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A Numerical Approach For Solving A Fractional Order COVID-19 Model Under Caputo Derivative*

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Abstract

This paper deals with a numerical method for solving a fractional-order model for the COVID-19 epidemic. In the considered fractional model, which is described by a nonlinear system of fractional order differential equations (SFODEs), the fractional derivatives are considered in the Caputo sense. The numerical solution in this paper is based on the combination of Block-pulse functions and Chebyshev polynomials. Also, convergence of the numerical method is studied. Furthermore, a numerical example is presented to demonstrate the applicability and the efficiency of the method.

1 Introduction

At the end of 2019, the outbreak of an infectious disease in Wuhan city in China attracted worldwide attention. At first, there was no idea it would spread to other parts of the world, but the world health organization (WHO) declared it as a pandemic in February 2020 and named it COVID-19. This virus because of its crown-like appearance is called coronavirus. Coronavirus disease (COVID-19), which is caused by the SARS-CoV-2 virus [28], has not been formerly identified in humans. Bats, snakes, or pangolins have been cited as possible sources of the outbreak [23, 34], but there is currently no certainty. Loss of smell or taste, fever, cough, and tiredness are the most common symptoms of the disease. In some patients, symptoms such as sore throat, headache, skin rash, diarrhea, aches, and pains, or discoloration of the fingers or toes, and redness or irritation of the eyes have also been reported. According to WHO, after China, Iran and Italy announced the outbreak of the virus in March 2020 with 43 and 29 deaths, respectively, and as the end of the following month, 44045 deaths of COVID-19 were reported in 129 countries. The governments made great efforts to control the disease, but for reasons such as the unknown behavior of the virus, the high rate of transmission, and the ineffectiveness of available treatments, the virus spread in many countries. To date, the outbreak of the virus has been reported in 175 countries, resulting in a large number of casualties, social anomalies as well as individual traumas, and huge financial losses. It is not yet clear whether surviving COVID-19 infection means long-term immunity and how long it will last if immunity is achieved. The study of the COVID-19 epidemic, as other diseases, has attracted the attention of researchers in the fields of mathematics, biology, epidemiology, pharmacy, and chemistry. The important role of mathematical models in describing and analyzing diseases has made them powerful tools for determining effective and appropriate strategies in the prediction, prevention, and treatment of diseases. Hence, various mathematical models have been formulated and developed for analyzing the dynamics of COVID-19 [1, 2, 4, 10, 25, 31]. Zeb et al. in

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[31] proposed a mathematical model for COVID-19 infection for analyzing the impact of isolation as follows:

$$\begin{cases} \frac{dX_1}{dt} = \mu - \delta X_1 - \beta X_1(X_2 + X_3), \\ \frac{dX_2}{dt} = \beta X_1(X_2 + X_3) - \rho X_2 - (\delta + \tau)X_2, \\ \frac{dX_3}{dt} = \rho X_2 - \sigma X_3 - \delta X_3, \\ \frac{dX_4}{dt} = \tau X_2 + \sigma X_3 - \kappa X_4 - \delta X_4, \\ \frac{dX_5}{dt} = \kappa X_4 - \delta X_5, \end{cases} \tag{1}$$

with the initial conditions

$$X_i(0) = x_{i(0)}, \quad i = 1, \ldots, 5. \tag{2}$$

Description of $X_i(t)(i = 1, \ldots, 5)$ as different classes of population and the parameters of the model are considered as follows:

X_1: *Disease susceptible population*,
X_2: *Population exposed to the disease*,
X_3: *Population of infected people*,
X_4: *Isolated population*,
X_5: *Recovered population from the disease*,
$\mathcal{N} = I + R + S + U + V$: *Total population*,
$\mu = \delta\mathcal{N}$: *Rate of recruitment*,
δ: *Natural death rate plus disease-related death rate*,
κ: *Rate at which isolated persons become recovered.*
τ: *Rate at which exposed people become isolated*,
σ: *Rate at which infected people are added to isolated individuals*,
ρ: *Rate at which exposed population moves to infected one*,
β: *Rate at which susceptible move to infected and exposed class.*

The positivity and stability analysis of the model have been studied in [31]. In recent decades, fractional calculus as a part of mathematical analysis has found a valuable role in describing phenomena in medicine, physics, economics, and engineering. Since some properties of many dynamical systems such as the past history or hereditary cannot be described by differential equations of integer order, fractional models of these systems have been considered extensively by many researchers [5, 8, 9, 20, 22, 26]. Fractional models in the study of diseases are important because they can help physicians to prescribe appropriate treatment or medication for each patient with different choices for fractional derivatives. The fractional models of COVID-19 have been studied by some researchers in [3, 6, 12, 14, 15, 19, 24, 27, 29, 30, 32, 33]. Here, we consider model (1) with the Caputo fractional derivatives as [33]

$$\begin{cases} {}^C_0\mathcal{D}^\alpha_x X_1 = \mu_\alpha - \delta_\alpha X_1 - \beta_\alpha X_1(X_2 + X_3), \\ {}^C_0\mathcal{D}^\alpha_x X_2 = \beta_\alpha X_1(X_2 + X_3) - \rho_\alpha X_2 - (\delta_\alpha + \tau_\alpha)X_2, \\ {}^C_0\mathcal{D}^\alpha_x X_3 = \rho_\alpha X_2 - \sigma_\alpha X_3 - \delta_\alpha X_3, \\ {}^C_0\mathcal{D}^\alpha_x X_4 = \tau_\alpha X_2 + \sigma_\alpha X_3 - \kappa_\alpha X_4 - \delta_\alpha X_4, \\ {}^C_0\mathcal{D}^\alpha_x X_5 = \kappa_\alpha X_4 - \delta_\alpha X_5, \end{cases} \tag{3}$$

with the same initial conditions in (2) and $0 < \alpha \leq 1$. Some analytical aspects of model (3) such as positivity, boundedness, equilibria, the basic reproduction number and stability analysis have been investigated in [33]. Our purpose is to present a numerical method based on the combination of Block-pulse functions (BPFs) and the first kind Chebyshev polynomials (FKCPs) for solving system (3). This paper is organized as follows: In the next section, the required concepts are introduced. In Section 3, the HCBPM is implemented for solving system (3). In Section 4, the convergence analysis of the method is proved. Also, an illustrative example is given in Section 5 and numerical results are reported in this section.

2 Preliminaries

In this section, we mention some concepts of fractional calculus. Furthermore, definitions and properties of hybrid BPFs and FKCPs are reviewed. Also, required operational matrices are presented.

