D_q^\alpha z(r))}{\phi_1(K_q(r))}\right] D_q\delta(r)$$
$$\leq -\delta(qr) e_q^{R_q(r)} a_3(r) + \frac{D_q\delta(r)}{\delta(r)} w_q(r) - \frac{\delta(qr)\mu\Gamma_q(1-\alpha) l_2 w_q^2(r)}{\delta^2(r) e_q^{R_q(r)} a_1(r)},$$

$$D_q w_q(r) \leq -\delta(qr) e_q^{R_q(r)} a_3(r) + \frac{\left(D_q\delta(r)\right)^2 e_q^{R_q(r)} a_1(r)}{4\mu l_2 \delta(qr)\Gamma_q(1-\alpha)}.$$

and, q-integrating, we have

$$\int_{r_1}^{r} D_q w_q(s) d_q s \leq -\int_{r_1}^{r}\left[\delta(qs) e_q^{R_q(s)} a_3(s) - \frac{\left(D_q\delta(s)\right)^2 e_q^{R_q(s)} a_1(s)}{4\mu l_2\delta(qs)\Gamma_q(1-\alpha)}\right] d_q s,$$

$$w_q(r) \leq w_q(r_1) - \int_{r_1}^{r}\left[\delta(qs) e_q^{R_q(s)} a_3(s) - \frac{\left(D_q\delta(s)\right)^2 e_q^{R_q(s)} a_1(s)}{4\mu l_2\delta(qs)\Gamma_q(1-\alpha)}\right] d_q s.$$

This implies that

$$\int_{r_1}^{r}\left[l_1\delta(qs) e_q^{R_q(s)} a_3(s) - \frac{\left(D_q\delta(s)\right)^2 e_q^{R_q(s)} a_1(s)}{4 l_2\delta(qs)\Gamma_q(1-\alpha)}\right] d_q s \leq w_q(r_1),$$

which contradicts (6). ■

Theorem 2 *Suppose that* $(A_1) - (A_4)$ *and (5) are satisfied. Suppose*

$$\limsup_{r\to\infty}\int_{r_1}^{r}\left[l_1\delta(qs) e_q^{R_q(s)} a_3(s) - \frac{\left(D_q\delta(s)\right)^2 e_q^{R_q(s)} a_1(s)}{4 l_2\delta(qs)\Gamma_q(1-\alpha)}\right] d_q s = \infty. \tag{10}$$

Then all the solutions of (1) are oscillatory.

Proof. *Let* $y(r)$ *be a non-oscillatory solution in (1). We may also suppose that* $y(r) > 0$, $y(qr) > 0$ *and* $y(r-\sigma) > 0$ *in* $[r_0, \infty)$. *Using a similar way of that in Theorem 1, one could get that (8) and* $D_q^\alpha z > 0$ *for* $r \geq r_1$. *Define the function* $\omega_q(r)$:

$$\omega_q(r) = \delta(r)\left(\frac{e_q^{R_q(r)} a_1(r)\varphi_2(D_q^\alpha z(r))}{K_q(r)}\right).$$

Then $\omega_q(r) > 0$, *we get*

$$D_q \omega_q(r) = \delta(qr) D_q\left[\frac{e_q^{R_q(r)} a_1(r)\varphi_2(D_q^\alpha z(r))}{K_q(r)}\right] + \left[\frac{e_q^{R_q(r)} a_1(r)\varphi_2(D_q^\alpha z(r))}{K_q(r)}\right] D_q\delta(r)$$
$$\leq -l_1\delta(qr) e_q^{R_q(r)} a_3(r) + \frac{D_q\delta(r)}{\delta(r)}\omega_q(r) - \frac{\delta(qr)\Gamma_q(1-\alpha) l_2\omega_q^2(r)}{\delta^2(r) e_q^{R_q(r)} a_1(r)},$$

$$D_q\omega_q(r) \leq -l_1\delta(qr) e_q^{R_q(r)} a_3(r) + \frac{\left(D_q\delta(r)\right)^2 e_q^{R_q(r)} a_1(r)}{4 l_2\delta(qr)\Gamma_q(1-\alpha)}$$

