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Oscillation Properties Of The Solutions Of A Second Order Differential Equation With Piecewise Constant Mixed Arguments*

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Abstract

We demonstrate the existence and uniqueness of solutions to a second order differential equation with piecewise constant mixed arguments. Moreover, we explore the properties of the equation's solutions, including oscillation and nonoscillation. Finally, we give an example that illustrates our oscillatory result.

1 Introduction

Ordinary differential equations with arguments having constant intervals have been investigated in recent years. Cooke and Wiener (1984) [1] were the first to investigate these equations. Some papers have also taken into account qualitative works such as oscillation, periodicity, and convergence of solutions of ordinary differential equations with piecewise constant arguments (EPCAs) (Aftabizadeh and Wiener 1988 [2]; Aftabizadeh et al. 1987 [3]; Akhmet 2008 [4]; Busenberg and Cooke 1982 [5]; Győri 1991 [6]; Győri and Ladas 1989 [7]; Huang 1990 [8]; Liang and Wang 2009 [9]; Muroya 2008 [10]; Pinto 2009 [11]; Yuan 2002 [12]). However, numerous kinds of research have been done on the qualitative characteristics of second order differential equations and second order differential equations with piecewise constant arguments.

Let us now take a brief look at the literature on second order differential equations with piecewise constant arguments.

Dai and Singh (1994) [13] published the solutions to different second differential equations describing motions of a spring-mass system perturbed by a piecewise constant force in the form of $f([t])$ or $f(x([t]))$. Then, in the continuous function, substitute the variable t. Then, Dai and Singh (1997) [14] substituted the continuous function $f(t)$ variable t in equation

$$mx'' + cx' + kx = f(t), \quad t > 0 \tag{1}$$

with the piecewise constant variable $\dfrac{[Nt]}{N}$, resulting in a piecewise constant system with the governing equation

$$mx'' + cx' + kx = g(\frac{[Nt]}{N}), \quad \frac{[Nt]}{N} \le t \le \frac{[Nt]+1}{N}.$$

They also demonstrated that, if the parameter N is large enough, this substitution is a good approximation to the supplied function $f(t)$ with argument t. Eq. (1) is a form of the equation that describes vibratory motion associated with simple and complicated dynamic systems in engineering applications. It can be simplified to the motion of a one-degree-of-freedom spring mass system where the time-dependent function $f(t)$ is known and expresses the continuous external force acting on the spring mass system. Later, in 1998 [15], they employed the piecewise constant technique to get approximate and numerical solutions to the driven Froude pendulum, which they had previously examined [8, 9]. They established this by using the piecewise constant approach. They demonstrated that approximate solutions for the nonlinear system can be found using the piecewise constant technique and that a numerical simulation of the oscillations of the driven

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Froude pendulum can be done on a computer easily. On the basis of numerical solutions, they investigated the pendulum's oscillating behavior.

Wiener and Lakshmikantham (1999) [16] studied the equation

$$x''(t) - a^2x(t) = bx([t-1]), \quad b \neq 0.$$

They discovered several sufficient conditions for oscillatory, nonoscillatory, and periodic solutions with period 3 in addition to finding explicit solutions of the equation. Yuan (2003) [17] investigated periodic, quasi periodic, and almost periodic solutions of the equation

$$x''(t) + p(t)x(t) = qx([t]) + f(t),$$

where $p : \mathbb{R} \longrightarrow \mathbb{R}, : t \longrightarrow p(t)$, is 1 periodic and continuous, q is a real constant which is different from zero and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Nieto and López (2005) [18] conducted a similar investigation into the existence of solutions for

$$\begin{aligned} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) &= \sigma(t), \quad t \in J = [0, T], \\ x(0) &= x(T), \\ x'(0) &= x'(T) + \lambda, \end{aligned}$$

for second-order functional differential equations of the form

$$\begin{aligned} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) &= 0, \quad t \in \mathbb{R}, \\ x(0) &= x(T), \\ x'(0^-) &= x'(T^+), \\ x'(s^+) &= x'(s^-) + 1. \end{aligned}$$

They applied Green's function to define the unique solution for a second-order functional differential equation with periodic boundary conditions and functional dependence expressed by a piecewise constant function under certain conditions. Moreover, the same authors (2012) [19] investigate the existence and uniqueness of solution to this problem, providing optimal conditions and calculating the exact expression of solutions.

$$\begin{aligned} x''(t) + ax'(t) + bx(t) + cx'([t]) + dx([t]) &= \sigma(t), \quad t \in J = [0, T], \\ x(T) &= x(\mu) + \varphi, \\ x'(T) &= x'(\mu) + \varphi. \end{aligned}$$

Buedo-Fernández et al. (2022) [20] study a class of nonlinear second-order functional differential equations with piecewise constant arguments

$$\begin{aligned} &x''(t) = g(t, x(t), x([t]), x'([t])), \quad t \in J = [0, T], \\ &x(0) = x(T), \\ &x'(0) = x'(T) + \lambda \end{aligned}$$

with applications to a thermostat that is controlled by the introduction of functional terms in the temperature and the speed of change of the temperature at some fixed instants

Bereketoğlu et al. (2011) [21] considered a class of second order differential equations with piecewise constant mixed arguments

$$x''(t) - a^2x(t) = bx([t-1]) + cx([t]) + dx([t+1]) = 0. \tag{2}$$

They show that the zero solution of the equation is a global attractor. They also look at some of the properties of the solutions of the equation, such as oscillation, nonoscillation, and periodicity. Oztepe (2017)

[22] obtained a second order mixed type impulsive differential equation with piecewise constant arguments by adding the impulse condition

$$\Delta x'(n) = \alpha x'(n)$$

to Eq. (2) and investigated the solutions' oscillation, non-oscillation, and periodicity.