2.1 Fractional calculus

Fractional calculus as a branch of mathematical analysis deals with integral and derivative operators of any positive real order. Several definitions have been presented for these operators by Caputo, Hadamard, Riemann-Liouville, Grünwald-Letnikov, and others which we remind two accepted and common definitions in the following.

Definition 1 ([22]) *The Riemann-Liouville non-integer integral operator* ${}_0\mathcal{I}_x^{\alpha}$ *(of order* $\alpha > 0$*), for a function* $z \in L^1(a, b)$ *is presented as*

$$({}_0\mathcal{I}_x^{\alpha} z)(x) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x \frac{z(\varsigma)}{(x-\varsigma)^{1-\alpha}} \, \mathrm{d}\varsigma, & \alpha > 0, \\ z(x), & \alpha = 0. \end{cases}$$

Definition 2 ([22]) *For* $x > 0$*, the Caputo non-integer derivative of order* $\alpha > 0$ *is expressed as*

$$({}_0^C\mathcal{D}_x^{\alpha} z)(x) = ({}_0\mathcal{I}_x^{n-\alpha} \, {}_0^C\mathcal{D}_x^{n} z)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\varsigma)^{n-\alpha-1} z^{(n)}(\varsigma) \, \mathrm{d}\varsigma, n-1 < \alpha \le n \in \mathbb{N}.$$

Property 1 *For two fractional operators defined above, the following properties yield:*

(a) $({}_0^C\mathcal{D}_x^{\alpha_1} \, {}_0^C\mathcal{D}_x^{\alpha_2} z)(x) = ({}_0^C\mathcal{D}_x^{\alpha_1+\alpha_2} z)(x)$,

(b) $({}_0\mathcal{I}_x^{\alpha_1} \, {}_0\mathcal{I}_x^{\alpha_2} z)(x) = ({}_0\mathcal{I}_x^{\alpha_2} \, {}_0\mathcal{I}_x^{\alpha_1} z)(x) = ({}_0\mathcal{I}_x^{\alpha_1+\alpha_2} z)(x)$,

(c)
$$\left({}_0^C\mathcal{D}_x^{\alpha} x^{\gamma}\right) = \begin{cases} 0 & \gamma \in \mathbb{Z}^+ \text{ and } \gamma < \alpha, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} & o. \ w. \end{cases}$$

(d) $({}_0\mathcal{I}_x^{\alpha} \, {}_0^C\mathcal{D}_x^{\alpha} z)(x) = z(x) - \sum_{j=0}^{\lceil\alpha\rceil-1} \frac{x^j}{j!} z^{(j)}(0^+), \quad n-1 < \alpha \le n \in \mathbb{N}.$

2.2 Hybrid Functions

Definition 3 *The sets of BPFs* $\beta_s(x)$ *in the interval* $[0, b_f)$ *are defined as*

$$\beta_s(x) = \begin{cases} 1, & \frac{s-1}{S} b_f \le x < \frac{s}{S} b_f, \\ 0, & o.w., \end{cases}$$

where $S \in \mathbb{N}$ *and* $s = 1, 2, \ldots, S$ *is the order of BPFs. Set* $\{\beta_s(x)\}$ *has orthogonality, disjointness, and completeness properties. We consider the vector of BPFs as* $B(x) = [\beta_1(x), \beta_2(x), \ldots, \beta_s(x)]^T$.

Definition 4 ([21]) *A sets of hybrid BPFs and FKCPs* $\omega_{sr}(t)$ *on the interval* $[0, b_f)$ *are defined as*

$$\omega_{sr}(x) = \begin{cases} T_r\left(\frac{2S}{b_f} x - 2s + 1\right), & \frac{s-1}{S} b_f \le x < \frac{s}{S} b_f, \\ 0, & \text{otherwise.}, \end{cases}$$

where T_r *indicates FKCPs of order* $r = 0, 1, 2, \ldots, R-1, (R \in \mathbb{N})$*, and can be obtained by the following formulas:*

$$T_0(x) = 1, \quad T_1(x) = x, \quad \text{and} \ \ T_{r+1}(x) = 2xT_r(x) - T_{r-1}(x), \quad x \in [-1, 1].$$

The FKCPs are orthogonal with respect to the non-constant weight function $w(x) = (1-x^2)^{\frac{-1}{2}}$. In the following, without losing generality, we assume that $b_f = 1$. Any function $z(x) \in L^2[0,1)$ can be expanded in terms of the basis functions $\omega_{sr}(x)$ as follows [11]:

$$z(x) = \sum_{s=1}^{\infty}\sum_{r=0}^{\infty} z_{sr}\,\omega_{sr}(x), \tag{4}$$

where the hybrid coefficients z_{sr} are calculated as

$$z_{sr} = \frac{\langle z, \omega_{sr}\rangle_w}{\langle \omega_{sr}, \omega_{sr}\rangle_w}, \qquad s = 1,2,\ldots, \quad r = 0,1,\ldots, \tag{5}$$

where $\langle u, v\rangle_w = \int_0^1 u(x)v(x)w(x)\,\mathrm{d}(x)$ denotes the inner product.

Theorem 1 ([13]) *Let Θ be a strictly convex normed space and Z be a finite dimensional subspace of Θ. Then, each $z \in \Theta$ has a unique best approximation in Z.*

Corollary 1 *Let*

$$\mathrm{Z} = span\{\omega_{10}(x), \ldots, \omega_{1(R-1)}(x), \omega_{20}(x), \ldots, \omega_{2(R-1)}(x), \ldots, \omega_{S0}(x), \ldots, \omega_{S(R-1)}(x)\},$$

where Z is a finite dimensional subspace of Θ. Since the Hilbert space Θ is strictly convex and considering Theorem 1, we conclude that the hybrid series of z in Eq. (4) can be truncated by $\hat{z}$ as

$$z(x) \simeq \hat{z}(x) = z_{SR}(x) = \sum_{s=1}^{S}\sum_{r=0}^{R-1} z_{sr}\,\omega_{sr}(x) = \mathbf{Z}^T\boldsymbol{\Omega}(x) = \boldsymbol{\Omega}^T(x)\mathbf{Z}, \tag{6}$$

where

$$\boldsymbol{\Omega}(x) = [\omega_{10}(x), \ldots, \omega_{1(R-1)}(x), \omega_{20}(x), \ldots, \omega_{2(R-1)}(x), \ldots, \omega_{S0}(x), \ldots, \omega_{S(R-1)}(x)]^T,$$

and

$$\mathbf{Z} = [z_{10}, z_{11}, \ldots, z_{1(R-1)}, z_{20}, z_{21}, \ldots, z_{2(R-1)}, \ldots, z_{S0}, z_{S1}, \ldots, z_{S(R-1)}]^T.$$

