Applied Mathematics E-Notes, 25(2025), 31-44 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

A Novel Methodology For Determining Row And Column Ranks Of Tropical Matrices*

Amutha Boopathy†, Perumal Ramachandran‡

Received 5 October 2023

Abstract

In the modern era, tropical cryptographic algorithms can be effectively deciphered through the application of tropical linear systems. The solvability of these systems is determined by assessing their ranks and determinant. In this paper, we suggest a method for finding the column rank and row rank of any tropical matrix using the advanced discrepancy method. This method can also be applied to check the solvability of tropical linear systems. Unlike classical algebra, in tropical algebra there are many ranks for a tropical matrix namely kapranov rank, barvinok rank, column rank, row rank and tropical rank. We have also investigated various ranks of certain special tropical matrices.

1 Introduction

Imre Simon [1], a Brazilian mathematician and computer scientist had introduced tropical geometry in 1978. The terminology 'tropical' was coined by French mathematicians. In tropical geometry, tropical semirings are quite relevant. Semirings [2] featuring a base set (subset of the system of real numbers) and in operation of tropical algebra, addition is the maximum or minimum, multiplication is the classical addition, have been around since the late 1950s [3]. Based on the binary operation, there are two tropical semirings. In the min-plus tropical semiring, adding two elements will result in a minimum of those two elements. In max-plus tropical semiring, the addition of two elements is obtained by finding the maximum of that two elements. Multiplying two elements in both cases is obtained by adding them classically [4, 5, 1, 6]. Max-plus semiring is isomorphic to the min-plus semiring and both are idempotent semirings [7]. Tropical notion is rising exponentially due to its simplicity and proximity to algebraic geometry [5, 8]. As a result tropical semirings are applied in many domains including computer science, linear algebra, number theory, automata theory, language theory, control theory, cryptography [9] and operations research [3, 10]. Tropical semirings plays an important role in linear algebra, especially in solving the linear systems [11, 12]. We intend to decide the behavior of rank and determinant of some matrices over the tropical semirings. Tropical addition is denoted by '$\oplus$' and the tropical multiplication is denoted by '$\odot$'. The paper is structured as follows. We give some basic definitions in Section 2 that are relevant to this study. The determinants of some special tropical matrices were presented in Section 3. The different types of ranks for some special tropical matrices were discussed in Section 4. The advanced discrepancy method and the method for finding the column rank and row rank of tropical matrices were presented in Section 5.

2 Preliminaries

In this section, we discuss some preliminaries required to understand this paper.

Definition 1 ([2]) *A set S (non-empty) with two binary operations namely addition $(+)$ and multiplication $(\cdot)$ is called a semiring, if it satisfy the following axioms:*

*Mathematics Subject Classifications: 12K10, 14M25, 14N10, 90C24.

†Department of Mathematics, St. Joseph's College of Engineering, OMR, Chennai-600 119, Tamilnadu, India

‡Department of Mathematics, College of Engineering and Technology, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur-603 203, Tamilnadu, India

1. *S is a commutative monoid under the operation addition $(+)$ with '0' as the unique identity element.*
2. *S is a monoid under the operation multiplication $(\cdot)$ with a unique identity element denoted by '1'.*
3. *$u \cdot (v + w) = u \cdot v + u \cdot w$ and $(v + w) \cdot u = v \cdot u + w \cdot u \ \forall\ u, v, w \in S$.*
4. *$u \cdot 0 = 0$ and $0 \cdot u = 0 \ \forall\ u \in S$.*
5. *Identity elements under the two operations should not be the same. i.e. $0 \neq 1$.*

Definition 2 ([10]) *Max-plus semiring is a semiring R where the operations $\oplus$ and $\odot$ denotes max-plus addition and max-plus multiplication respectively.*

1. *$u \oplus v = v \oplus u \ \forall\ u, v \in R$.*
2. *$(u \oplus v) \oplus w = u \oplus (v \oplus w)$ and $(u \odot v) \odot w = u \odot (v \odot w) \ \forall\ u, v, w \in R$.*
3. *$u \odot (v \oplus w) = (u \odot v) \oplus (u \odot w) \ \forall\ u, v, w \in R$.*
4. *$\exists\ e \in R \ \forall\ u \in R$ such that $e \oplus u = u \oplus e = u$ (Here the additive identity is '$-\infty$').*

2.1 Matrices over the Max-Plus Semiring

The collection of all matrices with entries from the tropical semiring R with 'm' rows and 'n' columns is denoted as $M_{m\times n}(R)$. Let P be a matrix defined over the max-plus semiring. Every j^{th} column of the matrix P is denoted by 'P_j'. Every ij^{th} element of $P \in M_{m\times n}(R)$ is denoted by 'p_{ij}' and transpose of the matrix P is denoted by 'P^T'. Let $P = (p_{ij}) \in M_{m\times n}(R)$, $Q = (q_{ij}) \in M_{m\times n}(R)$, $T = (t_{ij}) \in M_{n\times l}(R)$ and $\alpha \in S$ [4, 12]. Then,

$$P \oplus Q = (\max((p_{ij}), (q_{ij})))_{m\times n},$$

$$P \odot T = \sum_{k=1}^{n} \max\left((p_{ik}) + (t_{kj})\right)_{m\times l},$$

$$\alpha \odot P = (\alpha + (p_{ij}))_{m\times n}.$$

Definition 3 ([13]) *The linear system $T \odot x = u$ is said to be a tropical linear system, if the entries of the linear system are all from either max-plus semiring $R = (S \cup \{-\infty\}, \oplus, \odot)$ or min-plus semiring $R' = (S \cup \{\infty\}, \oplus, \odot)$.*

