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The Kirchhoff Equation By Ricceri's Approach*

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Abstract

In this paper, we study the Kirchhoff equation

$$\left[1+\int_{\mathbb{R}^3}\left(|\nabla u|^2+u^2\right)dx\right](-\Delta u+u)=\lambda\alpha(x)f(u) \text{ in } \mathbb{R}^3, \tag{$\mathcal{K}$}$$

where $\alpha\in L^{\infty}(\mathbb{R}^3)\cap L^{6/(5-q)}(\mathbb{R}^3)$ for some $q\in(0,1)$, and the continuous function $f:\mathbb{R}\to\mathbb{R}$ is sublinear at infinity and superlinear at zero. We prove the existence of at least two non-trivial solutions when $\lambda>0$ is large enough and a non-existence result when $\lambda>0$ is small. We also get the stability of ($\mathcal{K}$) for an arbitrary subcritical perturbation of the equation.

1 Introduction

The following Kirchhoff equation

$$\left[1+\int_{\mathbb{R}^3}\left(|\nabla u|^2+u^2\right)dx\right](-\Delta u+u)=g(x,u) \text{ in } \mathbb{R}^3, \tag{1}$$

has been widely studied in recent years [1, 2, 3, 7, 8, 11, 13, 20, 21, 24, 26]. Equation (1) is well-understood for the model nonlinearity $g(x,s)=|s|^{p-1}s$, where $p>0$, various existence and multiplicity results are available for (1) in the case $1<p<5$. Non-existence results can be found in the papers of Li and Sun [21].

Besides the model nonlinearity $g(x,s)=|s|^{p-1}s$, important contributions can be found in the theory of the Kirchhoff equation when the right-hand side nonlinearity is more general, verifying various growth assumptions near the origin and at infinity. We recall two such classes of nonlinearities (for simplicity, we consider only the autonomous case $g=g(x,\cdot)$):

(AR) $g\in C(\mathbb{R},\mathbb{R})$ verifies the global Ambrosetti-Rabinowitz growth assumption, i.e., there exists $\mu>4$ such that

$$0<\mu G(s)\le sg(s) \text{ for all } s\in\mathbb{R}\setminus\{0\}, \tag{2}$$

where $G(s)=\int_0^s g(t)dt$. Note that (2) implies the superlinearity at infinity of g, i.e., there exist c, $s_0>0$ such that $|g(s)|\ge c|s|^{\mu-1}$ for all $|s|\ge s_0$. Up to some further technicalities, by the standard symmetric mountain pass theorem one can prove that (1) has infinitely many nontrivial solutions (see [20]).

(BL) $g\in C(\mathbb{R},\mathbb{R})$ verifies the Berestycki-Lions growth assumptions, i.e.,

- $-\infty\le\limsup_{s\to\infty}\frac{g(s)}{s}=0$;
- $-\infty<\liminf_{s\to 0^+}\frac{g(s)}{s^5}\le\limsup_{s\to 0^+}\frac{g(s)}{s}=-m<0$;
- There exists $s_0\in\mathbb{R}$ such that $G(s_0)>0$.

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In this case, Azzollini [1, 2] proved the existence of at least a nontrivial solution $u \in H^1(\mathbb{R}^3)$ for the equation (1) via suitable truncation and monotonicity arguments. Multiple solutions are obtained by Azzollini et al. [3].

The purpose of the present paper is to describe a new phenomenon for Kirchhoff equations by considering the non-autonomous eigenvalue problem

$$\left[1 + \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx\right](-\Delta u + u) = \lambda\alpha(x)f(u) \text{ in } \mathbb{R}^3, \tag{3}$$

where $\lambda > 0$ is a parameter, $\alpha \in L^\infty(\mathbb{R}^3)$, and the continuous nonlinearity $f : \mathbb{R} \to \mathbb{R}$ verifies the assumptions

(f1) $\lim_{|s|\to\infty} \frac{f(s)}{s} = 0$;

(f2) $\lim_{s\to 0} \frac{f(s)}{s} = 0$;

(f3) there exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$.

When the nonlinearity has a parameter, the situation changes, we want to show the existence of at least two nontrivial solutions for larger values of $\lambda > 0$. To state our main theorem, we consider a perturbed form of the equation (3) as follows:

$$\left[1 + \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx\right](-\Delta u + u) = \lambda\alpha(x)f(u) + \theta\beta(x)g(u) \text{ in } \mathbb{R}^3, \tag{4}$$

where $\theta \in \mathbb{R}$, $\beta \in L^\infty(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$, while $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that for some $c > 0$ and $1 < p < 5$, one has

(g1) $|g(s)| \le c(|s| + |s|^p)$ for all $s \in \mathbb{R}$.

