Applied Mathematics E-Notes, 24(2024), 494-499 © ISSN 1607-2510
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On The Spread Of Polynomials And Its Derivatives*

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Received 20 October 2023

Abstract

In this work, we give some lower and upper bounds for the spread of two polynomials f and g. Also, we present certain inequalities for the spread of their derivatives $f^{(k)}$ and $g^{(k)}$, where $k \in \mathbb{N}$.

1 Introduction

Let A be a complex $n \times n$ matrix, and let $\lambda_1, \cdots, \lambda_n$ be its eigenvalues. The spread of the matrix A is defined by

$$sp(A) = \max_{i,j} |\lambda_i - \lambda_j|. \tag{1}$$

This concept was introduced for the first time by L. Mirsky [7]. Numerous contributions related to the spread of a matrix were made by various people, including Brauer and Mewborn [1], E. Deutsch [2], R. Drnovšek [3], and A. Frakis [4], [5].

In analogy with (1) for a complex polynomial

$$f(z) = z^n - a_1 z^{n-1} + a_2 z^{n-2} - \cdots + (-1)^n a_n$$

with zeros $z_1, \cdots, z_n$, *the spread* of the polynomial f, denoted by $sp(f)$, or in some literature by *the span* of f see [6] and [8], is defined by $sp(f) = \max_{i,j} |z_i - z_j|$. In [1] Brauer and Mewborn proved that if all the zeros of f are real, then

$$sp(f) \leq M(f),$$

where

$$M(f) = \left(2\left(1 - \frac{1}{n}\right)a_1^2 - 4a_2\right)^{1/2}.$$

Let A and B be two complex $n \times n$ matrices with eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\omega_1, \ldots, \omega_n$, respectively. Let $\mathcal{D}_\mathcal{A}$ and $\mathcal{D}_\mathcal{B}$ denote the smallest discs containing all the eigenvalues of A and B, respectively. Let

$$f(z) = z^n - a_1 z^{n-1} + a_2 z^{n-2} - \cdots + (-1)^n a_n$$

and

$$g(z) = z^n - b_1 z^{n-1} + b_2 z^{n-2} - \cdots + (-1)^n b_n$$

be the characteristic polynomials of A and B, respectively. We assume throughout this paper that $n \geq 2$, $\mathcal{D}_\mathcal{A} \not\subset \mathcal{D}_\mathcal{B}$, $\mathcal{D}_\mathcal{B} \not\subset \mathcal{D}_\mathcal{A}$, $\mathcal{D}_\mathcal{A} \cap \mathcal{D}_\mathcal{B} = \emptyset$ and all the eigenvalues of A and B are real numbers. It is well known that

$$a_1 = \sum_{i=1}^{n} \lambda_i \quad and \quad a_2 = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j.$$

Let $m_A = \frac{tr(A)}{n}$ with $tr(A)$ denotes *the trace* of the matrix A.

The spread of f and g is defined as

$$sp(f, g) = \max_{i,j} |\lambda_i - \omega_j|.$$

*Mathematics Subject Classifications: 15A18, 15A15, 12D10.

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2 Bounds for the Spread of Polynomials

Theorem 1 *Let f and g be as described above. Then*

$$sp(f,g) \leq \left(\frac{3n}{2(n+1)} \big(M^2(f) + M^2(g) \big) + \frac{3}{n+1}(a_1 - b_1)^2 \right)^{1/2}.$$

For the proof of this theorem we need the following lemma.