In this paper, motivated by the papers [18] and [21], we consider the following second order differential equation with piecewise constant arguments

$$\begin{aligned} &x''(t) + ax'(t) + bx(t) + c_1x'([t-1]) + c_2x'([t]) + c_3x'([t+1]) \\ &+d_1x([t-1]) + d_2x([t]) + d_3x([t+1]) = 0, \end{aligned} \tag{3}$$

where a, b, c_1, c_2, c_3, d_1, d_2, $d_3 \in \mathbb{R}$ and $[.]$ denotes the greatest integer function.

Eq. (3) arises as the piecewise form of the following model of an inverted pendulum stabilized by time-delayed proportional-derivative feedback in [23] if $a = -\omega_n^2$, $c_1 = -k_d$, $d_1 = -k_p$, $b = c_2 = c_3 = d_2 = d_3 = 0$,

$$\ddot{\theta}(t) - \omega_n^2\theta(t) = f(t),$$

where θ is the vertical displacement angle, ω_n is the natural angular frequency of the system hung upside down and $f(t)$ is the control torque

$$f(t) = -k_p\theta(t-\tau) - k_d\dot{\theta}(t-\tau),$$

where k_p and k_d are the proportional and the derivative control gains and τ is the reaction time delay.

2 Existence and Uniqueness of Solutions

Definition 1 *A function $x(t)$ defined on $[0,\infty)$ is said to be a solution of second order differential equation with piecewise constant arguments (3) if it satisfies the following conditions:*

(i) $x(t)$ is continuously differentiable on $[0,\infty)$,

(ii) $x'(t)$ exists and continuous on $[0,\infty)$,

(iii) $x''(t)$ exists on $[0,\infty)$ with the possible exception of the points $[t] \in [0,\infty)$, where one-sided derivatives exist,

(iv) $x(t)$ satisfies Eq. (3) on each interval $[n, n+1)$ with integral endpoints.

Theorem 1 *The solution of Eq. (3) with the initial conditions*

$$x(-1) = p_{-1},\ x'(-1) = q_{-1},\ x(0) = p_0,\ x'(0) = q_0,\ \ p_{-1}, q_{-1}, p_0, q_0 \in \mathbb{R} \tag{4}$$

is

$$\begin{aligned} x(t) = &\, p_{n-1}f_1(t-n) + q_{n-1}f_2(t-n) + p_nf_3(t-n) \\ &+ q_nf_4(t-n) + p_{n+1}f_5(t-n) + q_{n+1}f_6(t-n), \quad t \in [n, n+1), \end{aligned} \tag{5}$$

where

$$f_1(s) = \begin{cases} \frac{d_1}{a}(1 - s - e^{-as}) & \text{if} \quad b = 0,\ a \neq 0, \\ \frac{d_1}{2}s^2 & \text{if} \quad b = 0,\ a = 0, \\ -\frac{d_1}{b}[1 - (1 + \frac{a}{2}s)e^{-\frac{a}{2}s}] & \text{if} \quad b \neq 0,\ a^2 = 4b, \\ \frac{d_1}{b}[\frac{1}{\alpha-\beta}(\alpha e^{\beta s} - \beta e^{\alpha s}) - 1] & \text{if} \quad b \neq 0,\ a^2 > 4b, \\ \frac{d_1}{b}[-1 + e^{-\frac{a}{2}s}(cosKs + \frac{a}{2K}sinKs)] & \text{if} \quad b \neq 0,\ a^2 < 4b, \end{cases}$$

$$f_2(s)=\begin{cases}\frac{c_1}{a}(1-s-e^{-as}) & if \quad b=0,\ a\neq 0,\\ \frac{c_1}{2}s^2 & if \quad b=0,\ a=0,\\ -\frac{c_1}{b}[1-(1+\frac{a}{2}s)e^{-\frac{a}{2}s}] & if \quad b\neq 0,\ a^2=4b,\\ \frac{c_1}{b}[\frac{1}{\alpha-\beta}(\alpha e^{\beta s}-\beta e^{\alpha s})-1] & if \quad b\neq 0,\ a^2>4b,\\ \frac{c_1}{b}[-1+e^{-\frac{a}{2}s}(cosKs+\frac{a}{2K}sinKs)] & if \quad b\neq 0,\ a^2<4b,\end{cases}$$