and, q-integrating, we have

$$\int_{r_1}^{r} D_q\omega_q(s)d_qs \leq -\int_{r_1}^{r}\left[l_1\delta(qs)e_q^{R_q(s)}a_3(s) - \frac{(D_q\delta(s))^2 e_q^{R_q(s)}a_1(s)}{4l_2\delta(qs)\Gamma_q(1-\alpha)}\right]d_qs,$$

$$\omega_q(r) \leq \omega_q(r_1) - \int_{r_1}^{r}\left[l_1\delta(qs)e_q^{R_q(s)}a_3(s) - \frac{(D_q\delta(s))^2 e_q^{R_q(s)}a_1(s)}{4l_2\delta(qs)\Gamma_q(1-\alpha)}\right]d_qs.$$

This implies that

$$\int_{r_1}^{r}\left[l_1\delta(qs)e_q^{R_q(s)}a_3(s) - \frac{(D_q\delta(s))^2 e_q^{R_q(s)}a_1(s)}{4l_2\delta(qs)\Gamma_q(1-\alpha)}\right]d_qs < \omega_q(r_1),$$

which contradicts (10). ■

In the follow-up, we address several novel oscillation criteria of (1) taking advantage of the conditions of Philo's type.

The mapping $B \in C(\mathbb{D}; R)$ is classified as the class $\mathbb{P}$ if there is a mapping $b \in C(\mathbb{D}_0, R)$ such that

$$-\frac{\partial B(r,s)}{\partial s} = b(r,s)\sqrt{B(r,s)},$$

for all $(r,s) \in \mathbb{D}_0$.

Theorem 3 *Assume that $(A_1) - (A_4)$ are satisfied and that there is a mapping $B \in C(\mathbb{D}; R)$ which belongs to the class $\mathbb{P}$, where $\mathbb{D}_0 = \{(r,s) : r > s \geq r_0\}$ and $\mathbb{D} = \{(r,s) : r \geq s \geq r_0\}$, such that*

(T_1) $B(r,r) = 0$ for $r \geq r_0, B(r,s) > 0$ on $\mathbb{D}_0$;

(T_2) B contains a continuous as well as either zero or negative partial derivative on $\mathbb{D}_0$ and meets the following

$$\limsup_{r\to\infty}\frac{1}{B(r,r_1)}\int_{r_1}^{r} B(r,s)\left[l_1\delta(qs)e_q^{R_q(s)}a_3(s) - \frac{(D_q\delta(s))^2 e_q^{R_q(s)}a_1(s)}{4l_2\delta(qs)\Gamma_q(1-\alpha)}\right]d_qs = \infty. \tag{11}$$

Then each solution in (1) is oscillating.

Proof. *Assume that the solution $y(r)$ is a non oscillatory solution of the problem (1). We may also suppose that $y(r) > 0$, $y(qr) > 0$ and $y(r-\sigma) > 0$ in $[r_0, \infty)$. Proceeding as in the proof of Theorem 2, we obtain*

$$D_q\omega_q(r) \leq -l_1\delta(qr)e_q^{R_q(r)}a_3(r) + \frac{(D_q\delta(r))^2 e_q^{R_q(r)}a_1(r)}{4l_2\delta(qr)\Gamma_q(1-\alpha)}.$$

Multiplying on both sides by $B(r,s)$, and q-integrating from r_1 to r for $r \in [r_0, \infty)$, we get

$$\begin{aligned} & \int_{r_1}^{r} B(r,s)\left[l_1\delta(qs)e_q^{R_q(s)}a_3(s) - \frac{(D_q\delta(s))^2 e_q^{R_q(s)}a_1(s)}{4l_2\delta(qs)\Gamma_q(1-\alpha)}\right]d_qs \\ \leq \;& -[B(r,s)\omega_q(s)]_{r_1}^{r} + \int_{r_1}^{r} D_qB(r,s)\omega_q(s)d_qs \\ < \;& B(r,r_1)\omega_q(r_1). \end{aligned}$$