Definition 4 ([14]) *Let $R = (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ be the real number system under the max-plus algebra. Let P and Q be two $m \times n$ matrices over the real numbers under the tropical addition and tropical multiplication, we say $P \leq Q \iff p_{ij} \leq q_{ij} \ \forall\ i, j$.*

Definition 5 ([13]) *In max-plus semiring, a vector $q = (q_1, q_2, \cdots, q_m)$ is said to be a normal vector or regular vector, if $q_j \neq -\infty$, $\forall\ 1 \leq j \leq m$. In min-plus semiring, $q = (q_1, q_2, \cdots, q_m)$ is a regular vector if $q_j \neq \infty$, $\forall\ 1 \leq j \leq m$.*

Definition 6 ([15]) *A solution y^* of the max-linear system $T \odot y = u$ is called the principal solution of the max. linear system, if $y_i \leq y^*$, where y_i's are the solutions of the max. linear system.*

Definition 7 ([16]) *A matrix $Z \in M_{n\times n}(R)$ is said to be a circulant matrix, if it is of the form,*

$$\begin{bmatrix} z_0 & z_{n-1} & z_{n-2} & \cdots & z_1 \\ z_1 & z_0 & z_{n-1} & \cdots & z_2 \\ z_2 & z_1 & z_0 & \cdots & z_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{n-1} & z_{n-2} & z_2 & \cdots & z_0 \end{bmatrix},$$

where $z_0, z_1, \cdots, z_{n-1} \in R$.

Definition 8 ([17]) *The tropical determinant* $\Gamma(T)$ *of the matrix* $T \in M_{n\times n}(R)$ *is obtained by,*

$$\Gamma(T) = \bigoplus_{\sigma \in \daleth_n} T_{1\sigma(1)} \odot T_{2\sigma(2)} \odot \cdots \odot T_{n\sigma(n)},$$

where, the $\daleth_n$ *denotes the symmetric group with* n *elements.*

Definition 9 *A matrix* $T \in M_{n\times n}(R)$ *is singular, if* $\max(T_{1\sigma(1)} \odot T_{2\sigma(2)} \odot \cdots \odot T_{n\sigma(n)})$ *attains at least two times.*

$$\Gamma(T) = \sum_{i}^{n} T_{i\sigma_1(i)} = \sum_{i}^{n} T_{i\sigma_2(i)}.$$

All the other matrices are non-singular.

Definition 10 ([18, 19]) *The barvinok rank of the tropical matrix* $T \in M_{n\times n}(R)$ *is the smallest* k *such that,*

$$T = \bigoplus_{i=1}^{k} u_i \odot {v_i}^T,$$

for the column vectors u_i, v_i*. Barvinok rank of the matrix* T *is denoted as* $B.R(T)$*.*

Definition 11 ([17, 20]) *The kapranov rank of a tropical matrix* $T \in M_{n\times n}(R)$ *is the smallest classical rank of the set of matrices over some valued field whose tropicalization equals* T*. It is denoted as* $K.R(T)$*.*

Definition 12 ([19, 21]) *The tropical rank of a tropical matrix* $A \in M_{n\times n}(R)$ *is the largest order of the tropically non-singular square submatrix. It is denoted as* $T.R(A)$*.*

Definition 13 ([17]) *A column rank* ($clrank(T)$) *of a tropical matrix* $T \in M_{n\times n}(R)$ *is the number of columns that are tropically independent.*

Definition 14 ([17]) *A row rank* ($rwrank(T)$) *of a tropical matrix* $T \in M_{n\times n}(R)$ *is the number of rows that are tropically independent.*

Definition 15 *A matrix* $T \in M_{c\times d}(R)$ *is said to be a column-wise arithmetic matrix with a constant term* $a \in R$ *and a difference* $k \in R$ *if,*

$$\begin{bmatrix} a+k & a+2k & \cdots & a+dk \\ a+(d+1)k & a+(d+2)k & \cdots & a+2dk \\ \vdots & \vdots & \vdots & \vdots \\ a+((c-1)d+1)k & a+((c-1)d+2)k & \cdots & a+cdk \end{bmatrix}.$$

Definition 16 *A matrix* $T \in M_{c\times d}(R)$ *is said to be a row-wise arithmetic matrix with a constant term* $a \in R$ *and a difference* $k \in R$ *if,*

$$\begin{bmatrix} a+k & a+(c+1)k & \cdots & a+((d-1)c+1)k \\ a+2k & a+(c+2)k & \cdots & a+((d-1)c+2)k \\ \vdots & \vdots & \vdots & \vdots \\ a+ck & a+2ck & \cdots & a+cdk \end{bmatrix}.$$