$$f_3(s)=\begin{cases}1+\frac{d_2}{a}(1-s-e^{-as}) & if \quad b=0,\ a\neq 0,\\ 1+\frac{d_2}{2}s^2 & if \quad b=0,\ a=0,\\ (1+\frac{d_2}{b})(1+\frac{a}{2}s)e^{-\frac{a}{2}s}-\frac{d_2}{b} & if \quad b\neq 0,\ a^2=4b,\\ (1+\frac{d_2}{b})\frac{1}{\alpha-\beta}(\alpha e^{\beta s}-\beta e^{\alpha s})-\frac{d_2}{b} & if \quad b\neq 0,\ a^2>4b,\\ (1+\frac{d_2}{b})e^{-\frac{a}{2}s}(cosKs+\frac{a}{2K}sinKs)-\frac{d_2}{b} & if \quad b\neq 0,\ a^2<4b,\end{cases}$$

$$f_4(x)\begin{cases}\frac{1}{a}\left(1-e^{-as}+c_2\left(1-s-e^{-as}\right)\right) & if\, b=0,\ a\neq 0,\\ \frac{c_2}{2}s^2-s & if\, b=0,\ a=0,\\ \left[\frac{c_2}{b}\left(1+\frac{a}{2}s\right)+s\right]e^{\frac{-a}{2}s}-\frac{c_2}{b} & if\, b\neq 0,\ a^2=4b,\\ \frac{1}{\alpha-\beta}\left[\left(1-\frac{\beta d_2}{b}\right)e^{\alpha s}+\left(\frac{\alpha c_2}{b}-1\right)e^{\beta s}-\frac{c_2}{b}\right] & if\, b\neq 0,\ a^2>4b,\\ e^{\frac{-a}{2}s}\left(\frac{c_2}{b}\cos Ks+\frac{1+\frac{ac_2}{2b}}{K}\sin Ks\right)-\frac{c_2}{b} & if\, b\neq 0,\ a^2<4b,\end{cases}$$

$$f_5(s)=\begin{cases}\frac{d_3}{a}(1-s-e^{-as}) & if \quad b=0,\ a\neq 0,\\ \frac{d_3}{2}s^2 & if \quad b=0,\ a=0,\\ -\frac{d_3}{b}[1-(1+\frac{a}{2}s)e^{-\frac{a}{2}s}] & if \quad b\neq 0,\ a^2=4b,\\ \frac{d_3}{b}[\frac{1}{\alpha-\beta}(\alpha e^{\beta s}-\beta e^{\alpha s})-1] & if \quad b\neq 0,\ a^2>4b,\\ \frac{d_3}{b}[-1+e^{-\frac{a}{2}s}(cosKs+\frac{a}{2K}sinKs)] & if \quad b\neq 0,\ a^2<4b,\end{cases}$$

$$f_6(s)=\begin{cases}\frac{c_3}{a}(1-s-e^{-as}) & if \quad b=0,\ a\neq 0,\\ \frac{c_3}{2}s^2 & if \quad b=0,\ a=0,\\ -\frac{c_3}{b}[1-(1+\frac{a}{2}s)e^{-\frac{a}{2}s}] & if \quad b\neq 0,\ a^2=4b,\\ \frac{c_3}{b}[\frac{1}{\alpha-\beta}(\alpha e^{\beta s}-\beta e^{\alpha s})-1] & if \quad b\neq 0,\ a^2>4b,\\ \frac{c_3}{b}[-1+e^{-\frac{a}{2}s}(cosKs+\frac{a}{2K}sinKs)] & if \quad b\neq 0,\ a^2<4b\end{cases}$$

with

$$K=\sqrt{b-\frac{a^2}{4}},\quad \alpha=-\frac{a}{2}+\sqrt{(\frac{a}{2})^2-b}\quad if \quad \beta=-\frac{a}{2}-\sqrt{(\frac{a}{2})^2-b}$$

and (p_n,q_n) *is the solution of the following difference system*

$$\begin{aligned}p_{n+1}&=A_1p_{n-1}+B_1q_{n-1}+C_1p_n+D_1q_n,\\ q_{n+1}&=A_2p_{n-1}+B_2q_{n-1}+C_2p_n+D_2q_n.\end{aligned}\tag{6}$$

The characteristic equation of Eq.(2) is the following

$$\begin{aligned}&\lambda^4+(-C_1-D_2)\lambda^3+(C_1D_2-C_2D_1-A_1-B_2)\lambda^2\\ &\quad+(A_1D_2-A_2D_1+B_2C_1-B_1C_2)\lambda+A_1B_2-A_2B_1=0,\end{aligned}\tag{7}$$

here

$$
\begin{aligned}
A_1 &= \frac{(1-K_6')K_1 + K_1'K_6}{(1-K_5)(1-K_6') - K_5'K_6},\\
B_1 &= \frac{(1-K_6')K_2 + K_2'K_6}{(1-K_5)(1-K_6') - K_5'K_6},\\
C_1 &= \frac{(1-K_6')K_3 + K_3'K_6}{(1-K_5)(1-K_6') - K_5'K_6},\\
D_1 &= \frac{(1-K_6')K_4 + K_4'K_6}{(1-K_5)(1-K_6') - K_5'K_6},\\
A_2 &= \frac{(1-K_5)K_1' + K_1K_5'}{(1-K_5)(1-K_6') - K_5'K_6},\\
B_2 &= \frac{(1-K_5)K_2' + K_2K_5'}{(1-K_5)(1-K_6') - K_5'K_6},\\
C_2 &= \frac{(1-K_5)K_3' + K_3K_5'}{(1-K_5)(1-K_6') - K_5'K_6},\\
D_2 &= \frac{(1-K_5)K_4' + K_4K_5'}{(1-K_5)(1-K_6') - K_5'K_6}.
\end{aligned} \tag{8}
$$

with $K_1 = f_1(1),\ K_2 = f_2(1),\ K_3 = f_3(1),\ K_4 = f_4(1),\ K_5 = f_5(1),\ K_6 = f_1(6),\ K_1' = f_1'(1),\ K_2' = f_2'(1),\ K_3' = f_3'(1),\ K_4' = f_4'(1),\ K_5' = f_5'(1),\ K_6' = f_1'(6).$

