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On A Number Theoretic Inequality Of Ramanujan*

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Abstract

We present an upper bound for the number of divisors of a natural number. Our result refines an inequality due to Ramanujan.

1 Introduction and Results

Let

$$n = \prod_{j=1}^{k} {p_j}^{a_j} \tag{1}$$

be the canonical representation of a natural number $n > 1$ as a product of prime powers. Here, $p_1 < p_2 < \cdots < p_k$ are primes and $a_1, a_2, \ldots, a_k$ are positive integers. We denote by $d(n)$ the number of positive divisors of n. Then we have

$$d(n) = \prod_{j=1}^{k} (a_j + 1). \tag{2}$$

In his famous paper on highly composite numbers, Ramanujan [7] proved in 1915 the inequality

$$d(n) \le \frac{\big((1/k)\log(p_1 \cdots p_k n)\big)^k}{\log(p_1)\cdots\log(p_k)}; \tag{3}$$

see also Berndt [1, p. 79]. The aim of this note is to show that under the assumption that $(a_j)_{1\le j\le k}$ is an increasing sequence, we obtain a better and simpler upper bound for $d(n)$.

Theorem 1 *If $a_1 \le \cdots \le a_k$, then*

$$d(n) \le \Big(1 + \frac{\log(n)}{\log(p_1) + \cdots + \log(p_k)}\Big)^k. \tag{4}$$

The sign of equality holds if and only if $a_1 = \cdots = a_k$.

The following result of Chebyshev plays an important role in our proof.

Proposition 1 *If x_j and y_j $(j = 1, ..., k)$ are real numbers such that $x_1 \le \cdots \le x_k$ and $y_1 \le \cdots \le y_k$, then*

$$\sum_{j=1}^{k} x_j \sum_{j=1}^{k} y_j \le k \sum_{j=1}^{k} x_j y_j. \tag{5}$$

Equality holds in (5) if and only if $x_1 = \cdots = x_k$ or $y_1 = \cdots = y_k$.

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A proof of Chebyshev's inequality and various related results can be found in Hardy et al. [4, Section 2.17] and Mitrinović [6, Section 2.5].

Proof of Theorem 1. From (1) we obtain

$$1 + \frac{\log(n)}{\log(p_1) + \cdots + \log(p_k)} = \frac{\sum_{j=1}^{k}(a_j+1)\log(p_j)}{\sum_{j=1}^{k}\log(p_j)}. \tag{6}$$

In Proposition 1, we set

$$x_j = a_j + 1 \quad \text{and} \quad y_j = \log(p_j) \quad (j = 1, \ldots, k).$$

Then we conclude from (5) that

$$\sum_{j=1}^{k}(a_j+1)\log(p_j) \geq \frac{1}{k}\sum_{j=1}^{k}(a_j+1)\sum_{j=1}^{k}\log(p_j). \tag{7}$$

Next, we apply the arithmetic mean - geometric mean inequality and (2). This yields

$$\frac{1}{k}\sum_{j=1}^{k}(a_j+1) \geq \prod_{j=1}^{k}(a_j+1)^{1/k} = d(n)^{1/k}. \tag{8}$$

Combining (6), (7) and (8) leads to (4).
Since the sign of equality holds in (7) and (8) if and only if $a_1 = \cdots = a_k$, we conclude that equality is valid in (4) if and only if $a_1 = \cdots = a_k$. ■

Remark 1 *(i) The inequality*

$$1 + \frac{\log(n)}{\log(p_1) + \cdots + \log(p_k)} \leq \frac{(1/k)\log(p_1 \cdots p_k n)}{(\log(p_1) \cdots \log(p_k))^{1/k}}$$

is equivalent to

$$\Big(\prod_{j=1}^{k}\log(p_j)\Big)^{1/k} \leq \frac{1}{k}\sum_{j=1}^{k}\log(p_j).$$

It follows that (4) improves (3) unless $p_1 = \cdots = p_k$.

(ii) Using the well-known number theoretic functions

$$\gamma(n) = \mathrm{rad}(n) = \prod_{p|n} p \quad \text{and} \quad \omega(n) = \sum_{p|n} 1$$

we can write (4) in the form

$$d(n) \leq \Big(1 + \frac{\log(n)}{\log(\gamma(n))}\Big)^{\omega(n)}. \tag{9}$$

The ratio

$$\lambda(n) = \frac{\log(n)}{\log(\gamma(n))}$$

itself is a well-studied arithmetic function known as "the index of composition of the integer $n \geq 2$"; see De Koninck and Luca [3, Chapter 16]. The following counterpart of (4) was recently published by De Koninck and Letendre [2]:

$$d(n) < \Big(1 + \frac{\log(n)}{\omega(n)\log(\omega(n))}\Big)^{\omega(n)}. \tag{10}$$

This inequality holds for all n with $\omega(n) \geq 74$. See also Letendre [5].

(iii) *In order to compare the upper bounds presented in* (9) *and* (10) *we define the function*

$$R(n) = \frac{\omega(n)^{\omega(n)}}{\gamma(n)}.$$

It follows that (9) *improves* (10) *if and only if* $R(n) < 1$. *Computer calculations give that the set* $\{2, 3, \ldots, 10^5\}$ *has exactly* 99833 *elements such that* $R(n) < 1$. *The first number with* $R(n) > 1$ *is* $n = 210$. *We have* $R(210) = 1.219\ldots$.

(iv) *Inequality* (4) *is valid if the exponents in* (1) *are increasing. It is remarkable that this condition holds for a large number of positive integers, including the set of square-free integers which has the natural density* $6/\pi^2 \approx 61\%$. *It might be of interest to determine the exact density of positive integers which satisfy the assumptions of Theorem 1.*

(v) *In view of* (9) *and* (10), *it is natural to look for functions* $a(n)$ *and* $b(n)$ *such that*

$$\Big(1 + \frac{\log(n)}{\log(a(n))}\Big)^{\omega(n)} \le d(n) \le \Big(1 + \frac{\log(n)}{\log(b(n))}\Big)^{\omega(n)}. \qquad (11)$$

Since $2^{\omega(n)} \le d(n)$ *for* $n \ge 2$, *we conclude that the left-hand side of* (11) *holds with* $a(n) = n$ *for* $n \ge 2$. *Moreover, the right-hand side of* (11) *is valid with* $b(n) = \gamma(n)$ *for all* n *which satisfy the assumptions of Theorem 1 and with* $b(n) = \omega(n)^{\omega(n)}$ *for all* n *with* $\omega(n) \ge 74$, *according to* (10).

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