Definition 17 *A matrix* $T \in M_{c\times d}(R)$ *is called a* ϕ*-diagonal matrix if* $t_{ii} = \phi$ *for* $1 \le i \le c$ *and all other elements are zero.*

$$\begin{bmatrix} \phi & 0 & 0 & \cdots & 0 \\ 0 & \phi & 0 & \cdots & 0 \\ 0 & 0 & \phi & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \phi \end{bmatrix}.$$

Definition 18 *A matrix $T \in M_{c\times d}(R)$ is said to be a $J^l_{c\times d}$ matrix, if the entries of matrix T are all 'l'.*

$$\begin{bmatrix} l & l & l & \cdots & l \\ l & l & l & \cdots & l \\ l & l & l & \cdots & l \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l & l & l & \cdots & l \end{bmatrix}.$$

Definition 19 *Let $Z \in M_{c\times c}(R)$ be a circulant matrix and $T \in M_{c\times d}(R)$ is said to be an arithmetic circulant matrix, if the entries $z_0, z_1, \cdots, z_{n-1}$ are $1, 2, \cdots, c$ respectively*

$$\begin{bmatrix} 1 & c & c-1 & \cdots & 2 \\ 2 & 1 & c & \cdots & 3 \\ 3 & 2 & 1 & \cdots & 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c & c-1 & c-2 & \cdots & 1 \end{bmatrix}.$$

3 Determinant of the Tropical Matrices

The tropical determinant of some special matrices like the row-wise arithmetic matrix, column-wise arithmetic matrix, ϕ-diagonal matrix, $J^l_{n\times n}$ matrix and circulant arithmetic matrix were presented in this section.

Proposition 1 *Let $T \in M_{n\times n}(R)$ be a row-wise arithmetic matrix or column-wise arithmetic matrix. Then*

$$\Gamma(T) = Cla(tra(T)),$$

where, $Cla(tra(T))$ denotes the classical trace of the tropical matrix.

Proof.

$$\begin{aligned}\Gamma(T) &= \bigoplus_{\sigma\in \daleth_n} T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)} \\ &= \max_{\sigma\in \daleth_n}\{T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)}\}.\end{aligned}$$

In a row-wise arithmetic matrix and a column-wise arithmetic matrix, $T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)}$ are equal $\forall\ \sigma \in \daleth_n$.

$$T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)} = c, \forall\ \sigma \in \daleth_n.$$

Obviously, an identity permutation will satisfy this equality.

$$\begin{aligned}\max_{\sigma\in \daleth_n}\{T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)}\} &= c \\ &= t_{11} \odot t_{22} \odot \cdots t_{nn} \\ &= Cla(tra(T)).\end{aligned}$$

■

Proposition 2 *Let $T \in M_{n\times n}(R)$ be a row-wise arithmetic matrix or a column-wise arithmetic matrix. Then*

$$\Gamma(T) = \frac{2na + (n^2+1)nk}{2}.$$

Proof. By Proposition 1 we have,

$$\begin{aligned} Cla(tra(T)) &= \sum_{i=1}^{n} t_{ii} \\ &= (a+k) + (a+(n+2)k) + \cdots + a + (n(n-1)+n)k \\ &= \frac{2na + (n^2+1)nk}{2}. \end{aligned}$$

where, k is the difference and a is the constant. ■

Remark 1 *Every row-wise and column-wise arithmetic matrices are singular.*

Proposition 3 *Let $T \in M_{n\times n}(R)$ be a $J^l_{n\times n}$ matrix then $\Gamma(T) = nl$.*

Proof. We know that the matrix $J^l_{n\times n}$ is both row-wise and column-wise arithmetic. By Proposition 1, $\Gamma(T) = Cla(tra(T))$. Also, in $J^l_{n\times n}$, all the elements are l. Hence,

$$Cla(tra(T)) = nl.$$

Then $\Gamma(T) = nl$. ■

Proposition 4 *Let $T \in M_{n\times n}(R)$ be a ϕ-diagonal matrix. Then either $\Gamma(T) = n\phi$ or $\Gamma(T) = 0$.*

Proof. Let $t_{ii} = \phi \;\; \forall\, 1 \le i \le n$ and $t_{ij} = 0 \;\forall\, 1 \le i \ne j \le n$. There are two cases with respect to ϕ :

1. If $\phi \le 0$,

$$\begin{aligned} \Gamma(T) &= \bigoplus_{\sigma \in \beth_n} \{T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)}\} \\ &= \max\{T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)}\} \\ &= 0. \end{aligned}$$

2. If $\phi > 0$, then identity permutation will be the maximum.

$$\begin{aligned} \Gamma(T) &= \bigoplus_{\sigma \in \beth_n} \{T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)}\} \\ &= \max\{T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)}\} \\ &= t_{11} + t_{22} + \cdots + t_{nn} \\ &= n\phi. \end{aligned}$$