Therefore,

$$\frac{1}{B(r,r_1)}\int_{r_1}^{r} B(r,s)\left[l_1\delta(qs)e_q^{R_q(s)}a_3(s) - \frac{(D_q\delta(s))^2 e_q^{R_q(s)}a_1(s)}{4l_2\delta(qs)\Gamma_q(1-\alpha)}\right]d_qs < \omega_q(r_1) < \infty,$$

which contradicts to (11). ■

At this stage several sufficient conditions are presented for the oscillation of (1) under the assumption $\int_{r_0}^{\infty} \frac{1}{a_1(r)} d_q r < \infty$.

Theorem 4 *Assume that* (A_1)–(A_4) *and (6) are satisfied. In addition, suppose that*

$$\int_T^r g^{-1}\left[\frac{1}{e_q^{R_q(s)}a_1(s)}\int_T^s e_q^{R_q(\varsigma)}a_3(\varsigma)d_q\varsigma\right]d_qs = \infty.$$

Then each solution in (1) is oscillating or approaching zero.
Proof. *Assume that the solution* $y(r)$ *is a non oscillatory solution of the problem (1). We may also suppose that* $y(r) > 0$, $y(qr) > 0$ *and* $y(r-\sigma) > 0$ *in* $[r_0, \infty)$. *Using a similar way of the previously mentioned Theorem 1, one could observe that there are a couple of possibilities regarding the value* $D_q y(r)$. *First, if* $D_q y(r) > 0$ *we can proceed using the same approach that we applied in Theorem 1. Therefore, we may suppose that the value of* $D_q y(r)$ *is negative for* $r \geq r_1$. *Consequently,* $y(r)$ *is going to be decreasing as well as we will have that* $\lim_{r\to\infty} z(r) = b \geq 0$. *One must have that* $b = 0$, *otherwise,* $\varphi_1(K_q(r)) > b > 0$ *for* $r \geq T$. *Now, let us state the following mapping*

$$u_q(r) = e_q^{R_q(r)}a_1(r)\varphi_2(D_q^{\alpha}z(r)).$$

Then, from (1) for $r \geq T$, *we obtain*

$$D_q u_q(r) \leq -e_q^{R_q(r)}a_3(r)\varphi_1(K_q(r)) \leq -e_q^{R_q(r)}a_3(r)\varphi_1(b).$$

Hence, for $r \geq T$, *we have*

$$u_q(r) \leq u_q(T) - \varphi_1(b)\int_T^r e_q^{R_q(s)}a_3(s)d_qs.$$

Since $u_q(r) = e_q^{R_q(T)}a_1(T)\varphi_2(D_q^{\alpha}z(T)) < 0$, *q-integrating the last inequality from* T *to* r, *we have*

$$u_q(r) \leq e_q^{R_q(T)}a_1(T)\varphi_2(D_q^{\alpha}z(T)) - \varphi_1(b)\int_T^r e_q^{R_q(s)}a_3(s)d_qs,$$

$$\varphi_2(D_q^{\alpha}z(T)) \geq \frac{-\varphi_1(b)}{e_q^{R_q(T)}a_1(T)}\int_T^r e_q^{R_q(s)}a_3(s)d_qs,$$

$$D_q z(r) \geq \frac{-\varphi_1(b)}{\Gamma_q(1-\alpha)}\varphi_2^{-1}\left(\frac{1}{e_q^{R_q(T)}a_1(T)}\int_T^r e_q^{R_q(s)}a_3(s)d_qs\right).$$

Again, q-integrating, we obtain

$$z(r) \geq \frac{-\varphi_1(b)}{\Gamma_q(1-\alpha)}\varphi_2^{-1}\left(\frac{1}{e_q^{R_q(s)}a_1(s)}\int_T^s e_q^{R_q(\varsigma)}a_3(\varsigma)d_q\varsigma\right)d_qs. \tag{12}$$

Using the previously mentioned condition in (12), one could have that $z(r) \to -\infty$, *as* $r \to \infty$, *which contradicts the assumption* $z(r) > 0$ *for* $r \geq r_0$. *Therefore, the value* $b = 0$ *and, then,* $z(r) \to 0$ *as* $r \to \infty$. ∎

4 An Example

Here in the last section of this paper, an application for the above established results is presented.

