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# Some Variants Of Holomorphic Mean Value Theorem*

Purnima Pal†, Sushil Kumar Bhuiya‡

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**Abstract**

In this note, we propose some important results regarding the complex mean value theorem for holomorphic functions. These results are developed by using some real-valued auxiliary functions. The proposed results extend several existing results of the complex mean value theorem.

## 1 Introduction

The mean value theorem is a generalization of Rolle's theorem. These theorems have crucially been studied for analyzing various properties of real-valued functions. Moreover, it has several real-life applications. Consequently, the real-valued mean value theorem has been extended extensively [6]. The basic Rolle's and Mean Value theorem for real-valued function are given as follows.

**Theorem 1 (Rolle's Theorem [1])** *Let a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in $(a, b)$. If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.*

**Theorem 2 (Lagrange's Mean Value Theorem [1])** *Let a function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in $(a, b)$. Then there exists a point $c \in (a, b)$ such that*

$$f(b) - f(a) = f'(c)(b - a).$$

*If $f(a) = f(b)$, then the Lagrange's Mean Value theorem reduces to Rolle's theorem.*

It is well known that Rolle's Theorem is not valid for holomorphic functions of complex variable. For example, the function $f(z) = e^z - 1$ has value 0 at $z = 2k\pi i$ for every $k \in \mathbb{Z}$, but $f'(z) = e^z$ has no zeros in the complex plane $\mathbb{C}$. In 1992, Evard and Jafari [4] went around this difficulty by working with the real and imaginary parts of a holomorphic function. Using the standard notation $z = x+iy$ for $z \in \mathbb{C}$, where $x = \Re(z)$ and $y = \Im(z)$ are the real and imaginary parts of $z$, respectively. They have proposed the following theorems by defining the open line segment joining two complex numbers $a$ and $b$ as $]a, b[\ = \{a+t(b-a) : t \in\ (0, 1)\}$.

**Theorem 3 (Complex Rolle's Theorem [4])** *Let $f$ be a holomorphic function defined on an open convex subset $D_f$ of $\mathbb{C}$. Let $a, b \in D_f$ be such that $f(a) = f(b) = 0$ and $a \neq b$. Then there exists $z_1, z_2 \in\ ]a, b[$ such that*

$$\Re(f'(z_1)) = 0 \quad and \quad \Im(f'(z_2)) = 0.$$

**Theorem 4 (Complex Mean Value Theorem [4])** *Let $f$ be a holomorphic function defined on an open convex subset $D_f$ of $\mathbb{C}$. Let $a$ and $b$ be two distinct points in $D_f$. Then there exists $z_1, z_2 \in\ ]a, b[$ such that*

$$\Re(f'(z_1)) = \Re\left(\frac{f(b) - f(a)}{b - a}\right) \quad and \quad \Im(f'(z_2)) = \Im\left(\frac{f(b) - f(a)}{b - a}\right).$$

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†Department of Mathematics, Murshidabad University, Berhampore, West Bengal 742101, India

‡Department of Mathematics, Krishnath College (Murshidabad University), Berhampore, West Bengal 742101, India (Corresponding Author)

## 2 Some Variants of Mean Value Theorem

Several authors have generalized Lagrange's mean value theorem for real-valued and complex valued functions. In this section, we mention a few of them. In 1958, Flett [5], have proved the following variant of the Mean Value Theorem.

**Theorem 5 (Flett's Theorem [5])** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on $[a,b]$ and $f'(a) = f'(b)$. Then there exists a point $c \in (a,b)$ such that*

$$f(c) - f(a) = f'(c)(c-a). \tag{1}$$

In 1998, Sahoo and Riedel (see [9]) gave a generalization of Flett's theorem, where they remove the boundary condition on the derivative of $f$.

**Theorem 6 (Sahoo-Riedel's Theorem [9])** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on $[a,b]$. Then there exists a point $c \in (a,b)$ such that*

$$f(c) - f(a) = f'(c)(c-a) - \frac{1}{2}\frac{f'(b) - f'(a)}{b-a}(c-a)^2. \tag{2}$$

In 2012, Mohapatra [7] generalized Sahoo-Riedel's theorem using two functions $f$ and $g$.