■

Proposition 5 *Let $T \in M_{n\times n}(R)$ be a circulant arithmetic matrix. Then $\Gamma(T) = n^2$.*

Proof. In a circulant arithmetic matrix,

$$\begin{aligned} \Gamma(T) &= \bigoplus_{\sigma \in \beth_n} \{T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)}\} \\ &= \max\{T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)}\} \\ &= t_{12} + t_{23} + t_{34} \cdots + t_{(n-1)n} + t_{1n} \\ &= n^2. \end{aligned}$$

■

Proposition 6 *Let $T \in M_{n\times n}(R)$ be a circulant matrix. Then $\Gamma(T) = nk$, $\forall\ 0 \le i \le n-1$.*

Proof. Let the elements of the circulant matrix be $c_0, c_1, \cdots, c_{n-1}$ and let $\max\{c_i\} = k,\ \forall\ 0 \le i \le n-1$.

$$\begin{aligned}\Gamma(T) &= \bigoplus_{\sigma\in\beth_n}\{T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)}\}\\ &= \max\{T_{1\sigma(1)} \odot T_{2\sigma(2)} \cdots \odot T_{n\sigma(n)}\}.\end{aligned}$$

Suppose $k = t_{ij}$ for any one of $1 \le i, j \le n$. Then

$$t_{ij} + t_{(i+1)(j+1)} + t_{(i+2)(j+2)} + \cdots + t_{nl} + t_{1(l+1)} + \cdots + t_{rn} = nk.$$

■

4 Ranks of the Tropical Matrices

In this section, we discuss the barvinok rank, kapranov rank and the tropical rank of some special matrices.

4.1 Barvinok Rank

Proposition 7 *Let $T \in M_{n\times n}(R)$ be a row-wise arithmetic matrix. Then $B.R(T) = 1$.*

Proof. Choose $u_1 = T_1$ and $v_1^T = [0\ 1\ 2\ 3 \cdots n-1]$. Then $u_1 \odot v_1^T = T$. It implies that

$$B.R(T) = 1,$$

where $k = 1$. ■

Proposition 8 *Let $T \in M_{n\times n}(R)$ be a column-wise arithmetic matrix. Then $B.R(T) = 1$.*

Proof. Let $u_1 = T_1$ and $v_1^T = [0\ n\ 2n\ 3n \cdots n(n-1)]$. Then

$$T = \bigoplus_{i=1}^{k} u_i \odot {v_i}^T.$$

It implies that

$$B.R(T) = 1,$$

where $k = 1$. ■

Proposition 9 *Let $T \in M_{n\times n}(R)$ be a ϕ-diagonal matrix with $\phi \ne 0$. Then $B.R(T) > 1$.*

Proof. Assume that the matrix $T \in M_{n\times n}(R)$ is a ϕ-diagonal matrix. Suppose that $B.R(T) = 1$. Then

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \odot \begin{bmatrix} b_1 & b_2 & b_3 \cdots & b_n \end{bmatrix} = T,$$

$$a_1 + b_1 = \phi,\ a_2 + b_1 = 0, \cdots,\ a_n + b_1 = 0, \tag{1}$$
$$a_1 + b_2 = 0,\ a_2 + b_2 = \phi, \cdots,\ a_n + b_2 = 0, \tag{2}$$
$$\vdots$$
$$a_1 + b_n = 0,\ a_2 + b_n = 0, \cdots,\ a_n + b_n = \phi.$$

By subtracting the equations (1) and (2), then

$$b_1 - b_2 = \phi \ \text{ and } \ b_1 - b_2 = -\phi.$$

It follows that $\phi = -\phi$, which is a contradiction. Hence $B.R(T) > 1$. ■

Corollary 1 *Let $T \in M_{2\times 2}(R)$ be a ϕ-diagonal matrix with $\phi \neq 0$. Then $B.R(T) = 2$.*

Proposition 10 *Let $T \in M_{n\times n}(R)$ be a $J^l_{n\times n}$ matrix. Then $B.R(T) = 1$.*

Proof. Let

$$u_1 = \begin{bmatrix} l \\ l \\ \vdots \\ l \end{bmatrix}$$

and v_1^T be a zero vector or choose u_1 as a zero vector and

$$v_1 = \begin{bmatrix} l \\ l \\ \vdots \\ l \end{bmatrix},$$

$$T = \bigoplus_{i=1}^{k} u_i \odot {v_i}^T.$$

Then $B.R(T) = 1$, where $k = 1$. ■

Proposition 11 *Let $T \in M_{n\times n}(R)$ be a circulant arithmetic matrix. Then $B.R(T) > 1$.*

Proof. Assume that the matrix $T \in M_{n\times n}(R)$ is a circulant arithmetic matrix. Suppose that $B.R(T) = 1$. Then