Lemma 1 *Let $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$ and $\omega_n \leq \omega_{n-1} \leq \cdots \leq \omega_1$ be real numbers. Then*

$$\frac{n+1}{3} \max_{i,j} (\lambda_i - \omega_j)^2 \leq \sum_{i,j=1}^{n} (\lambda_i - \omega_j)^2. \tag{2}$$

Proof. We may assume, without loss of generality, that $\max_{i,j} |\lambda_i - \omega_j| = (\lambda_1 - \omega_n)$. It is well known that

$$\frac{(a+b+c)^2}{3} \leq a^2 + b^2 + c^2 \quad \text{for } a,b,c \geq 0. \tag{3}$$

We have

$$\begin{aligned}
\sum_{i,j=1}^{n} (\lambda_i - \omega_j)^2 &\geq (\lambda_1 - \omega_n)^2 + \sum_{i=2}^{n-1} \big((\lambda_1 - \lambda_i)^2 + (\lambda_i - \omega_i)^2 + (\omega_i - \omega_n)^2 \big) \\
&\geq (\lambda_1 - \omega_n)^2 + \frac{n-2}{3} (\lambda_1 - \omega_n)^2 \quad \text{(by the inequality (3))} \\
&= \frac{n+1}{3} (\lambda_1 - \omega_n)^2.
\end{aligned}$$

■

Now we present the proof of Theorem 1.

Proof of Theorem 1. From the previous lemma we deduce that

$$\frac{n+1}{3} sp^2(f,g) \leq \sum_{i,j=1}^{n} |\lambda_i - \omega_j|^2.$$

On the other hand, we have

$$\sum_{i,j=1}^{n} (\lambda_i - \omega_j)^2 = n \sum_{i=1}^{n} \lambda_i^2 + n \sum_{j=1}^{n} \omega_j^2 - 2 \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \omega_j.$$

Using the fact that $\sum_{i=1}^{n} \lambda_i^2 = a_1^2 - 2a_2$, it follows that

$$\begin{aligned}
\sum_{i,j=1}^{n} (\lambda_i - \omega_j)^2 &= n a_1^2 - 2n a_2 + n b_1^2 - 2n b_2 - 2 a_1 b_1 \\
&= \frac{n}{2} \big(M^2(f) + M^2(g) \big) + (a_1 - b_1)^2.
\end{aligned}$$

Applying the inequality (2) gives the required result. ■

Lemma 2 ([4]) *If $z_1, z_2, \cdots, z_n$ are complex numbers satisfying the condition $\sum_{i=1}^{n} z_i = 0$, then*

$$|z_i|^2 \leq \frac{n-1}{n} \sum_{j=1}^{n} |z_j|^2, \qquad i = 1, 2, \cdots, n. \tag{4}$$

Proof. See [4]. ■

Theorem 2 *Let f and g be the two polynomials as described above. Then*

$$sp(f,g) \leq \left(\frac{n-1}{2n}\right)^{1/2} \left(\sqrt{M(f)} + \sqrt{M(g)}\right) + \frac{1}{n}|a_1 - b_1|.$$

Proof. Let $\lambda_1, \cdots, \lambda_n$ and $\omega_1, \cdots, \omega_n$ be the zeros of f and g, respectively. Taking $z_i = (\lambda_i - m_A)$ in the inequality (4), it follows that

$$|\lambda_i - m_A| \leq \sqrt{\frac{n-1}{n}\sum_{j=1}^{n}|\lambda_j - m_A|^2}.$$

Furthermore,

$$\begin{aligned}\sum_{i=1}^{n}(\lambda_i - m_A)^2 &= \sum_{i=1}^{n}\left(\lambda_i^2 - 2m_A\lambda_i + (m_A)^2\right)\\ &= \sum_{i=1}^{n}\lambda_i^2 - \frac{(trA)^2}{n}\\ &= \left(1 - \frac{1}{n}\right)a_1^2 - 2a_2 = \frac{1}{2}M(f).\end{aligned}$$

Hence $|\lambda_i - m_A| \leq \sqrt{\frac{n-1}{2n}M(f)}$. On another hand, we have

$$\begin{aligned}|\lambda_i - \omega_j| &\leq |\lambda_i - m_A| + |\omega_j - m_B| + |m_A - m_B|\\ &\leq \sqrt{\frac{n-1}{2n}M(f)} + \sqrt{\frac{n-1}{2n}M(g)} + \frac{1}{n}|a_1 - b_1|.\end{aligned}$$