**Theorem 7 ([7])** *Let $f, g : [a,b] \to \mathbb{R}$ be differentiable functions on $[a,b]$. Then there exists a point $c \in (a,b)$ such that*

$$\begin{aligned} & [g(b) - g(a)]g'(b)[f(c) - f(a) - f'(c)(c-a)] \\ = \; & [f'(b) - f'(a)][g(c) - g(a)]\left[\tfrac{1}{2}(g(c) - g(a)) - g'(c)(c-a)\right]. \end{aligned}$$

In 2020, Lozada-Cruz (see [6]) proved the following variant of Sahoo-Riedel's theorem.

**Theorem 8 ([6])** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on $[a,b]$. Then there exists a point $c \in (a,b)$ such that*

$$f(c) - f(a) = f'(c)(c-a) - \frac{n-1}{n}\frac{f'(b) - f'(a)}{(b-a)^{n-1}}(c-a)^n, \quad n \in \mathbb{N}. \tag{3}$$

The next result is also proved by Lozada-Cruz in 2020 (see [6]), which is a variant of Theorem 7.

**Theorem 9 ([6])** *Let $f, g : [a,b] \to \mathbb{R}$ be differentiable functions on $[a,b]$. Then there exists a point $c \in (a,b)$ such that*

$$\begin{aligned} & [g(b) - g(a)]^{n-1}g'(b)[f(c) - f(a) - f'(c)(c-a)] \\ = \; & [f'(b) - f'(a)][g(c) - g(a)]^{n-1}\left[\tfrac{1}{n}(g(c) - g(a)) - g'(c)(c-a)\right], \quad n \geq 2. \end{aligned}$$

On the other hand, in 1997, Myers [8] proved a slight variant of Flett's theorem.

**Theorem 10 (Myers' Theorem [8])** *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on $[a,b]$ and $f'(a) = f'(b)$. Then there exists a point $c \in (a,b)$ such that*

$$f(b) - f(c) = f'(c)(b-c). \tag{4}$$

In 2012, Cakmak and Tiryaki (see [2]) proved a slight variant of Sahoo-Riedel's theorem. Cakmak and Tiryaki's theorem is also a generalization of Myers' theorem.

**Theorem 11 (Cakmak-Tiryaki's Theorem [2])** *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on $[a, b]$. Then there exists a point $c \in (a, b)$ such that*

$$f(b) - f(c) = f'(c)(b - c) + \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(b - c)^2. \tag{5}$$

In 2012, Mohapatra [7] generalized the theorem of Cakmak-Tiryaki's by considering two functions $f$ and $g$.

**Theorem 12 ([7])** *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions on $[a, b]$. Then there exists a point $c \in (a, b)$ such that*

$$\begin{aligned} & [g(b) - g(a)]g'(a)[f(b) - f(c) - f'(c)(b - c)] \\ = \quad & [f'(b) - f'(a)][g(c) - g(b)]\left[\tfrac{1}{2}(g(b) - g(c)) - g'(c)(b - c)\right]. \end{aligned}$$

In 2020, Lozada-Cruz [6] have proved the following variant of Cakmak-Tiryaki's theorem.

**Theorem 13 ([6])** *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on $[a, b]$. Then there exists a point $c \in (a, b)$ such that*

$$f(b) - f(c) = f'(c)(b - c) + \frac{n - 1}{n}\frac{f'(b) - f'(a)}{(b - a)^{n-1}}(b - c)^n, \quad n \in \mathbb{N}. \tag{6}$$

In 2020, Lozada-Cruz (see [6]) proposed the following result which is a variant of Theorem 12.

**Theorem 14 ([6])** *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable functions on $[a, b]$. Then there exists a point $c \in (a, b)$ such that*

$$\begin{aligned} & [g(a) - g(b)]^{n-1}g'(a)[f(b) - f(c) - f'(c)(b - c)] \\ & \quad = [f'(b) - f'(a)][g(c) - g(b)]^{n-1}\left[\tfrac{1}{n}(g(c) - g(b)) + g'(c)(b - c)\right], \quad n \geq 2. \end{aligned}$$

In general, the Flett's theorem is not valid for complex valued function. Let us consider the function $f(z) = e^z - z$ to illustrate it. The function $f$ is holomorphic and $f'(2k\pi i) = e^{2k\pi i} - 1 = 0 = f'(0)$ for all integer $k$. In spite of that the equation

$$f(z) - f(0) = f'(z)z$$

has no solution on the interval $(0, 2\pi i)$. Thus, the Flett's theorem fails in the complex domain. However, the following two results are the holomorphic version of Sahoo-Riedel's and Cakmak-Tiryaki's theorems, respectively. Moreover, the holomorphic version of Flett's and Mayer's theorem can also be reduced from these two theorems.