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \odot \begin{bmatrix} b_1 & b_2 & b_3 \cdots & b_n \end{bmatrix} = T,$$

$$a_1 + b_1 = 1,\ a_2 + b_1 = n, \cdots,\ a_n + b_1 = 2, \tag{3}$$

$$a_1 + b_2 = 2,\ a_2 + b_2 = 1, \cdots,\ a_n + b_2 = 3, \tag{4}$$

$$\vdots$$

$$a_1 + b_n = n,\ a_2 + b_n = n - 1, \cdots,\ a_n + b_n = 1.$$

By subtracting the equations (3) and (4), then

$$b_1 - b_2 = 1 - n,\ b_1 - b_2 = n - 1.$$

It follows that $1 - n = n - 1$, which is a contradiction. Hence, $B.R(T) > 1$. ■

Corollary 2 *Let $T \in M_{2\times 2}(R)$ be a circulant arithmetic matrix. Then $B.R(T) = 2$.*

4.2 Kapranov Rank

Proposition 12 *Let $T \in M_{n\times n}(R)$ be a coloumn-wise arithmetic matrix. Then $K.R(T) = 1$.*

Proof. Assume that the given matrix is coloumn-wise arithmetic. Choose the classical matrix $(C.M)$ with tropicalization T.

$$C.M = \begin{bmatrix} t^{a+k} & t^{a+2k} & \cdots & t^{a+dk} \\ t^{a+(d+1)k} & t^{a+(d+2)k} & \cdots & t^{a+2dk} \\ \vdots & \vdots & \vdots & \vdots \\ t^{a+((c-1)d+1)k} & t^{a+((c-1)d+2)k} & t^{a+((c-1)d+3)k} & t^{a+cdk} \end{bmatrix}.$$

Tropicalization of the classical matrix is the matrix T and the classical rank of the classical matrix is 1. Hence, $K.R(T) = 1$. ■

Proposition 13 *Let $T \in M_{n\times n}(R)$ be a row-wise arithmetic matrix. Then $K.R(T) = 1$.*

Proof. Choose $C.M = [c^{t_{ij}}]$. Clearly the tropicalization of the classical matrix is the matrix T whose classical rank is 1. Hence, $K.R(T) = 1$. ■

Proposition 14 *Let $T \in M_{n\times n}(R)$ be a ϕ-diagonal matrix. Then $K.R(T) = 2$.*

Proof. Choose the classical matrix $(C.M)$ as,

$$C.M = \begin{bmatrix} t^{\phi} & 1 & t^{\phi}+a_3 & t^{\phi}+a_4 & \cdots & t^{\phi}+a_n \\ 1 & t^{\phi} & 1+a_3t^{\phi} & 1+a_4t^{\phi} & \cdots & 1+a_nt^{\phi} \\ t^{\phi}-a_3 & 1 & t^{\phi} & t^{\phi}-a_3+a_4 & \cdots & t^{\phi}-a_3+a_n \\ t^{\phi}-a_4 & 1 & t^{\phi}-a_4+a_3 & t^{\phi} & \cdots & t^{\phi}-a_4+a_n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t^{\phi}-a_n & 1 & t^{\phi}-a_n+a_3 & t^{\phi}-a_n+a_4 & \cdots & t^{\phi} \end{bmatrix}.$$

The above classical matrix has a classical rank 2. Hence, $K.R(T) = 2$. ■

Proposition 15 *Let $T \in M_{n\times n}(R)$ be any matrix. Then $K.R(T) \leq B.R(T)$.*

Corollary 3 *Let $T \in M_{n\times n}(R)$ be a ϕ-diagonal matrix. Then $B.R(T) \geq 2$.*

Proposition 16 *Let $T \in M_{n\times n}(R)$ be a $J^l_{n\times n}$ matrix. Then $K.R(T) = 1$.*

Proof. Choose the classical matrix as,

$$\begin{bmatrix} t^l & t^l & \cdots & t^l \\ t^l & t^l & \cdots & t^l \\ \vdots & \vdots & \vdots & \vdots \\ t^l & t^l & \cdots & t^l \end{bmatrix}.$$

Then, the classical rank of the classical matrix is 1. Hence, $K.R(J_{n\times n}(l)) = 1$. ■

4.3 Tropical Rank

Proposition 17 *Let $T \in M_{n\times n}(R)$ be a row-wise arithmetic matrix. Then $T.R(T)$=1.*

Proof. All the submatrices of the matrix T are singular except the 1×1 submatrices. Hence, $T.R(T) = 1$. ■

Proposition 18 *Let $T \in M_{n\times n}(R)$ be a column-wise arithmetic matrix. Then $T.R(T) = 1$.*

Proof. The non-singular submatrices of T are 1×1 submatrices. Therefore, $T.R(T) = 1$. ∎

Proposition 19 *Let $T \in M_{n\times n}(R)$ be a ϕ-diagonal matrix with $\phi < 0$. Then $T.R(T) = 2$.*

Proof. Let T be a ϕ-diagonal matrix with $\phi < 0$. Then T has a 2×2 submatrix which is the largest non-singular matrix. Hence, $T.R(T) = 2$. ∎

Proposition 20 *Let $T \in M_{n\times n}(R)$ be a ϕ-diagonal matrix with $\phi > 0$. Then $T.R(T) = n - 1$.*

Proof. Let T be a ϕ-diagonal matrix with $\phi > 0$. Clearly, it has a $n - 1 \times n - 1$ submatrix which is the largest non-singular matrix. Therefore, $T.R(T) = n - 1$. ∎