2.3 Operational Matrices

2.3.1 Operational Matrix of Integration of the Vector $\boldsymbol{\Omega}$

Let $\lambda = SR$. The integration of the vector $\boldsymbol{\Omega}(x)$ can be expanded as [17]

$$\int_0^x \boldsymbol{\Omega}(\zeta)\,\mathrm{d}\zeta \simeq \boldsymbol{\Pi}_{\lambda\times\lambda}\,\boldsymbol{\Omega}(x),$$

where the operational matrix of integration $\boldsymbol{\Pi}_{\lambda\times\lambda}$ is presented as

$$\boldsymbol{\Pi}_{\lambda\times\lambda} = \begin{bmatrix} \mathrm{N} & \mathrm{M} & \mathrm{M} & \ldots & \mathrm{M} \\ \mathbf{0} & \mathrm{N} & \mathrm{M} & \ldots & \mathrm{M} \\ \mathbf{0} & \mathbf{0} & \mathrm{N} & \ldots & \mathrm{M} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathrm{N} \end{bmatrix},$$

where M and N have the following forms:

$$\mathrm{M} = \frac{b_f}{S}\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \frac{-1}{3} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^R-1}{2R(R-2)} & 0 & 0 & \ldots & 0 \end{bmatrix}_{R\times R},$$

$$
\mathrm{N} = \frac{b_f}{2S}\begin{bmatrix}
1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\
\frac{-1}{4} & 0 & \frac{1}{4} & 0 & \dots & 0 & 0 & 0 \\
\frac{-1}{3} & \frac{-1}{2} & 0 & \frac{1}{6} & \dots & 0 & 0 & 0 \\
\frac{1}{8} & 0 & \frac{-1}{4} & 0 & \frac{1}{8} & \dots & 0 & 0 \\
\frac{-1}{15} & 0 & 0 & \frac{-1}{6} & 0 & \frac{1}{10} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
\frac{(-1)^{R-1}}{(R-1)(R-3)} & 0 & 0 & 0 & \dots & \frac{-1}{2(R-3)} & 0 & \frac{1}{2(R-1)} \\
\frac{(-1)^{R}}{R(R-2)} & 0 & 0 & 0 & \dots & 0 & \frac{-1}{2(R-2)} & 0
\end{bmatrix}_{R\times R}.
$$

2.3.2 Operational Matrix of the Fractional Integration

We remind that the vector $\mathbf{\Omega}(x)$ can be approximated as

$$\mathbf{\Omega}(x) \simeq \mathbf{\Psi}_{\lambda\times\lambda} B(x), \tag{7}$$

in which

I) $\mathbf{\Psi}_{\lambda\times\lambda} = [\mathbf{\Omega}(\tau_1) \quad \mathbf{\Omega}(\tau_2) \quad \dots \quad \mathbf{\Omega}(\tau_\lambda)]$, where $\tau_j = \frac{2j-1}{2\lambda}$; $j = 1, 2, \dots, \lambda$.

II) $B(x)$ is the vector of BPFs.

We define

$$({}_0\mathcal{I}_x^\alpha \mathbf{\Omega})(x) \simeq \mathbf{\Pi}^\alpha_{\lambda\times\lambda}\mathbf{\Omega}(x). \tag{8}$$

where $\mathbf{\Pi}^\alpha_{\lambda\times\lambda}$ is the operational matrix for the fractional integration. Using Eq. (7), we can write

$$({}_0\mathcal{I}_x^\alpha \mathbf{\Omega})(x) \simeq ({}_0\mathcal{I}_x^\alpha \mathbf{\Psi}_{\lambda\times\lambda} B)(x) = \mathbf{\Psi}_{\lambda\times\lambda}({}_0\mathcal{I}_x^\alpha B)(x). \tag{9}$$

From Eqs. (8) and (9), we get

$$\mathbf{\Pi}^\alpha_{\lambda\times\lambda}\mathbf{\Omega}(x) \simeq \mathbf{\Psi}_{\lambda\times\lambda}({}_0\mathcal{I}_x^\alpha B)(x). \tag{10}$$

Using Definition 1, the Riemann-Liouville fractional integral of the BPFs can be obtained as [16]

$$({}_0\mathcal{I}_x^\alpha B)(x) \simeq F^\alpha B(x), \tag{11}$$

where F^α is the Block-pulse operational matrix of fractional order integration and has the following shape:

$$
F^\alpha = \frac{1}{\lambda^\alpha\,\Gamma(\alpha+2)}\begin{bmatrix}
1 & \varsigma_1 & \varsigma_2 & \dots & \varsigma_{\lambda-1} \\
0 & 1 & \varsigma_1 & \dots & \varsigma_{\lambda-2} \\
0 & 0 & 1 & \dots & \varsigma_{\lambda-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 1
\end{bmatrix}.
$$

with $\varsigma_j = (j+1)^{\alpha+1} - 2j^{\alpha+1} + (j-1)^{\alpha+1},\ j = 1, 2, \dots, \lambda-1$.

By substituting Eq. (11) in Eq. (10) and considering Eq. (7), we will have

$$\mathbf{\Pi}^\alpha_{\lambda\times\lambda}\mathbf{\Psi}_{\lambda\times\lambda}B(x) \simeq \mathbf{\Psi}_{\lambda\times\lambda}F^\alpha B(x). \tag{12}$$

And finally, $\mathbf{\Pi}^\alpha_{\lambda\times\lambda}$ is obtained as

$$\mathbf{\Pi}^\alpha_{\lambda\times\lambda} \simeq \mathbf{\Psi}_{\lambda\times\lambda}F^\alpha\mathbf{\Psi}^{-1}_{\lambda\times\lambda}. \tag{13}$$

3 Method Implementation

In the current section, we consider system (3) under the Caputo fractional derivative. We need to approximate ${}_0^C\mathcal{D}_x^\alpha z(x)$ by hybrid functions as [18]

$$ {}_0^C\mathcal{D}_x^\alpha z(x) \simeq \mathbf{Z}^T\mathbf{\Omega}(x), \tag{14} $$

where $\mathbf{Z} = [z_1, z_2, \ldots, z_\lambda]^T$ is the unknown vector.