In the every subsequent results, $\Re(\eta\bar{\omega})$ is denoted by $\langle \eta, \omega \rangle$ for any $\eta, \omega \in \mathbb{C}$.

**Theorem 15 (Holomorphic Sahoo-Riedel's Theorem [3])** *Let $f$ be a holomorphic function defined on an open convex subset $D_f$ of $\mathbb{C}$. Let $a$ and $b$ be two distinct points in $D_f$. Then there exists $z_1, z_2 \in \; ]a, b[$ such that*

$$\Re(f'(z_1)) = \frac{\langle b - a, f(z_1) - f(a) \rangle}{\langle b - a, z_1 - a \rangle} + \frac{1}{2}\frac{\Re(f'(b) - f'(a))}{b - a}(z_1 - a) \tag{7}$$

*and*

$$\Im(f'(z_2)) = \frac{\langle b - a, -i[f(z_2) - f(a)] \rangle}{\langle b - a, z_2 - a \rangle} + \frac{1}{2}\frac{\Im(f'(b) - f'(a))}{b - a}(z_2 - a). \tag{8}$$

By considering $f'(a) = f'(b)$ in Theorem 15, we get the following complex version of Flett's mean value theorem.

**Corollary 1 (Holomorphic Flett's Theorem [3])** *Let $f$ be a holomorphic function defined on an open convex subset $D_f$ of $\mathbb{C}$. Let $a$ and $b$ be two distinct points in $D_f$ with $f'(a) = f'(b)$. Then there exists $z_1, z_2 \in\ ]a, b[$ such that*

$$\Re(f'(z_1)) = \frac{\langle b-a, f(z_1) - f(a)\rangle}{\langle b-a, z_1 - a\rangle} \quad \text{and} \quad \Im(f'(z_2)) = \frac{\langle b-a, -i[f(z_2) - f(a)]\rangle}{\langle b-a, z_2 - a\rangle}.$$

**Theorem 16 (Holomorphic Cakmak-Tiryaki's Theorem [2])** *Let $f$ be a holomorphic function defined on an open convex subset $D_f$ of $\mathbb{C}$. Let $a$ and $b$ be two distinct points in $D_f$. Then there exists $z_1, z_2 \in\ ]a, b[$ such that*

$$\Re(f'(z_1)) = \frac{\langle b-a, f(b) - f(z_1)\rangle}{\langle b-a, b - z_1\rangle} - \frac{1}{2}\frac{\Re(f'(b) - f'(a))}{b-a}(b - z_1) \tag{9}$$

*and*

$$\Im(f'(z_2)) = \frac{\langle b-a, -i[f(b) - f(z_2)]\rangle}{\langle b-a, b - z_2\rangle} - \frac{1}{2}\frac{\Im(f'(b) - f'(a))}{b-a}(b - z_2). \tag{10}$$

By employing the condition $f'(a) = f'(b)$ in Theorem 16, we obtain the following complex version of Myers' theorem.

**Corollary 2 (Holomorphic Myers' Theorem [2])** *Let $f$ be a holomorphic function defined on an open convex subset $D_f$ of $\mathbb{C}$. Let $a$ and $b$ be two distinct points in $D_f$ such that $f'(a) = f'(b)$. Then there exists $z_1, z_2 \in\ ]a, b[$ such that*

$$\Re(f'(z_1)) = \frac{\langle b-a, f(b) - f(z_1)\rangle}{\langle b-a, b - z_1\rangle}$$

*and*

$$\Im(f'(z_2)) = \frac{\langle b-a, -i[f(b) - f(z_2)]\rangle}{\langle b-a, b - z_2\rangle}.$$

The succeeding two results are the holomorphic extension of the Theorems 7 and 12, respectively.