Proposition 21 *Let $T \in M_{n\times n}(R)$ be a $J^l_{n\times n}$ matrix. Then $T.R(T) = 1$.*

Proof. All the submatrices of $J^l_{n\times n}$ are singular except the 1×1 submatrices. So, $T.R(T) = 1$. ∎

Proposition 22 *Let $T \in M_{n\times n}(R)$ be a circulant matrix. Then $T.R(T) = n - 1$.*

Proof. We know that, the largest non-singular submatrix of an circulant matrix has an order of $n - 1$. Hence, $T.R(T) = n - 1$. ∎

Corollary 4 *Let $T \in M_{n\times n}(R)$ be an arithmetic circulant matrix. Then $T.R(T) = n - 1$.*

5 Advanced Discrepancy Method

We used the advanced discrepancy method to check the existence and uniqueness of the solution of max-linear system. This method is more easier and faster than any other existing methods. Suppose the given max-linear system is $T \odot x = u$,

$$(t_{11} \odot x_1) \oplus (t_{12} \odot x_2) \oplus \cdots \oplus (t_{1n} \odot x_n) = u_1,$$

$$(t_{21} \odot x_1) \oplus (t_{22} \odot x_2) \oplus \cdots \oplus (t_{2n} \odot x_n) = u_2,$$

$$\vdots$$

$$(t_{m1} \odot x_1) \oplus (t_{m2} \odot x_2) \oplus \cdots \oplus (t_{mn} \odot x_n) = u_m.$$

The given system can also be represented as the following,

$$\max\{t_{11} \odot x_1, t_{12} \odot x_2, \cdots, t_{1n} \odot x_n\} = u_1,$$

$$\max\{t_{21} \odot x_1, t_{22} \odot x_2, \cdots, t_{2n} \odot x_n\} = u_2,$$

$$\vdots$$

$$\max\{t_{m1} \odot x_1, t_{m2} \odot x_2, \cdots, t_{mn} \odot x_n\} = u_m.$$

In advanced discrepancy, we define the 'discrepancy(D_{Tu})' as the following,

$$D_{Tu} = \begin{bmatrix} u - T_1 & u - T_2 & \cdots & u - T_n \end{bmatrix}.$$

Now, find the minimum element in each columns and draw a box around that element.

Theorem 1 (Existence theorem) *Let $T \in M_{m\times n}(R)$. The max-linear system $T \odot x = u$ has a solution, if and only if every row of D_{Tu} has at least one boxed element.*

Theorem 2 (Uniqueness theorem) *Let $T \odot x = u$ be a max-linear system. The system has a unique solution, if and only if every row of D_{Tu} has exactly one boxed element.*

Example 1 *Consider the max-linear system,*

$$\begin{bmatrix} 9 & 8 & 12 & 17 & 6 \\ 18 & 20 & 10 & 7 & 10 \\ 14 & 17 & 16 & 12 & 13 \\ 11 & 7 & 9 & 10 & 22 \end{bmatrix} \odot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 14 \\ 17 \end{bmatrix}$$

where,

$$D_{Tu} = \begin{bmatrix} -1 & 0 & \boxed{-4} & \boxed{-9} & 2 \\ \boxed{-6} & \boxed{-8} & 2 & 5 & 2 \\ 0 & -3 & -2 & 2 & 1 \\ 6 & 10 & 8 & 7 & \boxed{-5} \end{bmatrix}.$$

By Theorem 1, there is no boxed element in the third row. Thus, the system has no solution.

Example 2 *Consider the following max-linear system,*

$$\begin{bmatrix} 20 & 25 & 17 & 19 & 16 \\ 19 & 24 & 20 & 20 & 20 \\ 17 & 17 & 22 & 17 & 17 \\ 16 & 15 & 17 & 16 & 17 \end{bmatrix} \odot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 16 \\ 20 \\ 17 \\ 17 \end{bmatrix}$$

where,

$$D_{Tu} = \begin{bmatrix} \boxed{-4} & \boxed{-9} & -1 & \boxed{-3} & \boxed{0} \\ 1 & -4 & 0 & 0 & \boxed{0} \\ 0 & 0 & \boxed{-5} & 0 & \boxed{0} \\ 1 & 2 & 0 & 1 & \boxed{0} \end{bmatrix}.$$

Since every row has a boxed element, by Theorem 1, we can conclude that this system has a solution.

5.1 Finding the Column Rank and Row Rank of the Tropical Matrices Using Advanced Discrepancy Method

In this section, we suggest the method to find the column rank and row rank of any tropical matrix by using the advanced discrepancy method.

Step 1: For any given matrix $T = [(T_1)\ (T_2)\ \cdots\ (T_n)]$, construct a linear system

$$[(T_1)\ (T_2)\ \cdots\ (T_{n-1})] \odot X = T_n$$

where, T_i's are the corresponding columns of T.