Now, applying the operator ${}_0\mathcal{I}_x^\alpha$ to both sides of Eq. (14), we get

$$ {}_0\mathcal{I}_x^\alpha\, {}_0^C\mathcal{D}_x^\alpha z(x) \simeq \mathbf{Z}^T({}_0\mathcal{I}_x^\alpha\mathbf{\Omega})(x). \tag{15} $$

Using Eq. (8) and item (1) in property (1), the last equation can be written as

$$ z(x) \simeq \sum_{j=0}^{\lceil\alpha\rceil-1}\frac{x^j}{j!}z^{(j)}(0^+) + \mathbf{Z}^T\,\mathbf{\Pi}_{\lambda\times\lambda}^\alpha\mathbf{\Omega}(x), $$

where, in the particular case $\alpha \in (0,1]$, we will have

$$ z(x) \simeq z(0^+) + \mathbf{Z}^T\,\mathbf{\Pi}_{\lambda\times\lambda}^\alpha\mathbf{\Omega}(x) = \mathbf{Z}_{(\alpha)}^T\mathbf{\Omega}(x). \tag{16} $$

Furthermore, we can expand the term $u(x)v(x)$ in terms of the vector $\mathbf{\Omega}(x)$ as

$$ u(x)v(x) \simeq (\mathbf{U}_{(\alpha)}^T\mathbf{\Omega}(x))(\mathbf{V}_{(\alpha)}^T\mathbf{\Omega}(x)) = \mathbf{U}_{(\alpha)}^T\mathbf{\Omega}(x)\mathbf{\Omega}^T(x)\mathbf{V}_{(\alpha)} = \mathbf{U}_{(\alpha)}^T\tilde{\mathbf{V}}_{(\alpha)}\mathbf{\Omega}(x), \tag{17} $$

where the evaluation procedure of $\mathbf{\Omega}(x)\mathbf{\Omega}^T(x)$ and matrix $\tilde{\mathbf{V}}_{(\alpha)}$ are given by [21].

Now, we turn our attention to solving the system (3) with the initial conditions in (2). Using Eq. (14), we can write

$$ {}_0^C\mathcal{D}_x^\alpha X_i(x) \simeq \mathbf{X}_i^T\mathbf{\Omega}(x), \quad i = 1, \ldots, 5, \tag{18} $$

where the vector $\mathbf{X}_i = [x_{i,1}, x_{i,2}, \ldots, x_{i,\lambda}]^T$ is unknown. Also, from Eq. (16), we have

$$ X_i(x) \simeq \mathbf{X}_{i(\alpha)}^T\mathbf{\Omega}(x), \quad i = 1, \ldots, 5. \tag{19} $$

Now, by substituting Eqs. (17)–(19) into system (3) and replacing $\simeq$ with $=$, we obtain

$$ \begin{cases} \left(\mathbf{X}_1^T - \mathbf{M}^T + \delta^\alpha\mathbf{X}_{1(\alpha)}^T + \beta^\alpha\left(\mathbf{X}_{2(\alpha)} + \mathbf{X}_{3(\alpha)}\right)^T\tilde{\mathbf{X}}_{1(\alpha)}\right)\mathbf{\Omega}(x) = 0, \\ \left(\mathbf{X}_2^T - \beta^\alpha\left(\mathbf{X}_{2(\alpha)} + \mathbf{X}_{3(\alpha)}\right)^T\tilde{\mathbf{X}}_{1(\alpha)} + (\rho^\alpha + \delta^\alpha + \tau^\alpha)\mathbf{X}_{2(\alpha)}^T\right)\mathbf{\Omega}(x) = 0, \\ \left(\mathbf{X}_3^T - \rho^\alpha\mathbf{X}_{2(\alpha)}^T + (\sigma^\alpha + \delta^\alpha)\mathbf{X}_{3(\alpha)}^T\right)\mathbf{\Omega}(x) = 0, \\ \left(\mathbf{X}_4^T - \tau^\alpha\mathbf{X}_{2(\alpha)}^T - \sigma^\alpha\mathbf{X}_{3(\alpha)}^T + (\kappa^\alpha + \delta^\alpha)\mathbf{X}_{4(\alpha)}^T\right)\mathbf{\Omega}(x) = 0, \\ \left(\mathbf{X}_5^T - \kappa^\alpha\mathbf{X}_{4(\alpha)}^T + \delta^\alpha\mathbf{X}_{5(\alpha)}^T\right)\mathbf{\Omega}(x) = 0, \end{cases} \tag{20} $$

where $\mu^\alpha = \mathbf{M}^T\mathbf{\Omega}(x)$. The unknown vectors $\mathbf{X}_i$, $i = 1, \ldots, 5$ can be calculated by solving the nonlinear system of algebraic equation obtained from collocated system (20) at the points $x_k = \frac{2k-1}{2\lambda}$, $k = 1, 2, \ldots, \lambda$.

4 Convergence Analysis

The current section is devoted to the study of the convergence of the proposed method. For this purpose, it is necessary to present some required definitions and lemmas.

Definition 5 ([7]) *Let $w(x)$ be a weight function on the interval $(-1,1)$. The weighted L^p-norm for $1 \leq p < \infty$ is defined as*

$$\|z\|_{L^p_w(-1,1)} = \left(\int_{-1}^{1} |z(x)|^p w(x)\,\mathrm{d}x \right)^{1/p}.$$

For $p = \infty$, we define

$$\|z\|_{L^\infty_w(-1,1)} = \sup_{-1 \leq x \leq 1} |z(x)| = \|z\|_{L^\infty(-1,1)}.$$

Definition 6 ([7]) *Consider the Chebyshev weight $w(x) = (1-x^2)^{-1/2}$. Let γ be a non-negative integer. A Sobolev space is the vector space of the functions $z \in L^2_w(-1,1)$ where $z^{(m)} \in L^2_w(-1,1)$, for $0 \leq m \leq \gamma$. Furthermore, the norm of the Sobolev space $H^\gamma_w(-1,1)$ is defined as*

$$\|z\|_{H^\gamma_w(-1,1)} = \left(\sum_{m=0}^{\gamma} \|z^{(i)}\|^2_{L^2_w(-1,1)} \right)^{1/2}.$$

Definition 7 ([7]) *Let (a,b) be a bounded interval in $\mathbb{R}$. For each function $z \in H^1_w(a,b)$ the following inequality (called the Sobolev inequality) holds:*

$$\|z\|_{L^\infty(a,b)} \leq \left(\frac{1}{b-a} + 2 \right)^{1/2} \|z\|^{1/2}_{L^2_w(a,b)} \|z\|^{1/2}_{H^1_w(a,b)}$$

Lemma 1 ([7]) *Let $\{T_r\}_{r=0}$ be the sequence of Chebyshev polynomials and $P_J(x) = \sum_{k=0}^{J} t_k T_k(x)$ be the best polynomial approximation of degree J for $z \in L^2_w(-1,1)$. Then, for $\gamma \geq 0$, there exists a constant $\mathcal{C} > 0$ such that*

$$\|z - P_J\|_{L^2_w(-1,1)} \leq \mathcal{C} J^{-\gamma} \|z\|_{H^\gamma_w(-1,1)},$$

for all functions z in $H^\gamma_w(-1,1)$.