Proof. On the interval $[n, n+1)$, Eq. (3) can be rewritten as

$$
\begin{aligned}
&x''(t) + ax'(t) + bx(t) + c_1x'(n-1) + c_2x'(n) + c_3x'(n+1)\\
&+d_1x(n-1) + d_2x(n) + d_3x(n+1) = 0.
\end{aligned} \tag{9}
$$

Hence, it is obvious that solutions to the states of a and b are established as (5). For example, for $a \neq 0$, $b = 0$, Eq. (9) is the following

$$x''(t) + ax'(t) + c_1x'(n-1) + c_2x'(n) + c_3x'(n+1) + d_1x(n-1) + d_2x(n) + d_3x(n+1) = 0. \tag{10}$$

By finding homogeneous and particular solutions to Eq. (10), we get

$$
\begin{aligned}
x_n(t) = {} & \frac{d_1}{a}(1 - t + n - e^{-a(t-n)})p_{n-1} + \frac{c_1}{a}(1 - t + n - e^{-a(t-n)})q_{n-1}\\
& + (1 + \frac{d_2}{a}(1 - t + n - e^{-a(t-n)}))p_n + \frac{1}{a}(1 - e^{-a(t-n)}\\
& + c_2(1 - t + n - e^{-a(t-n)}))q_n + \frac{d_3}{a}(1 - t + n - e^{-a(t-n)})p_{n+1}\\
& + \frac{c_3}{a}(1 - t + n - e^{-a(t-n)})q_{n+1}.
\end{aligned} \tag{11}
$$

In other cases, it can be done similarly. So, the solution of Eq. (9) on the interval $[n, n+1)$ is founded as

$$x_n(t) = p_{n-1}f_1(t-n) + q_{n-1}f_2(t-n) + p_nf_3(t-n) + q_nf_4(t-n) + p_{n+1}f_5(t-n) + q_{n+1}f_6(t-n). \tag{12}$$

Similarly, on the interval $[n+1, n+2)$, we find

$$
\begin{aligned}
x_{n+1}(t) \quad = \quad & p_nf_1(t-n-1) + q_nf_2(t-n-1) + p_{n+1}f_3(t-n-1) + q_{n+1}f_4(t-n-1\\
& +p_{n+2}f_5(t-n-1) + q_{n+2}f_6(t-n-1).
\end{aligned} \tag{13}
$$

Because $x(t)$ is continuous at $t = n+1$, it must hold that

$$\lim_{t\to n+1} x_n(t) = \lim_{t\to n+1} x_{n+1}(t).$$

Therefore, from (12) and (13), we get

$$p_{n+1} = \frac{1}{1-K_5}(p_{n-1}K_1 + q_{n-1}K_2 + p_nK_3 + q_nK_4 + q_{n+1}K_6), \quad n \in \mathbb{Z}^+. \tag{14}$$

On the other hand, the derivative of (12) is

$$\begin{aligned} x_n'(t) &= p_{n-1}f_1'(t-n) + p_nf_2'(t-n) + p_{n+1}f_3'(t-n) \\ &\quad + q_{n-1}f_4'(t-n) + q_nf_5'(t-n) + q_{n+1}f_6'(t-n), \quad t \in [n, n+1). \end{aligned} \tag{15}$$

On the interval $[n+1, n+2)$, (15) is following as

$$\begin{aligned} x_{n+1}'(t) &= p_nf_1'(t-n-1) + p_{n+1}f_2'(t-n-1) + p_{n+2}f_3'(t-n-1) \\ &\quad + q_nf_4'(t-n-1) + q_{n+1}f_5'(t-n-1) + q_{n+1}f_6'(t-n-1), \quad t \in [n+1, n+2). \end{aligned} \tag{16}$$

Since $x'(t)$ is continuous at $t = n+1$, it must hold that

$$\lim_{t\to n+1} x_n'(t) = \lim_{t\to n+1} x_{n+1}'(t).$$

Hence, from (15) and (16), we obtain

$$q_{n+1} = \frac{1}{1-K_6'}(p_{n-1}K_1' + q_{n-1}K_2' + p_nK_3' + q_nK_4 + p_{n+1}K_5'), \quad n \in \mathbb{Z}^+, \tag{17}$$

where $x(n) = p_n$, $x'(n) = q_n$, $x(n-1) = p_{n-1}$, $x'(n-1) = q_{n-1}$. Therefore, from (14) and (17), we obtain the system

$$p_{n+1} = A_1p_{n-1} + B_1q_{n-1} + C_1p_n + D_1q_n,$$

$$q_{n+1} = A_2p_{n-1} + B_2q_{n-1} + C_2p_n + D_2q_n,$$

where A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2 are given as in (8).