**Theorem 17 ([7])** *Let $f$ and $g$ be holomorphic functions on a convex open domain $D_f$ of $\mathbb{C}$. Then there exists $z_1, z_2 \in\ ]a, b[$ such that*

$$\begin{aligned} & [\Re(g'(b))]\langle b-a, g(b) - g(a)\rangle \left[\frac{\langle b-a, f(z_1) - f(a)\rangle}{\langle b-a, z_1 - a\rangle} - \Re(f'(z_1))\right] \\ = \ & [\Re(f'(b) - f'(a))]\langle b-a, g(z_1) - g(a)\rangle \times \left[\frac{1}{2}\frac{\langle b-a, g(z_1) - g(a)\rangle}{\langle b-a, z_1 - a\rangle} - \Re(g'(z_1))\right] \end{aligned}$$

*and*

$$\begin{aligned} & [\Im(g'(b))]\langle b-a, -i[g(b) - g(a)]\rangle \left[\frac{\langle b-a, -i[f(z_2) - f(a)]\rangle}{\langle b-a, z_2 - a\rangle} - \Im(f'(z_2))\right] \\ = \ & [\Im(f'(b) - f'(a))]\langle b-a, -i[g(z_2) - g(a)]\rangle \times \left[\frac{1}{2}\frac{\langle b-a, -i[g(z_2) - g(a)]\rangle}{\langle b-a, z_2 - a\rangle} - \Im(g'(z_2))\right]. \end{aligned}$$

**Theorem 18 ([7])** *Let $f$ and $g$ be holomorphic functions on a convex open domain $D_f$ of $\mathbb{C}$. Then there exists $z_1, z_2 \in\ ]a, b[$ such that*

$$\begin{aligned} & [\Re(g'(a))]\langle b-a, g(b) - g(a)\rangle \left[\frac{\langle b-a, f(b) - f(z_1)\rangle}{\langle b-a, b - z_1\rangle} - \Re(f'(z_1))\right] \\ = \ & [\Re(f'(b) - f'(a))]\langle b-a, g(z_1) - g(b)\rangle \times \left[\frac{1}{2}\frac{\langle b-a, g(b) - g(z_1)\rangle}{\langle b-a, b - z_1\rangle} - \Re(g'(z_1))\right] \end{aligned}$$

*and*

$$\begin{aligned}&[\Im(g'(a))]\langle b-a,-i[g(b)-g(a)]\rangle\left[\frac{\langle b-a,-i[f(b)-f(z_2)]\rangle}{\langle b-a,b-z_2\rangle}-\Im(f'(z_2))\right]\\ =\ &[\Im(f'(b)-f'(a))]\big\langle b-a,-i[g(z_2)-g(b)]\big\rangle\times\left[\frac{1}{2}\frac{\langle b-a,-i[g(b)-g(z_2)]\rangle}{\langle b-a,b-z_2\rangle}-\Im(g'(z_2))\right].\end{aligned}$$

## 3 Main Results

In this section, we shall provide some variants of the mean value theorem for the holomorphic function. The subsequent result is a variant of Theorem 15 (Holomorphic Sahoo-Riedel's theorem).

**Theorem 19** *Let $f$ be a holomorphic function defined on an open convex subset $D_f$ of $\mathbb{C}$. Let $a$ and $b$ be two distinct points in $D_f$. Then there exists $z_1, z_2 \in\ ]a,b[$ such that*

$$\Re(f'(z_1)) = \frac{\langle b-a, f(z_1)-f(a)\rangle}{\langle b-a, z_1-a\rangle} + \frac{n-1}{n}\frac{\Re(f'(b)-f'(a))}{(b-a)^{n-1}}(z_1-a)^{n-1} \tag{11}$$

*and*

$$\Im(f'(z_2)) = \frac{\langle b-a, -i[f(z_2)-f(a)]\rangle}{\langle b-a, z_2-a\rangle} + \frac{n-1}{n}\frac{\Im(f'(b)-f'(a))}{(b-a)^{n-1}}(z_2-a)^{n-1}, \quad \forall\ n\in\mathbb{N}. \tag{12}$$

**Proof.** Let us consider $u(z) = \Re(f(z))$ and $v(z) = \Im(f(z))$ for any $z \in D_f$. We now define the auxiliary function $\phi : [0,1] \to \mathbb{R}$ by

$$\phi(t) = \langle b-a, f(a+t(b-a))\rangle. \tag{13}$$

The equation (13) can also be written as the following form.