Then the desired result follows directly. ■

Theorem 3 *Let f and g be the two polynomials as described above. Then*

$$\sqrt{\frac{1}{n}}\left\{\frac{1}{2}M^2(f) + \frac{1}{2}M^2(g) + \frac{1}{n}(a_1 - b_1)^2\right\}^{1/2} \leq sp(f,g).$$

Proof. Let $\lambda_1, \cdots, \lambda_n$ and $\omega_1, \cdots, \omega_n$ be the zeros of f and g, respectively. For any indices i and j, we have

$$(\lambda_i - \omega_j) \leq sp(f,g).$$

Furthermore,

$$\sum_{i,j=1}^{n}(\lambda_i - \omega_j)^2 \leq n^2 sp^2(f,g).$$

Thus

$$\frac{1}{n}\left(\sum_{i=1}^{n}\lambda_i^2 + \sum_{j=1}^{n}\omega_j^2 - \frac{2\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{n}\omega_j}{n}\right) \leq sp^2(f,g).$$

Hence

$$\sqrt{\frac{1}{n}}\left(\frac{1}{2}M^2(f) + \frac{1}{2}M^2(g) + \frac{1}{n}(a_1 - b_1)^2\right)^{1/2} \leq sp(f,g).$$

■

Corollary 1 *Let f and g be the two polynomials as described above with real zeros $\lambda_1, \cdots, \lambda_n$ and $\omega_1, \cdots, \omega_n$, respectively. If $\lambda_i = \omega_i$ for $i = \{1, \cdots, p\}$, then*

$$\left(\frac{n}{2(n^2-p)} \left(M^2(f) + M^2(g) \right) + \frac{1}{n^2-p} (a_1 - b_1)^2 \right)^{1/2} \leq sp(f,g).$$

Proof. The proof is the same as the proof of the previous theorem using the relation

$$\sum_{i,j=1}^{n} (\lambda_i - \omega_j)^2 \leq (n^2-p) sp^2(f,g).$$

■

3 Upper Bounds for the Spread of Derivatives of Polynomials

Theorem 4 *Let f and g be the two polynomials as described above with real zeros $\lambda_1, \cdots, \lambda_n \geq 0$ and $\omega_1, \cdots, \omega_n \geq 0$, respectively, satisfying the conditions*

$$\frac{\sum_{i=1}^n \lambda_i}{n} \leq 1, \ \frac{\sum_{i=1}^n \omega_i}{n} \leq 1$$

and let $k \leq n-2$. Then

$$sp\big(f^{(k)}, g^{(k)}\big) \leq n-k.$$

Proof. Denote by $x_1 \leq x_2 \leq \cdots \leq x_{n-k}$ and $y_1 \leq y_2 \leq \cdots \leq y_{n-k}$ the real zeros of $f^{(k)}(z)$ and $g^{(k)}(z)$, respectively. We have

$$a_1 = \sum_{i=1}^{n} \lambda_i, \quad b_1 = \sum_{i=1}^{n} \omega_i.$$

Since $f^{(k)}(z) = z^{n-k} - \frac{(n-k)}{n} a_1 z^{n-k-1} + \cdots$, and $g^{(k)}(z) = z^{n-k} - \frac{(n-k)}{n} b_1 z^{n-k-1} + \cdots$, it follows that

$$\sum_{i=1}^{n-k} x_i = \frac{n-k}{n} a_1 \ \text{ and } \ \sum_{i=1}^{n-k} y_i = \frac{n-k}{n} b_1.$$

Observing that $x_i \geq 0, \ y_i \geq 0$ for all i. Then

$$\max_i x_i \leq \frac{n-k}{n} a_1 \leq n-k \ \text{ and } \ \max_i y_i \leq \frac{n-k}{n} b_1 \leq n-k.$$