If we denote

$$p_{n-1} = y_1(n), \quad q_{n-1} = y_2(n), \quad p_n = y_3(n), \quad q_n = y_4(n),$$

we get the difference system

$$Y(n+1) = AY(n) \tag{18}$$

with

$$Y(n) = \begin{pmatrix} y_1(n) \\ y_2(n) \\ y_3(n) \\ y_4(n) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \end{pmatrix}, \quad n \in \mathbb{Z}^+.$$

Thus, the solution of the difference system (18),

$$Y(n+1) = A^nY(0)$$

with

$$Y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{pmatrix}.$$

Then for $[n, n+1)$, the solution for the initial value problem and the characteristic equation of the matrix A^n are as in (5) and (7), respectively. ■

Now, let us look at the solutions of (7). We write the following general solution of Eq. (7), assuming that these roots are simple:

$$v_n = \lambda_1^n k_1 + \lambda_2^n k_2 + \lambda_3^n k_3 + \lambda_4^n k_4, \tag{19}$$

where $v_n = col(p_{n-1}, q_{n-1}, p_n, q_n)$ and $k_j = col(k_{ij})$, $i = 1, 2, 3, 4$ are variables that may be identified using appropriate initial or boundary conditions. If some λ_j is a multiple zero of Eq. (7), then the formula for v_n also contains n, n^2 or n^3 products of λ_j^n. Finally, by substituting the relevant components of the vectors v_n and v_{n+1} in Eq. (5), the solution $x(t)$ is achieved.

Theorem 2 *Eq. (3) with the conditions*

$$x(-1) = p_{-1}, \quad x(0) = p_0, \quad x(1) = p_1, \quad x(N-1) = p_{N-1} \tag{20}$$

and

$$x'(-1) = q_{-1}, \quad x'(0) = q_0, \quad x'(1) = q_1, \quad x'(N-1) = q_{N-1} \tag{21}$$

has a unique solution on $0 \le t < \infty$ *if* $N > 2$ *is an integer and both of the following hypotheses are satisfied:*

(i) The roots of Eq. (7) are nontrivial and distinct,

(ii)

$$\begin{aligned} &\lambda_1^N(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4) + \lambda_3^N(\lambda_1-\lambda_2)(\lambda_1-\lambda_4)(\lambda_2-\lambda_4) \\ &\neq \lambda_2^N(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)(\lambda_3-\lambda_4) + 4^N(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3). \end{aligned}$$

Proof. The first row of the vector equation (19) gives us

$$p_{n-1} = \lambda_1^n k_{11} + \lambda_2^n k_{12} + \lambda_3^n k_{13} + \lambda_4^n k_{14}. \tag{22}$$

We get following system by applying the conditions (20) to (22), respectively

$$\begin{aligned} k_{11} + k_{12} + k_{13} + k_{14} &= p_{-1}, \\ \lambda_1 k_{11} + \lambda_2 k_{12} + \lambda_3 k_{13} + \lambda_4 k_{14} &= p_0, \\ \lambda_1^2 k_{11} + \lambda_2^2 k_{12} + \lambda_3^2 k_{13} + \lambda_4^2 k_{14} &= p_1, \\ \lambda_1^N k_{11} + \lambda_2^N k_{12} + \lambda_3^N k_{13} + \lambda_4^N k_{14} &= p_{N-1}. \end{aligned} \tag{23}$$

On the other hand, the third row of the vector equation (19) gives us

$$p_n = \lambda_1^n k_{31} + \lambda_2^n k_{32} + \lambda_3^n k_{33} + \lambda_4^n k_{34}. \tag{24}$$

We obtain following system by applying the conditions (20) to (24), respectively

$$\begin{aligned} \lambda_1^{-1} k_{31} + \lambda_2^{-1} k_{32} + \lambda_3^{-1} k_{33} + \lambda_4^{-1} k_{34} &= p_{-1}, \\ k_{31} + k_{32} + k_{33} + k_{34} &= p_0, \\ \lambda_1 k_{31} + \lambda_2 k_{32} + \lambda_3 k_{33} + \lambda_4 k_{34} &= p_1, \\ \lambda_1^{N-1} k_{31} + \lambda_2^{N-1} k_{32} + \lambda_3^{N-1} k_{33} + \lambda_4^{N-1} k_{34} &= p_{N-1}. \end{aligned} \tag{25}$$

From hypothesis (ii), the determination of the coefficients of the systems (23) and (25) is different from zero. Hence, we can find k_{ij} and also p_{n-1} or p_n uniquely. Furthermore, once the values p_{n-1} or p_n have been found, we calculate p_n, p_{n+1} or p_{n-1}, p_{n+1}, respectively. Similarly, the second row of the vector equation (19) gives us

$$q_{n-1} = \lambda_1^n k_{21} + \lambda_2^n k_{22} + \lambda_3^n k_{33} + \lambda_4^n k_{44}. \tag{26}$$

Applying the conditions (21) to (26), we get q_{n-1}, q_n and q_{n+1} uniquely. Substituting p_{n-1}, p_n, p_{n+1}, q_{n-1}, q_n and q_{n+1} in Eq. (5), the unique solution $x_n(t)$ is obtained. ■

Let us now explore the four theorems that are based on the various situations of the characteristic roots. The proofs of theorems are excluded since they are related to the proof of Theorem 2.