Example 1 *Consider*

$$D_q\left(r\left(D_q^{\alpha}\left(y(r)+\frac{1}{2}y(r-2\pi)\right)\right)\right)+r^{-2}\left(D_q^{\alpha}\left(y(r)+\frac{1}{2}y(r-2\pi)\right)\right)$$
$$+\frac{e_q^{\frac{1}{8}}}{r\delta(qr)}\left(\int_0^r r^{-\alpha-1}\left(\frac{qs}{r};q\right)_{-\alpha-1}y(s)d_qs\right)=0, \tag{13}$$

for $r\in[2,\infty]$. *We have in (1)* $\alpha\in(0,1)$, $0<q<1$, $a_1(r)=r$, $a_2(r)=r^{-2}$, $a_3(r)=\frac{e_q^{\frac{1}{8}}}{r\delta(qr)}$, $\sigma=2\pi$, $\Lambda=\frac{1}{2}$, $r_0=2$, $\delta(s)=1$, $\mu=l_2=2$, *and* $\Gamma_q(1-\alpha)>0$. *Then,* $R_q(r)=\frac{-1}{2}[r^{-2}-2^{-2}]$. *Therefore,*

$$\int_{r_0}^{\infty}\varphi_2^{-1}\left(\frac{1}{a_1(s)e_q^{R_q(s)}}\right)d_q(s)=\int_2^{\infty}\left(\frac{1}{a_1(s)e_q^{R_q(s)}}\right)d_qs\ge e_q^{\frac{1}{8}}\int_2^{\infty}\frac{1}{s}d_q(s)=\infty.$$

Furthermore,

$$\limsup_{r\to\infty}\int_{r_0}^{r}\left[\delta(qs)e_q^{R_q(s)}a_3(s)-\frac{(D_q(\delta(s))^2e_q^{R_q(s)}a_1(s)}{4\mu l_2\delta(qs)\Gamma_q(1-\alpha)}\right]d_qs=\limsup_{r\to\infty}\int_2^r\frac{1}{s}d_q(s)=\infty.$$

Since all assumptions of Theorem 1 are satisfied, we verify that all solutions of (13) are oscillatory.

5 Conclusion

In this paper we established some sufficient conditions for the oscillation of solutions of nonlinear neutral fractional quantum difference equations of the form (1). The results are new, they are derived based on the integral averaging and Riccati techniques, and the effectiveness was illustrated when applied to an example.

References

[1] S. Abbas, M. Benchohra and G. N'Guérékata, Topics in Fractional Differential Equations, Springer, New York, 2012.

[2] B. Abdalla, On the oscillation of q-fractional difference equations, Adv. Difference Equ., 2017, 1–11.

[3] R. Agarwal, Certain fractional q-integrals and q-derivatives, Mathematical Proceedings Of The Cambridge Philosophical Society, 66(1969), 365–370.

[4] T. Abdeljawad and O. Alzabut, The fractional analogue for Gronwall-type inequality, J. Funct. Spaces, vol. 2013, Article ID 543839, 7 pages, 2013.

[5] T. Abdeljawad, J. Alzabut and D. Baleanu, A generalized q-fractional Gronwall inequality and its applications to nonlinear delay q-fractional difference systems, J. Inequal. Appl., (2016), 240.

[6] R. P. Agarwal, M. Bohner, S. R. Grace and D. O'Regan, Discrete Oscillation Theory, Hindawi Publishing Corporation, New York, NY, USA, 2005.

[7] H. Adiguzel, Oscillatory behavior of solutions of certain fractional difference equations, Adv. Difference Equ., 2018(2018), 1–13.

[8] S. R. Grace, R. P. Agarwal, P. J. Y. Wong and A. Zafer, On the oscillation of fractional differential equations, Fract. Calc. Appl. Anal., 15(2012), 222–231.