$$\phi(t) = \Re(b-a)u(a+t(b-a)) + \Im(b-a)v(a+t(b-a)) \quad \forall t\in[0,1].$$

Therefore, using the Cauchy-Riemann equations, we get

$$\phi'(t) = |b-a|^2\Re(f'(z)).$$

By applying Theorem 8 to the auxiliary function $\phi$ on $[0,1]$, we obtain

$$t_1\phi'(t_1) = \phi(t_1)-\phi(0) + \frac{n-1}{n}\frac{\phi'(1)-\phi'(0)}{(1-0)^{n-1}}(t_1-0)^n, \quad n\in\mathbb{N},$$

for some $t_1 \in (0,1)$. Thus,

$$\Re(f'(z_1)) = \frac{\phi(t_1)-\phi(0)}{t_1|b-a|^2} + \frac{n-1}{n}\frac{\phi'(1)-\phi'(0)}{|b-a|^2}t_1^{n-1}, \tag{14}$$

where $z_1 = a+t_1(b-a)$. Moreover, $z_1$ satisfy the condition $t_1|b-a|^2 = \langle b-a, z_1-a\rangle$ for $t_1\in(0,1)$. By using the equation (13) in the equation (14), we obtain the result which is described in the equation (11).

Letting $g = -if$, we have

$$\Re(g'(z)) = \frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y} = \Im(f'(z)).$$

Now, applying the result of the first part to $g$ and rewriting in terms of $f$, we obtain (12). Hence the proof. ∎

**Remark 1** *Note that:*

*(i) If $n = 1$ in Theorem 19, we get the Corollary 1 (Holomorphic Flett's Theorem).*

*(ii) If $n = 2$ in Theorem 19, we get Theorem 15 (Holomorphic Sahoo-Riedel's Theorem).*

The next theorem is a variant of Theorem 16 (Holomorphic Cakmak-Tiryaki's Theorem).

**Theorem 20** *Let $f$ be a holomorphic function defined on an open convex subset $D_f$ of $\mathbb{C}$. Let $a$ and $b$ be two distinct points in $D_f$. Then there exists $z_1, z_2 \in\ ]a, b[$ such that*

$$\Re(f'(z_1)) = \frac{\langle b-a, f(b)-f(z_1)\rangle}{\langle b-a, b-z_1\rangle} - \frac{n-1}{n}\frac{\Re(f'(b)-f'(a))}{(b-a)^{n-1}}(b-z_1)^{n-1} \tag{15}$$

*and*

$$\Im(f'(z_2)) = \frac{\langle b-a, -i[f(b)-f(z_2)]\rangle}{\langle b-a, b-z_2\rangle} - \frac{n-1}{n}\frac{\Im(f'(b)-f'(a))}{(b-a)^{n-1}}(b-z_2)^{n-1}, \quad \forall\ n \in \mathbb{N}. \tag{16}$$

**Proof.** Let us consider $u(z) = \Re(f(z))$ and $v(z) = \Im(f(z))$ for any $z \in D_f$. We now define the auxiliary function $\phi : [0,1] \to \mathbb{R}$ by

$$\phi(t) = \langle b-a, f(a+t(b-a))\rangle, \tag{17}$$

which is

$$\phi(t) = \Re(b-a)u(a+t(b-a)) + \Im(b-a)v(a+t(b-a)),$$

for every $t \in [0,1]$. Therefore, using the Cauchy-Riemann equations, we get

$$\phi'(t) = |b-a|^2\Re(f'(z)).$$

Now applying Theorem 13 to the auxiliary function $\phi$ on $[0,1]$, we obtain

$$(1-t_1)\phi'(t_1) = \phi(1) - \phi(t_1) + \frac{n-1}{n}\frac{\phi'(1)-\phi'(0)}{(1-0)^{n-1}}(1-t_1)^n, \quad n \in \mathbb{N},$$

for some $t_1 \in (0,1)$. Thus

$$\Re(f'(z_1)) = \frac{\phi(1)-\phi(t_1)}{(1-t_1)|b-a|^2} - \frac{n-1}{n}\frac{\phi'(1)-\phi'(0)}{|b-a|^2}(1-t_1)^{n-1}, \tag{18}$$

where $z_1 = a + t_1(b-a)$. Furthermore, $z_1$ satisfy the condition $(1-t_1)|b-a|^2 = \langle b-a, b-z_1\rangle$ for some $t_1 \in (0,1)$. Therefore, employing the equation (17) in the equation (18), we get the required result (15).