Theorem 3 *Let us assume that all characteristic roots are nontrivial and two of them are equal* ($\lambda_1 = \lambda_2$), *others are different from each other* ($\lambda_3 \neq \lambda_4$). *If*

$$\lambda_1^N(\lambda_3-\lambda_4)(\lambda_1^2(N-2)-(\lambda_3+\lambda_4)\lambda_1(N-1)+\lambda_3\lambda_4 N) \neq -\lambda_3^N(\lambda_1-\lambda_4)^2\lambda_1+\lambda_4^N(\lambda_1-\lambda_3)^2\lambda_1,$$

then Eq.(3) with the conditions (20) and (21) has a unique solution on $[0,\infty)$.

Theorem 4 *If the characteristic roots* λ_j *are nontrivial,* $\lambda_1=\lambda_2$, $\lambda_3=\lambda_4$ *and*

$$\lambda_1^N((N-2)\lambda_1^2\lambda_3+2(1-N)\lambda_1\lambda_3^2+N\lambda_3^3) \neq \lambda_3^N((2-N)\lambda_1\lambda_3^2+2(N-1)\lambda_1^2\lambda_3-N\lambda_1^3),$$

then Eq. (3) with the conditions (20) and (21) has a unique solution on $[0,\infty)$.

Theorem 5 *If the characteristic roots* λ_j *are nontrivial,* $\lambda_1=\lambda_2=\lambda_3$ *and*

$$\lambda_1^N((-N^2+3N-2)\lambda_1^2+2N(N-2)\lambda_1\lambda_4+N(1-N)\lambda_4^2) \neq -2\lambda_4^N\lambda_1^2,$$

then Eq. (3) with the conditions (20) and (21) has a unique solution on $[0,\infty)$.

Theorem 6 *If the characteristic roots* λ_j *are nontrivial,* $\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda$ *and*

$$N(2-3N+N^2)\lambda^N \neq 0,$$

then Eq. (3) with the conditions (20) and (21) has a unique solution on $[0,\infty)$.

3 Oscillatory and Nonoscillatory Results

Definition 2 *A function* $x(t)$ *defined on* $[0,\infty)$ *is called oscillatory if there exist two real valued sequences* $\{t_n\}_{n\geq 0}, \{\bar{t}_n\}_{n\geq 0} \subset [0,\infty)$ *such that* $t_n \to +\infty$, $\bar{t}_n \to +\infty$ *as* $n \to +\infty$ *and* $x(t_n)x(\bar{t}_n) \leq 0$ *for* $n \geq N$ *where* N *is sufficiently large. Otherwise, the solution is called nonoscillatory.*

Definition 3 *A solution* $x(t)$ *of Eq. (3) is said to oscillatory if* $x(t)$ *has arbitrarily large zeros. Otherwise, it is called to be nonoscillatory.*

Theorem 7 *Every solution of Eq. (3) oscillates if and only if its characteristic equation has no positive roots.*

Definition 4 ([24]) *If* $f(x)=0$ *is an equation of degree* n *with real coefficients, where* $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$, *then the number of positive roots is equal to or an even number less than the number of variations in the signs of successive terms. Multiple roots have to be counted by their multiplicity, and zero is not a positive root.*

For convenience, we denote

$$\begin{aligned}
\Delta_1 &= K_4K_5'-(-1+K_5)K_4'+K_6K_3'-K_3(-1+K_6'),\\
\Delta_2 &= -1+K_5+K_6K_5'-(-1+K_5)K_6',\\
\Delta_3 &= K_2K_5'-(-1+K_5)K_2'+K_6K_1'+K_4K_3'-K_3K_4'-K_1(-1+K_6'),\\
\Delta_4 &= K_4K_1'-K_3K_2'+K_2K_3'-K_1K_4',\\
\Delta_5 &= K_2K_1'-K_1K_2'
\end{aligned}$$

and K_1, K_2, K_3, K_4, K_5, K_6, K_1', K_2', K_3', K_4', K_5', K_6' are the same as in Theorem 1.

Theorem 8 *If one of the following conditions is satisfied, then Eq. (3) has oscillatory solution*

(i) $\Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 < 0,\ \Delta_4 > 0,\ \Delta_5 < 0$ *or* $\Delta_1 < 0,\ \Delta_2 < 0,\ \Delta_3 > 0,\ \Delta_4 < 0,\ \Delta_5 > 0,$

(ii) $\Delta_1 > 0,\ \Delta_2 < 0,\ \Delta_3 > 0,\ \Delta_4 < 0,\ \Delta_5 > 0$ *or* $\Delta_1 < 0,\ \Delta_2 > 0,\ \Delta_3 < 0,\ \Delta_4 > 0,\ \Delta_5 < 0,$

(iii) $\Delta_1 > 0,\ \Delta_2 < 0,\ \Delta_3 < 0,\ \Delta_4 < 0,\ \Delta_5 > 0$ *or* $\Delta_1 < 0,\ \Delta_2 > 0,\ \Delta_3 > 0,\ \Delta_4 > 0,\ \Delta_5 < 0,$