Step 2: Now, by using the advanced discrepancy method find the existence of the solution to the system

$$[(T_1)\ (T_2)\ \cdots\ (T_{n-1})] \odot X = T_n.$$

Step 3: Suppose the system $[(T_1)\ (T_2) \cdots\ (T_{n-1})] \odot X = T_n$ has a solution, that implies T_n is dependent on the other columns of T. Now we remove the n^{th} column of T also we conclude that $clrank(T) \leq n-1$.

Now, we reduce the system $[(T_1)\ (T_2)\ \cdots\ (T_{n-1})] \odot X = T_n$ to $[(T_1)\ (T_2)\ \cdots (T_{n-2})] \odot X = T_{n-1}$.

Step 4: Suppose the system has no solution then T_n is an independent column of T.

Now, modify the system $[(T_1)\ (T_2)\ \cdots\ (T_{n-1})] \odot X = T_n$ to $[(T_n)\ (T_1)\ (T_2)\ \cdots\ (T_{n-2})] \odot X = T_{n-1}$.

Step 5: If the system $[(T_n)\ (T_1)\ \cdots\ (T_{n-2})] \odot X = T_{n-1}$ has a solution then follow Step 3.

Step 6: If the system $[(T_n)\ (T_2)\ \cdots\ (T_{n-2})] \odot X = T_{n-1}$ has no solution then repeat Step 4.

Step 7: Repeat these steps until we get the system whose vector is the first column T_1. Now, the left matrix has the independent columns of the matrix T. The number of columns in the final matrix is the column rank of T.

By using a similar method, we can obtain the row rank of T.

Example 3 *Suppose we have to find the column rank of the following matrix T*

$$T = \begin{bmatrix} 9 & 1 & 7 & 8 & 8 \\ 10 & 12 & 12 & 7 & 11 \\ 15 & 17 & 17 & 13 & 16 \\ 9 & 2 & 7 & 8 & 8 \end{bmatrix}.$$

Now, we construct the following system

$$\begin{bmatrix} 9 & 1 & 7 & 8 \\ 10 & 12 & 12 & 7 \\ 15 & 17 & 17 & 13 \\ 9 & 2 & 7 & 8 \end{bmatrix} \odot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 16 \\ 8 \end{bmatrix}.$$

Here

$$T_1 = \begin{bmatrix} 9 \\ 10 \\ 15 \\ 9 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 \\ 12 \\ 17 \\ 7 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 7 \\ 12 \\ 17 \\ 7 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 8 \\ 7 \\ 13 \\ 8 \end{bmatrix} \quad \text{and} \quad T_5 = \begin{bmatrix} 8 \\ 11 \\ 16 \\ 8 \end{bmatrix}.$$

To check the solvability of the above system, we construct the discrepancy matrix D_{Tu}.

$$D_{Tu} = \begin{bmatrix} \boxed{-1} & 7 & 1 & 0 \\ 1 & \boxed{-1} & \boxed{-1} & 4 \\ 1 & \boxed{-1} & \boxed{-1} & 3 \\ \boxed{-1} & 6 & 1 & 0 \end{bmatrix}.$$

By the advanced discrepancy method, we can conclude that the system has a solution and T_5 is dependent on the remaining columns of T. Now we remove T_5 and the system is reduced to

$$\begin{bmatrix} 9 & 1 & 7 \\ 10 & 12 & 12 \\ 15 & 17 & 17 \\ 9 & 2 & 7 \end{bmatrix} \odot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 13 \\ 8 \end{bmatrix}$$

where,

$$D_{Tu} = \begin{bmatrix} -1 & 7 & 1 \\ \boxed{-3} & \boxed{-5} & \boxed{-5} \\ -2 & -4 & -4 \\ -1 & 6 & 1 \end{bmatrix}.$$

By the advanced discrepancy method we can conclude that the system has no solution. T_4 *is linearly independent and it cannot be reduced from the system. We permute the columns and now we get,*

$$\begin{bmatrix} 8 & 9 & 1 \\ 7 & 10 & 12 \\ 13 & 15 & 17 \\ 8 & 9 & 2 \end{bmatrix} \odot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 17 \\ 7 \end{bmatrix}.$$

Now, the discrepancy matrix of the above system is

$$D_{Tu} = \begin{bmatrix} \boxed{-1} & \boxed{-2} & 6 \\ 5 & 2 & \boxed{0} \\ 4 & 2 & \boxed{0} \\ \boxed{-1} & \boxed{-2} & 5 \end{bmatrix}.$$

By the advanced discrepancy method, the above system has a solution. Thus, T_3 *is dependent on the other columns of* T*. Now, we remove* T_3 *and reduce the system*

$$\begin{bmatrix} 8 & 9 \\ 7 & 10 \\ 13 & 15 \\ 8 & 9 \end{bmatrix} \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \\ 17 \\ 2 \end{bmatrix},$$

where, the discrepancy matrix is

$$\begin{bmatrix} \boxed{-7} & \boxed{-8} \\ 5 & 2 \\ 4 & 2 \\ -6 & -7 \end{bmatrix}.$$

By Theorem 1, the system never has a solution and T_2 *is linearly independent. The system is now reduced to*

$$\begin{bmatrix} 1 & 8 \\ 12 & 7 \\ 17 & 13 \\ 2 & 8 \end{bmatrix} \odot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \\ 15 \\ 9 \end{bmatrix},$$

$$D_{Tu} = \begin{bmatrix} 8 & \boxed{1} \\ \boxed{-2} & 3 \\ \boxed{-2} & 2 \\ 7 & \boxed{1} \end{bmatrix}.$$

By Theorem 1, this system has a solution and T_1 *is linearly dependent. Now we remove* T_1 *from the system and finally we obtain the matrix*

$$\begin{bmatrix} 1 & 8 \\ 12 & 7 \\ 17 & 13 \\ 2 & 8 \end{bmatrix}$$

where the columns are linearly independent. Thus, in this example $clrank(T) = 2$.