Lemma 2 ([7]) *Let $\gamma \geq 1$ and $1 \leq l \leq \gamma$. For $z \in H^\gamma_w(-1,1)$, the following inequality in higher order Sobolev norms holds:*

$$\|z - P_J\|_{H^1_w(-1,1)} \leq \mathcal{C} J^{2l-1/2-\gamma} \|z\|_{H^\gamma_w(-1,1)},$$

The following corollary is obviously obtained using Lemma 1, Lemma 2 and the Sobolev inequality.

Corollary 2 *Let $l = 1$ be defined in Lemma 2. With the assumptions of Lemma 1, and for $\gamma \geq 1$, there exists a constant $\mathcal{C}_0 > 0$ such that*

$$\|z - P_J\|_{L^\infty(-1,1)} \leq \mathcal{C}_0 J^{3/4-\gamma} \|z\|_{H^\gamma_w(-1,1)}.$$

Corollary 3 *Let $z \in H^\gamma_w[0,1)$ and $\hat{z} = z_{SR}$ be the polynomial approximation of z defined in Eq. (6). Then,*

$$\|z - \hat{z}\|_{L^\infty[0,1)} \leq \eta (SR)^{\frac{3}{4}-\gamma} \max_{1 \leq s \leq S} \|z\|_{H^\gamma_w(I_s)},$$

where $I_s = [\frac{s-1}{S}, \frac{s}{S})$ and η is a positive constant.

Proof. It can be clearly concluded from Corollary 2. ■

Now, by applying the operator ${}_0\mathcal{I}^{\alpha}$ to both sides of equations in the system (3), we obtain

$$\begin{cases} {}_0\mathcal{I}_x^{\alpha}\big({}_0^C\mathcal{D}_x^{\alpha}X_1\big) = {}_0\mathcal{I}_x^{\alpha}\big(\mu_\alpha - \delta_\alpha X_1 - \beta_\alpha X_1(X_2+X_3)\big), \\ {}_0\mathcal{I}_x^{\alpha}\big({}_0^C\mathcal{D}_x^{\alpha}X_2\big) = {}_0\mathcal{I}_x^{\alpha}\big(\beta_\alpha X_1(X_2+X_3) - \rho_\alpha X_2 - (\delta_\alpha+\tau_\alpha)X_2\big), \\ {}_0\mathcal{I}_x^{\alpha}\big({}_0^C\mathcal{D}_x^{\alpha}X_3\big) = {}_0\mathcal{I}_x^{\alpha}\big(\rho_\alpha X_2 - \sigma_\alpha X_3 - \delta_\alpha X_3\big), \\ {}_0\mathcal{I}_x^{\alpha}\big({}_0^C\mathcal{D}_x^{\alpha}X_4\big) = {}_0\mathcal{I}_x^{\alpha}\big(\tau_\alpha X_2 + \sigma_\alpha X_3 - \kappa_\alpha X_4 - \delta_\alpha X_4\big), \\ {}_0\mathcal{I}_x^{\alpha}\big({}_0^C\mathcal{D}_x^{\alpha}X_5\big) = {}_0\mathcal{I}_x^{\alpha}\big(\kappa_\alpha X_4 - \delta_\alpha X_5\big), \end{cases} \tag{21}$$

Using Definitions 1, 2 and Property 1, the system (21) can be written as

$$\begin{cases} X_1(x) - X_1(0^+) = \mu_\alpha \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-\zeta)^{\alpha-1}\, h_1(\zeta, X(\zeta))\,\mathrm{d}\zeta, \\ X_2(x) - X_2(0^+) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\zeta)^{\alpha-1}\, h_2(\zeta, X(\zeta))\,\mathrm{d}\zeta, \\ X_3(x) - X_3(0^+) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\zeta)^{\alpha-1}\, h_3(\zeta, X(\zeta))\,\mathrm{d}\zeta, \\ X_4(x) - X_4(0^+) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\zeta)^{\alpha-1}\, h_4(\zeta, X(\zeta))\,\mathrm{d}\zeta, \\ X_5(x) - X_5(0^+) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\zeta)^{\alpha-1}\, h_5(\zeta, X(\zeta))\,\mathrm{d}\zeta, \end{cases} \tag{22}$$

where

$$\begin{aligned} h_1(\zeta, X(\zeta)) &= -\delta_\alpha X_1(\zeta) - \beta_\alpha X_1(\zeta)\big(X_2(\zeta)+X_3(\zeta)\big), \\ h_2(\zeta, X(\zeta)) &= \beta_\alpha X_1(\zeta)\big(X_2(\zeta)+X_3(\zeta)\big) - \rho_\alpha X_2(\zeta) - (\delta_\alpha+\tau_\alpha)X_2(\zeta), \\ h_3(\zeta, X(\zeta)) &= \rho_\alpha X_2(\zeta) - \sigma_\alpha X_3(\zeta) - \delta_\alpha X_3(\zeta), \\ h_4(\zeta, X(\zeta)) &= \tau_\alpha X_2(\zeta) + \sigma_\alpha X_3(\zeta) - \kappa_\alpha X_4(\zeta) - \delta_\alpha X_4(\zeta), \\ h_5(\zeta, X(\zeta)) &= \kappa_\alpha X_4(\zeta) - \delta_\alpha X_5(\zeta), \end{aligned}$$

and

$$X(\zeta) = \big[X_1(\zeta), X_2(\zeta), X_3(\zeta), X_4(\zeta), X_5(\zeta)\big]^T.$$

We consider the matrix form of the system (22) as

$$X(x) = C(x) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-\zeta)^{\alpha-1}\, H(\zeta, X(\zeta))\,\mathrm{d}\zeta, \tag{23}$$

where

$$C(x) = \Big[\mu_\alpha \tfrac{x^\alpha}{\Gamma(\alpha+1)} + X_1(0^+), X_2(0^+), X_3(0^+), X_4(0^+), X_5(0^+)\Big]^T$$

and

$$H(\zeta, X(\zeta)) = \big[h_1(\zeta, X(\zeta)), h_2(\zeta, X(\zeta)), h_3(\zeta, X(\zeta)), h_4(\zeta, X(\zeta)), h_5(\zeta, X(\zeta))\big]^T.$$