(iv) $\Delta_1 > 0,\ \Delta_2 < 0,\ \Delta_3 < 0,\ \Delta_4 > 0,\ \Delta_5 > 0$ *or* $\Delta_1 < 0,\ \Delta_2 > 0,\ \Delta_3 > 0,\ \Delta_4 < 0,\ \Delta_5 < 0.$

Proof. Now, consider the following conditions

$$x(0) = p_0, \quad x(-1) = p_{-1} = p_0\lambda_1^{-1}, \quad x(1) = p_1 = p_0\lambda_1, \quad x(2) = p_2 = p_0\lambda_1^2.$$

Applying these conditions to (22), the coefficients k_{1i}, $i = 1, 2, 3, 4$ are found as

$$k_{11} = p_0, \quad k_{12} = k_{13} = k_{14} = 0$$

and therefore, (22) becomes

$$p_n = x(n) = p_0\lambda_1^n.$$

Hence,

$$x(n)x(n+1) = \lambda_1 p_0^2 \lambda_1^{2n}.$$

The characteristic equation (7) can be rewritten as a polynomial of λ

$$f(\lambda) = \lambda^4 + \alpha\lambda^3 + \beta\lambda^2 + \gamma\lambda + \delta \tag{27}$$

with $\alpha = -C_1 - D_2$, $\beta = C_1D_2 - C_2D_1 - A_1 - B_2$, $\gamma = A_1D_2 - A_2D_1 + B_2C_1 - B_1C_2$, $\delta = A_1B_2 - A_2B_1$ and A_i, B_i, C_i, D_i, $i = 1, 2$ are given in (8). To prove the oscillation of solutions, we need to show that there exists a unique negative root of the characteristic equation (7). For this reason, let us take the polynomial

$$f(-\lambda) = \lambda^4 - \alpha\lambda^3 + \beta\lambda^2 - \gamma\lambda + \delta.$$

Now, if hypothesis (i) is true, then we find that

$$\alpha = \frac{K_4K_5' - (-1+K_5)K_4' + K_6K_3' - K_3(-1+K_6')}{-1+K_5+K_6K_5' - (-1+K_5)K_6'} > 0,$$
$$\beta = \frac{K_2K_5' - (-1+K_5)K_2' + K_6K_1' + K_4K_3' - K_3K_4' - K_1(-1+K_6')}{-1+K_5+K_6K_5' - (-1+K_5)K_6'} < 0,$$
$$\gamma = \frac{K_4K_1' - K_3K_2' + K_2K_3' - K_1K_4'}{-1+K_5+K_6K_5' - (-1+K_5)K_6'} > 0,$$
$$\delta = \frac{K_2K_1' - K_1K_2'}{-1+K_5+K_6K_5' - (-1+K_5)K_6'} < 0.$$

From (ii), (iii) and (iv), it is obtained

$$\alpha < 0,\ \beta < 0,\ \gamma > 0,\ \delta < 0,$$
$$\alpha < 0,\ \beta > 0,\ \gamma > 0,\ \delta < 0,$$
$$\alpha < 0,\ \beta > 0,\ \gamma < 0,\ \delta < 0,$$

respectively. By using Descartes' rule of signs, we conclude that there exists a unique negative root of (27). Let us take λ_1 as this root. Hence

$$x(n)x(n+1) = \lambda_1 p_0^2 \lambda_1^{2n} < 0, \quad p_0 \neq 0,$$

the solution $x(t)$ of Eq. (3) has a zero in each interval $(n, n+1)$ and this yields the oscillatory solutions of Eq. (3). ■

Theorem 9 *If*

(i) $\Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 < 0,\ \Delta_4 > 0,\ \Delta_5 < 0$ *or* $\Delta_1 < 0,\ \Delta_2 < 0,\ \Delta_3 > 0,\ \Delta_4 < 0,\ \Delta_5 > 0,$

(ii) $\Delta_1 > 0,\ \Delta_2 < 0,\ \Delta_3 < 0,\ \Delta_4 > 0,\ \Delta_5 > 0$ *or* $\Delta_1 < 0,\ \Delta_2 > 0,\ \Delta_3 > 0,\ \Delta_4 < 0,\ \Delta_5 < 0,$

then there exist nonoscillatory solutions of Eq. (3).

Proof. Assume that (i) and (ii) are true, then the coefficients of the characteristic equation are found as

$$\alpha > 0, \quad \beta < 0, \quad \gamma > 0, \quad \delta < 0,$$

and

$$\alpha < 0, \quad \beta > 0, \quad \gamma < 0, \quad \delta < 0,$$

respectively. So considering (27) because of Descartes' rule of sign method, it is concluded that the characteristic equation has at least one positive root. Thus Eq. (3) has nonoscillatory solutions. ■

Theorem 10 *If one of the following conditions is satisfied, then there exist both oscillatory and nonoscillatory solutions of Eq. (3).*

(i) $\Delta_1 > 0,\ \Delta_2 < 0,\ \Delta_3 > 0,\ \Delta_4 > 0,\ \Delta_5 > 0$ *or* $\Delta_1 < 0,\ \Delta_2 > 0,\ \Delta_3 < 0,\ \Delta_4 < 0,\ \Delta_5 < 0,$