[9] W. Al-Salam, q-Analogues of Cauchy's formulas, Proc. Amer. Math. Soc., 17(1966), 616–621.

[10] M. Annaby and Z. Mansour, q-Fractional Calculus and Equations, Springer-Verlag, Berlin, 2012.

[11] R. Askey, The q-gamma and q-beta functions, Appl. Anal., 8(1978), 125–141.

[12] F. Atici and P. Eloe, Fractional q-calculus on a time scale, J. Nonlinear Math. Phys., 14(2007), 341–352.

[13] R. Bagley and P. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, Journal Of Rheology, 27(1983), 201–210.

[14] D. Chen and Oscillatory behavior of a class of fractional differential equations with damping, UPB Scientific Bulletin, Series A, 75(2013), 107–118.

[15] K. Chiu and T. Li, Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments, Math. Nachr., 292(2019), 2153–2164.

[16] V. Daftardar-Gejji, Fractional Calculus, Narosa Publishing House, New Delhi, 2013.

[17] A. De Sole and V. Kac, On integral representations of q-gamma and q-beta functions, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 9(2005), 11–29.

[18] K. Diethelm, The Analysis of Fractional Differential Equations: An Application-Oriented Exposition using Differential Pperators of Caputo Type, Springer, Berlin, 2010.

[19] J. Džurina, S. R. Grace, I. Jadlovská and T. Li, Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term, Math. Nachr., 293(2020), 910–922.

[20] T. Ernst, The History of q-Calculus and a New Method, U. U. D. M. Report 2000:16, Department of Mathematics, Uppsala University, 2000.

[21] G. Hardy, J. Littlewood and G. Polya, Inequalities, Cambridge University Press, Cambridge, 1952.

[22] T. Gayathri, M. Sathish Kumar and V. Sadhasivam, Hille and Nehari type oscillation criteria for conformable fractional differential equations, Iraqi Journal of Science, (2021), 578–587.

[23] D. Jackson, T. Fukuda, O. Dunn and E. Majors, On q-definite integrals, Quart. J. Pure Appl. Math., 41(1910), 193–203.

[24] M. Sathish Kumar, O. Bazighifan, K. Al-Shaqsi, F. Wannalookkhee and K. Nonlaopon, Symmetry and its role in oscillation of solutions of third-order differential equations, Symmetry, 13(2021), 1485.

[25] M. Sathish Kumar, R. Elayaraja, V. Ganesan, O. Bazighifan, K. Al-Shaqsi and K. Nonlaopon, Qualitative behavior of unbounded solutions of neutral differential equations of third-order, Fractal Fract., 5(2021), 95.

[26] V. Kac and P. Cheung, Quantum Calculus, Springer, New York, 2002.

[27] A. Kilbas, H. Srivastava and J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science, Amsterdam, 2006.

[28] T. Li, S. Frassu and G. Viglialoro, Combining effects ensuring boundedness in an attraction-repulsion chemotaxis model with production and consumption, Z Angew. Math. Phys., 74(2023), 1–21.

[29] T. Li and Y.V. Rogovchenko, On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations, Appl. Math. Lett., 105(2020), 1–7.

[30] K. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

[31] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Academic Press, San Diego, 1998.

[32] M. Rahimy, Applications of fractional differential equations, Appl. Math. Sci., 4(2010), 2453–2461.

[33] S. Rezapour, A. Imran, A. Hussain, F. Martínez, S. Etemad and M. K. A. Kaabar, Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs, Symmetry, 13(2021), 469.

[34] P. Rajković, S. Marinković and M. Stanković, Fractional integrals and derivatives in q-calculus, Appl. Anal. Discrete Math., 1(2007), 311–323.

[35] S. Samko, A. Kilbas and O. Marichev, Fractional Integrals and Derivatives, Gordon, 1993.

[36] P. Wang, D. Lassoued, S. Abbas, A. Zada and T. Li, On almost periodicity of solutions of second-order differential equations involving reflection of the argument, Adv. Difference Equ., (2019), 1–11.