Letting $g = -if$, we have

$$\Re(g'(z)) = \frac{\partial v(z)}{\partial x} = -\frac{\partial u(z)}{\partial y} = \Im(f'(z)).$$

Now, employing the result of the first part to $g$ and rewriting in terms of $f$, we obtain (16). This completes the proof. ∎

**Remark 2** *Note that:*

*(i) If $n = 1$ in Theorem 19, we get the Corollary 2 (Holomorphic Myers' Theorem).*

*(ii) If $n = 2$ in Theorem 19, we get Theorem 16 (Holomorphic Cakmak-Tiryaki's Theorem).*

In the next theorem, we shall provide a variant of Theorem 17.

**Theorem 21** *Let $f$ and $g$ be holomorphic on a convex open domain $D_f \subseteq \mathbb{C}$. Then there exists $z_1, z_2 \in\ ]a, b[$ such that $\forall\ n \geq 2$,*

$$\begin{aligned}&[\Re(g'(b))][\langle b-a, g(b)-g(a)\rangle]^{n-1}\left[\frac{\langle b-a, f(z_1)-f(a)\rangle}{\langle b-a, z_1-a\rangle} - \Re(f'(z_1))\right]\\ =\ &[\Re(f'(b)-f'(a))][\langle b-a, g(z_1)-g(a)\rangle]^{n-1} \times \left[\frac{1}{n}\frac{\langle b-a, g(z_1)-g(a)\rangle}{\langle b-a, z_1-a\rangle} - \Re(g'(z_1))\right]\end{aligned} \tag{19}$$

*and*

$$\begin{aligned} & [\Im(g'(b))][\langle b-a, -i[g(b)-g(a)]\rangle]^{n-1}\left[\frac{\langle b-a, -i[f(z_2)-f(a)]\rangle}{\langle b-a, z_2-a\rangle}-\Im(f'(z_2))\right] \\ = \quad & [\Im(f'(b)-f'(a))][\langle b-a, -i[g(z_2)-g(a)]\rangle]^{n-1} \\ & \times\left[\frac{1}{n}\frac{\langle b-a, -i[g(z_2)-g(a)]\rangle}{\langle b-a, z_2-a\rangle}-\Im(g'(z_2))\right]. \end{aligned} \tag{20}$$

**Proof.** Let us consider $u(z) = \Re(f(z))$, $v(z) = \Im(f(z))$, $\alpha(z) = \Re(g(z))$ and $\beta(z) = \Im(g(z))$ for every $z \in D_f$. We now define two auxiliary functions $\phi, \psi : [0,1] \to \mathbb{R}$ by

$$\phi(t) = \langle b-a, f(a+t(b-a))\rangle \quad \text{and} \quad \psi(t) = \langle b-a, g(a+t(b-a))\rangle, \tag{21}$$

which are

$$\phi(t) = \Re(b-a)u(a+t(b-a)) + \Im(b-a)v(a+t(b-a)), \tag{22}$$

$$\psi(t) = \Re(b-a)\alpha(a+t(b-a)) + \Im(b-a)\beta(a+t(b-a)), \tag{23}$$

for every $t \in [0,1]$. Therefore, using the Cauchy-Riemann equations, we get

$$\phi'(t) = |b-a|^2\Re(f'(z)) \quad \text{and} \quad \psi'(t) = |b-a|^2\Re(g'(z)). \tag{24}$$

Applying Theorem 9 to the auxiliary functions $\phi, \psi$ on $[0,1]$, we obtain

$$\begin{aligned} & [\psi(1)-\psi(0)]^{n-1}\psi'(1)[\phi(t_1)-\phi(0)-\phi'(t_1)(t_1-0)] \\ = \quad & [\phi'(1)-\phi'(0)][\psi(t_1)-\psi(0)]^{n-1}\left[\frac{1}{n}(\psi(t_1)-\psi(0))-\psi'(t_1)(t_1-0)\right], \quad n \geq 2, \end{aligned}$$

for some $t_1 \in (0,1)$. Thus,

$$\begin{aligned} & \psi'(1)[\psi(1)-\psi(0)]^{n-1}\left[\frac{\phi(t_1)-\phi(0)}{t_1|b-a|^2}-\Re(f'(z_1))\right] \\ = \quad & [\phi'(1)-\phi'(0)][\psi(t_1)-\psi(0)]^{n-1}\left[\frac{1}{n}\frac{\psi(t_1)-\psi(0)}{t_1|b-a|^2}-\Re(g'(z_1))\right], \end{aligned} \tag{25}$$