(ii) $\Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 < 0,\ \Delta_4 < 0,\ \Delta_5 < 0$ *or* $\Delta_1 < 0,\ \Delta_2 < 0,\ \Delta_3 > 0,\ \Delta_4 > 0,\ \Delta_5 > 0,$

(iii) $\Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 > 0,\ \Delta_4 < 0,\ \Delta_5 < 0$ *or* $\Delta_1 < 0,\ \Delta_2 < 0,\ \Delta_3 < 0, \Delta_4 > 0,\ \Delta_5 > 0,$

(iv) $\Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 > 0,\ \Delta_4 > 0,\ \Delta_5 < 0$ *or* $\Delta_1 < 0,\ \Delta_2 < 0, \Delta_3 < 0,\ \Delta_4 < 0,\ \Delta_5 > 0.$

Proof. The conditions imply the following cases, respectively,

$$\alpha < 0, \quad \beta < 0, \quad \gamma < 0, \quad \delta < 0,$$

$$\alpha > 0, \quad \beta < 0, \quad \gamma < 0, \quad \delta < 0,$$

$$\alpha > 0, \quad \beta > 0, \quad \gamma < 0, \quad \delta < 0,$$

$$\alpha > 0, \quad \beta > 0, \quad \gamma > 0, \quad \delta < 0.$$

Hence, from Descartes' rule of sign, we conclude that there exists a single positive root of (27). So, the other roots are negative or complex. Positive root generates nonoscillatory solutions, and others give us the oscillatory solutions of Eq. (3). ■

Theorem 11 *If one of the following conditions is satisfied,*

(i) $\Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 > 0,\ \Delta_4 > 0,\ \Delta_5 > 0,$

(ii) $\Delta_1 < 0,\ \Delta_2 < 0,\ \Delta_3 < 0,\ \Delta_4 < 0,\ \Delta_5 < 0,$

then every solution of Eq. (3) is oscillatory.

Proof. From each of the conditions, we get

$$\alpha > 0, \quad \beta > 0, \quad \gamma > 0, \quad \delta > 0.$$

So, because of Descartes' rule of sign rule, we see that Eq. (27) does not have any positive root. This means that all roots are negative or complex. Thus, these roots yield that every solution of Eq. (3) oscillates. ■

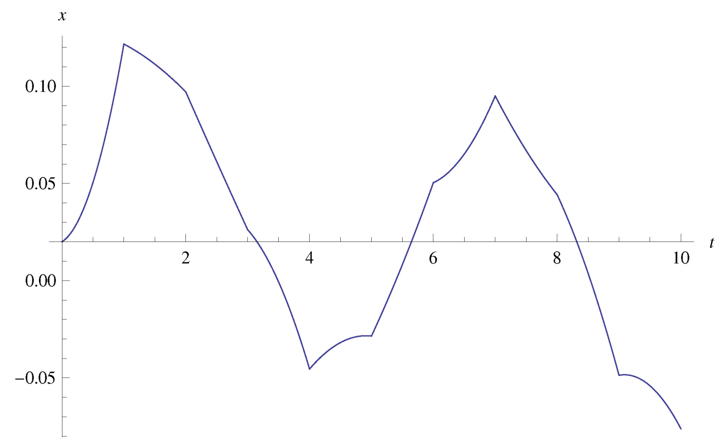

Figure 1: The solution $x(t)$ of Eq. (28) with $x(-1) = 0.1$, $x(0) = 0.02$, $x'(-1) = 0.1$, $x'(0) = 0.02$.

4 Example

We consider the second order differential equation with piecewise constant arguments

$$x''(t) + x'([t-1]) + 3x'([t]) - 4x'([t+1]) + 2x([t-1]) - x([t]) - 2x([t+1]) = 0. \tag{28}$$

Because of this hypothesis $\Delta_1 > 0$, $\Delta_2 < 0$, $\Delta_3 < 0$, $\Delta_4 > 0$, $\Delta_5 > 0$ of Theorem 8, we get $\alpha < 0$, $\beta > 0$, $\gamma < 0$, $\delta < 0$. So, the solution of Eq. (28) has oscillatory solutions. The solution $x_n(t)$ with initial conditions $x(-1) = 0.1$, $x(0) = 0.02$, $x'(-1) = 0.1$, $x'(0) = 0.02$ is shown in Figure 1.

5 Conclusion

We have proved that solutions to a class of second order differential equations with piecewise constant mixed arguments (3) exist and are unique. We have also looked into the properties of the equation's solutions, such as oscillation and nonoscillation. Moreover, Mathematica has been used to draw Figure 1. The figure shows the behavior of the solution $x(t)$, where $t \in [0, 10]$. Eq. (3) is a generalization of the equation studied in Bereketoglu et al. (2011) [21]. Taking $a = c_1 = c_2 = c_3 = 0$, $b = -a^2$, $d_1 = -b$, $d_2 = -c$, $d_3 = -d$ in Eq. (3), the equation (1.1) in Bereketoglu et al. (2011) [21] is obtained. Moreover, Eq. (3) develops from a model of stabilized inverted pendulum [23].

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