Theorem 2 *Let $\hat{X}(x) = X_{SR}(x)$ be the approximate solution obtained by the HCBPM and $X \in H_w^\gamma[0,1)$, for $\gamma \geq 1$, be the exact solution of Eq. (23). Let the function $h_i(\zeta, X(\zeta))$ satisfies the Lipschitz condition*

$$\big|h_i(\zeta, X(\zeta)) - h_i(\zeta, Y(\zeta))\big| \leq L_i\, \|X - Y\|_\infty, \tag{24}$$

where $L_i > 0\,(i = 1, \dots, 5)$ are Lipschitz constants. Moreover, assume that

$$\mathcal{A}_\alpha = \frac{1}{\Gamma(\alpha)} \sup_{0\leq x<1} \int_0^x (x-\zeta)^{\alpha-1}\,\mathrm{d}\zeta = \frac{1}{\Gamma(\alpha+1)}. \tag{25}$$

There exists a positive constant ϵ such that

$$\|E\|_\infty = \|X - \hat{X}\|_\infty \leq \mathcal{A}_\alpha \epsilon.$$

Proof. For $i = 1, \ldots, 5$, let $\hat{X}_i(x)$ be the approximate solution of the system (22). We consider the error terms as

$$e_i(x) = X_i(x) - \hat{X}_i(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\zeta)^{\alpha-1} h_i(\zeta, X(\zeta))\, \mathrm{d}\zeta, \quad i = 1, \ldots, 5. \tag{26}$$

Using the assumptions (24) and (25) and $0 \leq x < 1$, we obtain

$$|e_i(x)| \leq \mathcal{A}_\alpha L_i \, \|X_i - \hat{X}_i\|_\infty = \mathcal{A}_\alpha L_i \upsilon, \quad i = 1, \ldots, 5, \tag{27}$$

where $\upsilon = \max\limits_{1 \leq i \leq 5} \left\{ \|X_i - \hat{X}_i\|_\infty \right\}$. Let $L = \max\limits_{1 \leq i \leq 5} L_i$. Then

$$\|e_i\|_\infty \leq \mathcal{A}_\alpha L \, \upsilon \quad \text{for } i = 1, \ldots, 5. \tag{28}$$

Using Corollary 3, there exists positive constant η_i; $i = 1, \ldots, 5$ such that

$$\|X_i - \hat{X}_i\|_\infty \leq \eta_i \, (SR)^{\frac{3}{4}-\gamma} \max_{1 \leq s \leq S} \|X_i\|_{H_w^\gamma(I_s)} = \theta_i, \quad i = 1, \ldots, 5. \tag{29}$$

Now, by using Eqs. (28) and (29), we have

$$\|e_i\|_\infty \leq \mathcal{A}_\alpha L \max_{1 \leq i \leq 5} \{\theta_i\} = \mathcal{A}_\alpha \epsilon, \quad i = 1, \ldots, 5, \tag{30}$$

where $\epsilon = L \max\limits_{1 \leq i \leq 5} \{\theta_i\}$.

Defining $E(x) = X(x) - \hat{X}(x)$ and using (30), we will have

$$\|E\|_\infty = \|X - \hat{X}\|_\infty \leq \mathcal{A}_\alpha \epsilon.$$

■

5 Numerical Simulation

In this section, the model (3) is solved using the present method. The initial conditions and parameter values are considered as [33]

$$(X_1(0), X_2(0), X_3(0), X_4(0), X_5(0)) = (153, 55, 79, 68, 20),$$
$$\mu_\alpha = 0.145, \quad \delta_\alpha = 0.000411, \quad \beta_\alpha = 0.00038, \quad \rho_\alpha = 0.00211,$$
$$\tau_\alpha = 0.0021, \quad \sigma_\alpha = 0.0169, \quad \theta_\alpha = 0.0181.$$

Figure 1 shows the approximate solutions of the model for the given initial values for $\alpha = 1$. Since, the exact solution of the model is unknown, we have compared the solutions of the model obtained using the proposed method and the RK4 solutions in Table 1 on interval [0, 1] and Figure 2 on interval [0, 30]. The results show us that the obtained solutions using the HCBPM, which also easily used in the implementation, are in desired agreement with the RK4 solutions. Therefore, it can be concluded that the proposed method has the ability to predict the behavior of variables in the region under investigation. Also, we solved the model (3) using the peresent method for fractional derivatives. Figure 3 show the approximate solutions for the compartments $X_1(t)$, $X_2(t)$, $X_3(t)$, $X_4(t)$ and $X_5(t)$ for some values of α. All our computations were done using Matlab 2017a software.

6 Conclusions

In the present work, a hybrid functions method based on the combination of Chebyshev polynomials and Block-pulse functions was used for solving a fractional model of COVID-19 epidemic. The properties of selected hybrid functions were utilized to reduce the solution of the considered model to the solution of non-linear algebraic equations. Also, the convergence analysis of the method was studied. Finally, an illustrative example was given to demonstrate the validity and applicability of the proposed method.

t	$\|X_i(\text{RK4}) - X_i(\text{HCBPM})\|$				
	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
0.0	0	0	0	0	0
0.1	1.20E-11	4.42E-11	2.24E-11	2.84E-11	2.22E-11
0.2	5.50E-11	2.03E-11	2.55E-11	1.19E-11	8.20E-12
0.3	3.01E-12	1.53E-11	6.00E-12	1.89E-12	3.34E-11
0.4	5.80E-11	3.57E-11	5.10E-12	2.02E-11	2.37E-11
0.5	6.60E-11	5.77E-11	6.49E-12	4.07E-11	2.21E-11
0.6	6.60E-11	1.24E-11	6.10E-12	2.78E-11	9.95E-14
0.7	4.50E-11	6.10E-12	1.40E-11	3.55E-11	3.51E-11
0.8	3.50E-11	5.74E-11	1.80E-12	3.07E-11	2.16E-11
0.9	2.10E-11	1.40E-12	1.48E-11	1.83E-11	8.40E-12
1.0	3.01E-12	1.09E-12	4.07E-11	3.17E-11	2.61E-11

Table 1: Comparison of the solutions obtained by the present method and the RK4 solutions.