where $z_1 = a + t_1(b-a)$. Moreover, $z_1$ satisfy the condition $t_1|b-a|^2 = \langle b-a, z_1-a\rangle$ for $t_1 \in (0,1)$. By employing the equations (21)–(24), we have the following properties of $\phi$ and $\psi$

$$\begin{aligned} \phi(t_1) &= \langle b-a, f(z_1)\rangle, & \psi(t_1) &= \langle b-a, g(z_1)\rangle, \\ \phi(0) &= \langle b-a, f(a)\rangle, & \psi(0) &= \langle b-a, g(a)\rangle, \\ \phi'(0) &= |b-a|^2\Re(f'(a)), & \psi(1) &= \langle b-a, g(b)\rangle, \\ \phi'(1) &= |b-a|^2\Re(f'(b)), & \psi'(1) &= |b-a|^2\Re(g'(b)). \end{aligned}$$

Now, substituting these values in (25) yields (19). Letting $f_1 = -if$ and $g_1 = -ig$, we have

$$\Re(f_1'(z)) = \Im(f'(z)) \quad \text{and} \quad \Re(g_1'(z)) = \Im(g'(z)).$$

Applying the result of the first part to $f_1$ and $g_1$ and rewriting them in terms of $f$ and $g$, we obtain (20). Hence the proof. ■

**Remark 3** *Note that: if $n = 2$ in Theorem 21, we get Theorem 17. Also, putting $g(z) = z$ in Theorem 21 and noting that*

$$\frac{z-a}{b-a} = \frac{\langle b-a, z-a\rangle}{\langle b-a, b-a\rangle}, \quad \forall\, z \in\ ]a,b[,$$

*we get Theorem 19.*

In the following result, we shall give a variant of Theorem 18.

**Theorem 22** *Let $f$ and $g$ be holomorphic on a convex open domain $D_f \subseteq \mathbb{C}$. Then there exists $z_1, z_2 \in\ ]a,b[$ such that $\forall\ n \geq 2$,*

$$\begin{aligned}
& [\Re(g'(a))][\langle b-a, g(a)-g(b)\rangle]^{n-1}\left[\frac{\langle b-a, f(b)-f(z_1)\rangle}{\langle b-a, b-z_1\rangle} - \Re(f'(z_1))\right] \\
= \ & [\Re(f'(b)-f'(a))][\langle b-a, g(z_1)-g(b)\rangle]^{n-1}\left[\frac{1}{n}\frac{\langle b-a, g(z_1)-g(b)\rangle}{\langle b-a, b-z_1\rangle} + \Re(g'(z_1))\right]
\end{aligned} \tag{26}$$

*and*

$$\begin{aligned}
& [\Im(g'(a))][\langle b-a, -i[g(a)-g(b)]\rangle]^{n-1}\left[\frac{\langle b-a, -i[f(b)-f(z_2)]\rangle}{\langle b-a, b-z_2\rangle} - \Im(f'(z_2))\right] \\
= \ & [\Im(f'(b)-f'(a))][\langle b-a, -i[g(z_2)-g(b)]\rangle]^{n-1}\left[\frac{1}{n}\frac{\langle b-a, -i[g(z_2)-g(b)]\rangle}{\langle b-a, b-z_2\rangle} + \Im(g'(z_2))\right].
\end{aligned} \tag{27}$$

**Proof.** Let us consider $u(z) = \Re(f(z)),\ v(z) = \Im(f(z)),\ \alpha(z) = \Re(g(z)),\ \beta(z) = \Im(g(z))$ for every $z \in D_f$. We now define two auxiliary functions $\phi, \psi : [0,1] \to \mathbb{R}$ by

$$\phi(t) = \langle b-a, f(a+t(b-a))\rangle \quad \text{and} \quad \psi(t) = \langle b-a, g(a+t(b-a))\rangle, \tag{28}$$

which are

$$\phi(t) = \Re(b-a)u(a+t(b-a)) + \Im(b-a)v(a+t(b-a))$$

and

$$\psi(t) = \Re(b-a)\alpha(a+t(b-a)) + \Im(b-a)\beta(a+t(b-a)),$$

for every $t \in [0,1]$. Therefore, using the Cauchy-Riemann equations, we get

$$\phi'(t) = |b-a|^2\Re(f'(z)) \quad \text{and} \quad \psi'(t) = |b-a|^2\Re(g'(z)). \tag{29}$$