Remark 2 *To find the row rank of a matrix* B*, the above method can be used to find the* $clrank(B^T)$*, since* $clrank(B^T) = rwrank(B)$*.*

6 Conclusions

In this paper, we presented the determinant, barvinok rank, kapranov rank and tropical rank of some special matrices such as column-wise arithmetic matrix, row-wise arithmetic matrix, ϕ-diagonal matrix, $J^l_{n\times n}$ matrix and circulant matrix. We have discussed the advanced discrepancy method to check whether the max-linear system has a solution or not. Advanced discrepancy method is a rapid method used to solve tropical linear systems. We have suggested a scheme to find the column rank and row rank of any tropical matrices using the advanced discrepancy method. In future, we will try to attack the tropical cryptographic algorithms by using our results. Also, we will use novel method to find the solutions for tropical linear systems. Furthermore, we intend to present novel cryptographic algorithms in the future, which will be based on tropical ranks and determinants, using the results we have obtained.

Acknowledgment. The authors express their gratitude to the editor and reviewers for providing valuable comments and suggestions that significantly contributed to enhancing the presentation of this article. Special thanks are extended to the referees for their careful review of the manuscript and insightful feedback.

References

[1] I. Simon, Limited subsets of a free monoid, In 19th Annual Symposium on Foundations of Computer Science, IEEE Comput. Soc., (1978), 143–150.

[2] J. S Golan, Semirings and their Applications, Springer Science and Business Media, 2013.

[3] L. Aceto, Z. Ésik and A. Ingólfsdóttir, Equational theories of tropical semirings, Theoret. Comput. Sci., 298(2003), 417–469.

[4] D. Krob, Some Automata-Theoretic Aspects of Min-Max-Plus Semirings, Idempotency Analysis, 1998.

[5] D. Maclagan and B. Sturmfels, Introduction to tropical geometry, American Mathematical Society, Providence, RI, 2015.

[6] D. Speyer and B. Sturmfels, Tropical mathematics, Math. Mag., 82(2009), 163–173.

[7] M. Akian, S. Gaubert and A. Guterman, Linear independence over tropical semirings and beyond, Contemp. Math., 495(2009), 1–38.

[8] D. Joó and K. Mincheva, On the dimension of polynomial semirings, J. Algebra, 507(2018), 103–119.

[9] B. Amutha and R. Perumal, Public key exchange protocols based on tropical lower circulant and anti circulant matrices, AIMS math., 8(2023), 17307–17334.

[10] S. Gaubert and M. Plus, Methods and applications of (max,+) linear algebra, In Annual Symposium on Theoretical Aspects of Computer Science, (1997), 261–282.

[11] B. Amutha and R. Perumal, The General Maximal Solution of Some Matrices Over the Tropical Semiring, Asia Pac. j. math., 10(2023), 1–20.

[12] Z. Izhakian, Basics of linear algebra over the extended tropical semiring, Contemp. Math., 495(2009), 173–191.

[13] S. Jamshidvand, S. Ghalandarzadeh, A. Amiraslani and F. Olia, On the maximal solution of a linear system over tropical semirings, Math. Sci., 14(2020), 147–157.

[14] A. Guterman and Y. Shitov, Rank functions of tropical matrices. Linear Algebra Appl., 498(2016), 326–348.

[15] D. Grigoriev, Complexity of solving tropical linear systems, Comput. Complexity, 22(2013), 71–88.

[16] J. Plavka and S. Sergeev, Reachability of eigenspaces for interval circulant matrices in max-algebra, Linear Algebra Appl., 550(2018), 59–86.

[17] P. Guillon, Z. Izhakian, J. Mairesse and G. Merlet, The ultimate rank of tropical matrices, J. Algebra, 437(2015), 222–248.

[18] D. Cartwright and M. Chan, Three notions of tropical rank for symmetric matrices, Discrete Math. Theor. Comput. Sci., (2010), 203–214.

[19] M. Develin, F. Santos and B. Sturmfels, On the rank of a tropical matrix, Combinatorial and computational geometry, 52(2005), 213–242.

[20] J. Yu, Tropicalizing the positive semidefinite cone, Proc. Amer. Math. Soc., 143(2015), 1891–1895.

[21] Y. Shitov, Detecting matrices of combinatorial rank three, J. Combin. Theory Ser. A, 126(2014), 166–176.