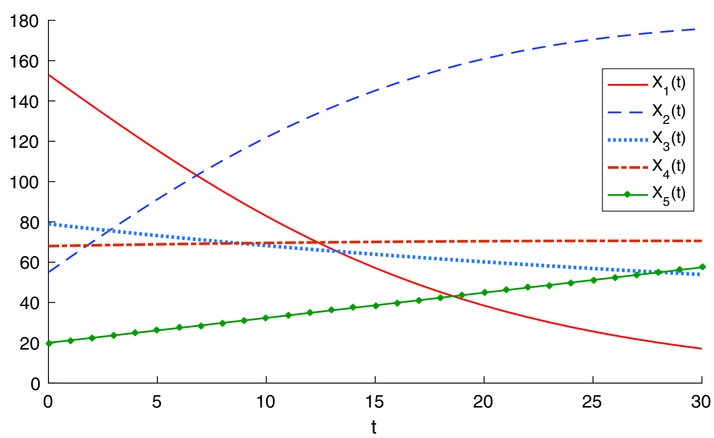
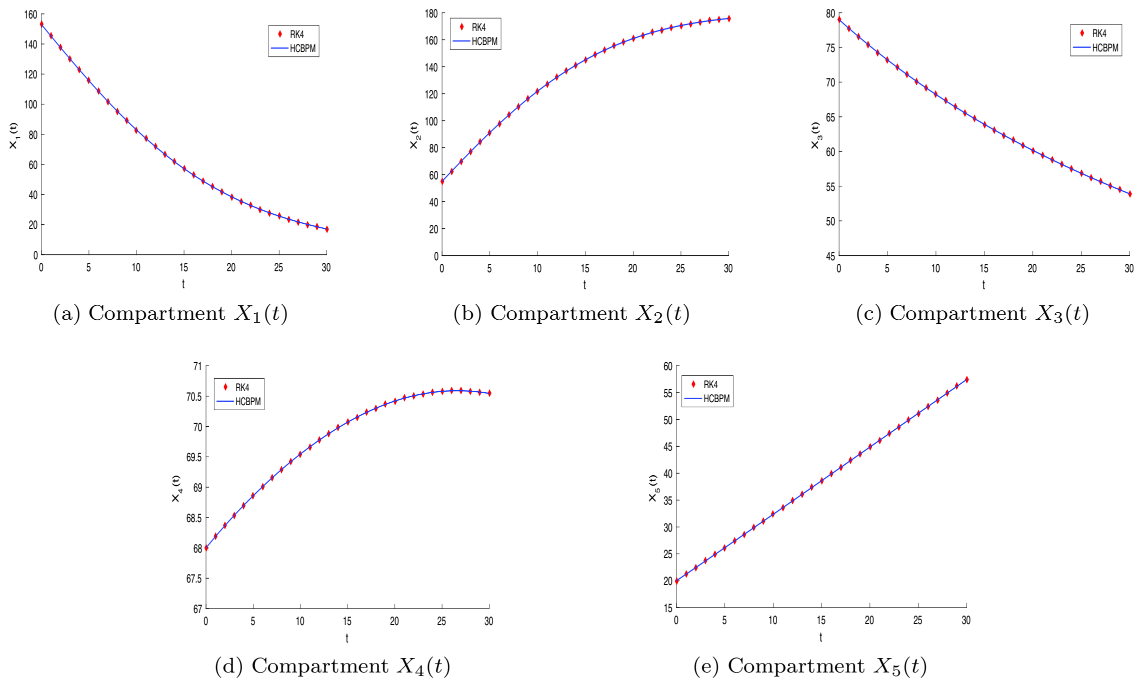


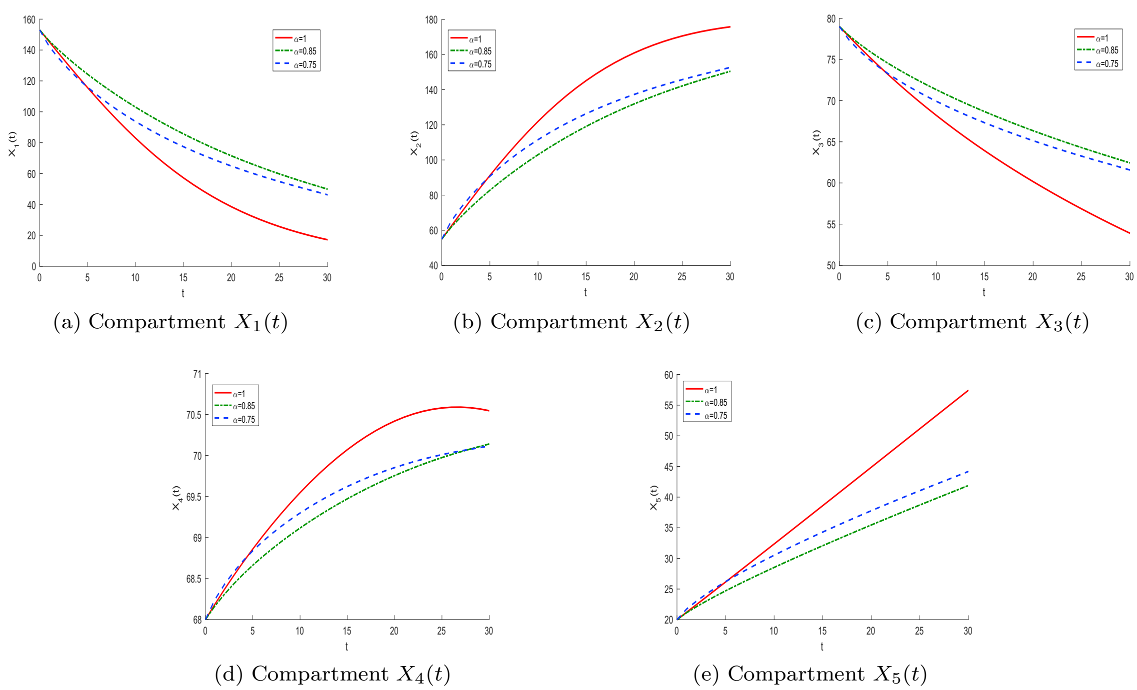
Figure 1: Plots of compartments $X_1(t)$, $X_2(t)$, $X_3(t)$, $X_4(t)$ and $X_5(t)$ for the given initial values of the model for $\alpha = 1$.



(a) Compartment $X_1(t)$ (b) Compartment $X_2(t)$ (c) Compartment $X_3(t)$

(d) Compartment $X_4(t)$ (e) Compartment $X_5(t)$

Figure 2: Intersection of the dual sets of order 0 in (a) and (b) (see [1, Lemma 4 and Theorem 3.3]) to yield (c).



(a) Compartment $X_1(t)$ (b) Compartment $X_2(t)$ (c) Compartment $X_3(t)$

(d) Compartment $X_4(t)$ (e) Compartment $X_5(t)$

Figure 3: The approximate solution for $\alpha = 1,\, 0.85,\, 0.75$.

References

[1] K. Abuasbeh, R. Shafqat, A. Alsinai and M. Awadalla, Analysis of the mathematical modelling of COVID-19 by using mild solution with delay Caputo operator, Symmetry, 15(2023), 286.

[2] N. Anggriani, M. Z. Ndii, R. Amelia, W. Suryaningrat and M. A. A. Pratama, A mathematical COVID-19 model considering asymptomatic and symptomatic classes with waning immunity, Alex. Eng. J., 61(2022), 113–124.