Applying Theorem 14 to the auxiliary functions $\phi, \psi$ on $[0,1]$, we obtain

$$\begin{aligned}
& [\psi(0)-\psi(1)]^{n-1}\psi'(0)[\phi(1)-\phi(t_1)-\phi'(t_1)(1-t_1)] \\
= \ & [\phi'(1)-\phi'(0)][\psi(t_1)-\psi(1)]^{n-1}\left[\frac{1}{n}(\psi(t_1)-\psi(1)) + \psi'(t_1)(1-t_1)\right], \quad n \geq 2,
\end{aligned}$$

for some $t_1 \in (0,1)$. Thus,

$$\begin{aligned}
& \psi'(0)[\psi(0)-\psi(1)]^{n-1}\left[\frac{\phi(1)-\phi(t_1)}{t_1|b-a|^2} - \Re(f'(z_1))\right] \\
= \ & [\phi'(1)-\phi'(0)][\psi(t_1)-\psi(1)]^{n-1}\left[\frac{1}{n}\frac{\psi(t_1)-\psi(1)}{t_1|b-a|^2} + \Re(g'(z_1))\right],
\end{aligned} \tag{30}$$

where $z_1 = a + t_1(b-a)$. Furthermore, $z_1$ satisfy the condition $(1-t_1)|b-a|^2 = \langle b-a, b-z_1\rangle$ for $t_1 \in (0,1)$. By employing the equations (28)–(29), we have the following properties of $\phi$ and $\psi$

$$\begin{array}{ll}
\phi(t_1) = \langle b-a, f(z_1)\rangle, & \psi(t_1) = \langle b-a, g(z_1)\rangle, \\
\phi(1) = \langle b-a, f(b)\rangle, & \psi(0) = \langle b-a, g(a)\rangle, \\
\phi'(0) = |b-a|^2\Re(f'(a)), & \psi(1) = \langle b-a, g(b)\rangle, \\
\phi'(1) = |b-a|^2\Re(f'(b)), & \psi'(0) = |b-a|^2\Re(g'(a)).
\end{array}$$

Now, substituting these values in (30) yields (26).

Letting $f_1 = -if$ and $g_1 = -ig$, we have

$$\Re(f_1'(z)) = \Im(f'(z)) \quad \text{and} \quad \Re(g_1'(z)) = \Im(g'(z)).$$

Applying the result of the first part to $f_1$ and $g_1$ and rewriting them in terms of $f$ and $g$, we obtain (27). Hence the proof. ∎

**Remark 4** *Note that: if $n = 2$ in Theorem 22, we get Theorem 18. Also, putting $g(z) = z$ in Theorem 22 and noting that*

$$\frac{b-z}{b-a} = \frac{\langle b-a, b-z\rangle}{\langle b-a, b-a\rangle}, \quad \forall\, z \in \ ]a, b[,$$

*we get Theorem 20.*

# 4 Conclusion

We have developed some important results regarding the complex mean value theorem for holomorphic functions in this paper. These results have been proposed by using some real-valued auxiliary functions. The developed results extend several existing results of complex mean value theorem. For future work on this topic, it will be interesting to obtain the same types of results for complex-valued functions of several variables.

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# References

[1] G. R. Bartle and D. R. Sherbert, Introduction to Real Analysis, John Wiley & Sons, Inc. 4th edition, 2000.

[2] D. Cakmak and A. Tiryaki, Mean value theorem for holomorphic functions, 2012(2012), 1–6.

[3] R. M. Davitt, R. C. Powers, T. Riedel and P. K. Sahoo, Flett's mean value theorem for holomorphic functions, Mathematics Magazine, 74(1999), 304–307.

[4] J. C. Evard and F. Jafari, A complex Rolle's theorem, The American Mathematical Monthly, (1992), 858–861.

[5] T. Flett, A mean value problem, Math. Gazette, 42(1958), 38–39.

[6] G. L. Cruz, Some variants of Lagrange's mean value theorem, Selecciones Matemáticas, (2022), 144–150.

[7] A. Mohapatra, Cauchy type generalizations of holomorphic mean value theorems, Electron. J. Diff. Eqn., 184(2012), 1–6.

[8] R. Myers, Some elementary results related to the mean value theorem, The Two-Year College Mathematics Journal, 8(1997), 51–53.

[9] P. Sahoo and T. Riedel, Mean Value Theorems and Functional Equations, NJ: World Scientific, River Edge, 1998.