We have

$$sp(f^{(k)}, g^{(k)}) = \max_i x_i - \min_j y_j \leq \max_i x_i$$

or

$$sp(f^{(k)}, g^{(k)}) = \max_j y_j - \min_i x_i \leq \max_j y_j.$$

Hence the desired result is obtained. ■

Theorem 5 *Let f and g be the two polynomials as described above with real zeros $\lambda_1, \cdots, \lambda_n \geq 0$ and $\omega_1, \cdots, \omega_n \geq 0$, respectively. Assume that $\left(\frac{a_1}{n}, \frac{b_1}{n}\right) \in D(0,1)$, where $D(0,1)$ is the closed unit disc about the origin and let $k \leq n-2$. Then*

$$sp\big(f^{(k)}, g^{(k)}\big) \leq \left(\frac{3(n-k)^3}{n-k+1} \right)^{1/2}.$$

Proof. Denote by $x_1 \leq x_2 \leq \cdots \leq x_{n-k}$ and $y_1 \leq y_2 \leq \cdots \leq y_{n-k}$ the real zeros of $f^{(k)}(z)$ and $g^{(k)}(z)$, respectively. We have

$$a_1 = \sum_{i=1}^{n} \lambda_i, \quad b_1 = \sum_{i=1}^{n} \omega_i, \quad a_2 = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j, \quad b_2 = \sum_{1 \leq i < j \leq n} \omega_i \omega_j,$$

and

$$\sum_{i=1}^{n-k} x_i = \frac{n-k}{n} a_1, \quad \sum_{i=1}^{n-k} y_i = \frac{n-k}{n} b_1.$$

Also

$$\sum_{1 \leq i < j \leq n-k} x_i x_j = \frac{(n-k)(n-k-1)}{n(n-1)} a_2, \qquad \sum_{1 \leq i < j \leq n-k} y_i y_j = \frac{(n-k)(n-k-1)}{n(n-1)} b_2.$$

We have

$$\begin{aligned}
\triangle &= \sum_{i,j=1}^{n-k} (x_i - y_j)^2 \\
&= (n-k) \sum_{i=1}^{n-k} x_i^2 + (n-k) \sum_{j=1}^{n-k} y_j^2 - 2 \sum_{i=1}^{n-k} x_i \sum_{j=1}^{n-k} y_j \\
&= (n-k) \left[\left(\sum_{i=1}^{n-k} x_i \right)^2 - 2 \sum_{1 \leq i < j \leq n-k} x_i x_j \right] \\
&+ (n-k) \left[\left(\sum_{i=1}^{n-k} y_j \right)^2 - 2 \sum_{1 \leq i < j \leq n-k} y_i y_j \right] - 2 \frac{(n-k)^2}{n^2} a_1 b_1 \\
&= (n-k) \left[\frac{(n-k)^2}{n^2} a_1^2 - 2 \frac{(n-k)(n-k-1)}{n(n-1)} a_2 \right] \\
&+ (n-k) \left[\frac{(n-k)^2}{n^2} b_1^2 - 2 \frac{(n-k)(n-k-1)}{n(n-1)} b_2 \right] - 2 \frac{(n-k)^2}{n^2} a_1 b_1 \\
&= \frac{(n-k)^3}{n^2} (a_1^2 + b_1^2) - \frac{2(n-k)^2(n-k-1)}{n(n-1)} (a_2 + b_2) - \frac{2(n-k)^2}{n^2} a_1 b_1 \\
&\leq (n-k)^3 \left(\frac{a_1^2}{n^2} + \frac{b_1^2}{n^2} \right) \\
&\leq (n-k)^3.
\end{aligned}$$

Using the inequality (2), it follows that

$$\triangle = \sum_{i,j=1}^{n-k} (x_i - y_j)^2 \geq \frac{(n-k+1)}{3} sp^2(f^{(k)}, g^{(k)}).$$

Hence the assertion follows immediately. ■

Acknowledgment. The author is grateful to the referee for his/her useful comments and suggestions.

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