[3] S. Arshad, I. Siddique, F. Nawaz, A. Shaheen and H. Khurshid, Dynamics of a fractional order mathematical model for COVID-19 epidemic transmission, Physica A: Statistical Mechanics and its Applications, 609(2023), 18 pp.

[4] W. S. Avusuglo, N. Bragazzi, A. Asgary, J. Orbinski, J. Wu and J. D. Kong, Leveraging an epidemic–economic mathematical model to assess human responses to COVID-19 policies and disease progression, Sci. Rep., 13(2023), 12842.

[5] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trojillo, Fractional Calculus Models and Numerical Methods, World Scientific, Berlin, 2012.

[6] S. Bhatter, K. Jangid, A. Abidemi, K. M. Owolabi and S. D. Purohit. A new fractional mathematical model to study the impact of vaccination on COVID-19 outbreaks, Decis. Anal. J., 6(2023), 100156.

[7] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, Spectral Methods in Fluid Dynamics, Springer Verlag, Berlin Heidelberg, 1988.

[8] W. C. Chen, Nonlinear dynamics and chaos in a fractional-order financial system, Chaos Soliton Fract., 36(2008), 1305–1314.

[9] S. A. David, C. Fischer and J. A. Tenreiro Machado, Fractional electronic circuit simulation of a nonlinear macroeconomic model, AEU Int. J. Electron. Commun., 84(2018), 210–220.

[10] S. Gao, P. Binod, C. W. Chukwu, T. Kwofie, S. Safdar, L. Newman, S. Choe, B. K. Datta, W. K. Attipoe, W. Zhang and P. Driessche, A mathematical model to assess the impact of testing and isolation compliance on the transmission of COVID-19, Infect. Dis. Model., 8(2023), 427–444.

[11] S. M. Hoseini, Analysis of linear proportional delay systems via hybrid functions method, Asian Journal of Control, 24(2022), 344–354.

[12] M. A. Khan and A. Atangana, Modeling the dynamics of novel coronavirus (2019-ncov) with fractional derivative, Alex. Eng. J., 59(2020), 2379–2389.

[13] E. Kreyszig, Introductory Functional Analysis with Applications, Wiley, New York, 1989.

[14] S. Kumar, J. Cao and M. Abdel-Aty, A novel mathematical approach of COVID-19 with non-singular fractional derivative, Chaos Solitons Fract., 139(2020), 110048.

[15] P. Kumar, V. S. Erturk and M. Murillo-Arcila, A new fractional mathematical modelling of COVID-19 with the availability of vaccine, Results in Physics, 24(2021), 104213.

[16] Y. Lu, Y. Tang, X. Zhang and S. Wang, Parameter identification of fractional order systems with nonzero initial conditions based on block pulse functions. Measurement, 158(2020), 107684.

[17] H. R. Marzban and M. Shahsiah, Solution of piecewise constant delay systems using hybrid of block-pulse and Chebyshev polynomials, Optim. Control Appl. Methods, 32(2011), 647–659.

[18] S. Mashayekhi and M. Razzaghi, Numerical solution of distributed order fractional differential equations by hybrid functions, J. Comput. Phys., 315(2016), 169–181.

[19] H. Mohammadi, S. Rezapour and A. Jajarmi, On the fractional SIRD mathematical model and control for the transmission of COVID-19: the first and the second waves of the disease in Iran and Japan, ISA transactions, 124(2022), 103–114.

[20] K. B. Oldham and J. Spanier, The Fractional Calculus-Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, London, 1974.

[21] P. Pirmohabbati, A. R. Sheikhani and A. A. Ziabari, Numerical Solution of Nonlinear Fractional Bratu Equation with Hybrid Method, Int. J. Appl. Comput. Math., 6(2020), 1–22.

[22] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.

[23] S. Rezapour, H. Mohammadi and M. E. Samei, SEIR epidemic model for COVID-19 transmission by Caputo derivative of fractional order, Adv. Differ. Equ., 2020(2020), 1–199.

[24] H. Singh, H. M. Srivastava, Z. Hammouch and K. S. Nisar, Numerical simulation and stability analysis for the fractional order dynamics of COVID-19, Results in physics, 20(2021), 103722.

[25] K. Shah and T. Abdeljawad, Study of a mathematical model of COVID-19 outbreak using some advanced analysis, Waves in Random and Complex Media, (2022), 1–18.

[26] H. G. Sun, Y. Zhang, W. Baleanu, W. Chen and Y. Q. Chen, A new collection of real world applications of fractional calculus in science and engineering, Commun. Nonlinear Sci. Numer. Simul., 64(2018), 213–231.

[27] N. H. Tuan, H. Mohammadi and S. Rezapour, A mathematical model for COVID-19 transmission by using the Caputo fractional derivative, Chaos Solitons Fract., 140(2020), 110107.

[28] World Health Organization, Coronavirus, World Health Organization, cited January 19, 2020. Available: https://www.who.int/health-topics/coronavirus.

[29] R. P. Yadav and R. Verma, A numerical simulation of fractional order mathematical modeling of COVID-19 disease in case of Wuhan China, Chaos Solitons Fract., 140(2020), 110124.

[30] R. Zarin, A. Khan, A. Yusuf, S. Abdel-Khalek and M. Inc, Analysis of fractional COVID-19 epidemic model under Caputo operator, Math. Methods Appl. Sci., 46(7) (2023), 7944–7964.

[31] A. Zeb, E. Alzahrani, V. S. Erturk and G. Zaman, Mathematical model for coronavirus disease 2019 (COVID-19) containing isolation class, BioMed research international, 2020.

[32] M. Sinan and N. H. Alharthi, Mathematical Analysis of Fractal-Fractional Mathematical Model of COVID-19, Fractal and Fractional, 7(2023), 358.

[33] Z. Zhang, A. Zeb, O. F. Egbelowo and V. S. Erturk, Dynamics of a fractional order mathematical model for COVID-19 epidemic, Adv. Differ. Equ., 2020(2020), 1–16.

[34] P. Zhou, X.-L. Yang, X.-G. Wang, B. Hu, L. Zhang, W. Zhang, H.-R. Si, Y. Zhang, B. Li, C.-L. Huang, H.-D. Chen, J. Chen, Y. Luo, H. Guo, R.-D. Jiang, M.-Q. Liu, Y. Chen, X.-R. Shen, X. Wang, X.-S. Zheng, K. Zhao, Q.-J. Chen, F. Deng, L.-L. Liu, B. Yan, F.-X. Zhan, Y.-Y. Wang, G.-F. Xiao and Z.-L. Shi, A pneumonia outbreak associated with a new coronavirus of probable bat origin, Nature, 579(2020), 270–273.