The main result reads as follows:

Theorem 1 *Let f, $g : \mathbb{R} \to \mathbb{R}$ be continuous functions which satisfy (f1)–(f3) and (g1), respectively, $\alpha \in L^\infty(\mathbb{R}^3) \cap L^{6/(5-q)}(\mathbb{R}^3)$ be a non-negative, non-zero, radially symmetric function for some $q \in (0, 1)$, and $\beta \in L^\infty(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ be a radially symmetric function. Then there exists $\lambda^* > 0$ such that for every $\lambda > \lambda^*$, there is $\delta > 0$ with the property that for every $\theta \in [-\delta, \delta]$, equation (4) has at least two distinct, radially symmetric, nontrivial solutions $u_i \in H^1(\mathbb{R}^3)$, $i \in \{1, 2\}$.*

Some remarks are in order.

Remark 1 *The proof of Theorem 1 shows that for every compact interval $[a, b] \subset (\lambda^*, \infty)$, there exists a number $\nu > 0$ such that for every $\lambda \in [a, b]$, the solutions $u_i \in H^1(\mathbb{R}^3)$, $i \in \{1, 2\}$ of (4) verify*

$$\|u_i\|_{H^1} \le \nu. \tag{5}$$

Remark 2 *A Strauss-type argument shows that the solutions in Theorem 1 are homoclinic, i.e., for every $\lambda > \lambda^*$, $\theta \in [-\delta, \delta]$, and $i \in \{1, 2\}$, we have $u_i(x) \to 0$ as $|x| \to \infty$.*

Example 1 *Typical nonlinearity which fulfills hypotheses (f1)–(f3) is $f(s) = \ln(1 + s^2)$.*

Remark 3 *(a) Property (f1) is a sublinearity growth assumption at infinity on f which complements the Ambrosetti–Rabinowitz-type assumption (2).*

(b) If (f1)–(f3) hold for f, then the function $g(s) = -s + f(s)$ verifies all the assumptions in (BL) whenever $1 < \max_{s\neq 0} \frac{2F(s)}{s^2}$. Consequently, the results of Azzollini [1] can be applied also for (3), guaranteeing the existence of at least one nontrivial solution when $\lambda = \alpha(x) = 1$, and $b > 0$ is sufficiently small.

Let $\theta = 0$ and according to hypotheses (f1)–(f3), one can define the number

$$c_f = \max_{s \neq 0} \frac{|f(s)|}{|s| + 4s^3} > 0. \tag{6}$$

The following non-existence result for the equation (3) is expected whenever $\lambda > 0$ is small enough. More precisely, we have

Theorem 2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies (f1)–(f3), and $\alpha \in L^\infty(\mathbb{R}^3)$. Then for every $\lambda \in [0, \|\alpha\|_\infty^{-1} c_f^{-1})$ (with convention $1/0 = +\infty$), problem (3) has only the solution $u = 0$.*

2 Preliminaries and Variational Framework

Notations

- For every $p \in [1, \infty]$, $\| \cdot \|_p$ denotes the usual norm of the Lebesgue space $L^p(\mathbb{R}^3)$.
- The standard Sobolev space $H^1(\mathbb{R}^3)$ is endowed with the norm $\|u\|_{H^1} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2 dx\right)^{1/2}$. Note that the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is continuous for every $p \in [2, 6]$; let $s_p > 0$ be the best Sobolev constant in the above embedding. $H^1_r(\mathbb{R}^3)$ denotes the radially symmetric functions of $H^1(\mathbb{R}^3)$. The embedding $H^1_r(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is compact for every $p \in (2, 6)$(see Willem's book [29]).

We are interested in the existence of weak solutions $u \in H^1_r(\mathbb{R}^3)$ for the equation (4), i.e.,

$$\begin{aligned} &\int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) dx + \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx\right)^2 \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) dx \\ = \;& \lambda \int_{\mathbb{R}^3} \alpha(x) f(u) v dx + \theta \int_{\mathbb{R}^3} \beta g(u) v dx, \end{aligned} \tag{7}$$

for all $v \in H^1_r(\mathbb{R}^3)$, whenever (f1)–(f3) and (g1) hold, $\alpha \in L^\infty(\mathbb{R}^3)$ and $\beta \in L^\infty(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$. First, (f1) and (f2) imply in particular that one can find a number $n_f > 0$ such that

$$|f(s)| \leq n_f |s| \text{ for all } s \in \mathbb{R}. \tag{8}$$

More precisely, we define the functional $I : H^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$\begin{aligned} I(u) \;=\; & \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx + \frac{1}{4} \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx\right)^2 \\ & -\lambda \int_{\mathbb{R}^3} \alpha(x) F(u) dx - \theta \int_{\mathbb{R}^3} \beta(x) G(u) dx, \end{aligned} \tag{9}$$

where $F(t) = \int_0^t f(s) ds$ and $G(t) = \int_0^t g(s) ds$, which is of class C^1 on $H^1_r(\mathbb{R}^3)$.

Since

$$H^1_r(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) : \gamma u = u \text{ for all } \gamma \in O(3)\},$$

the principle of symmetric criticality of Palais [27] implies that the critical points $u \in H^1_r(\mathbb{R}^3)$ of the functional $I|_{H^1_r(\mathbb{R}^3)}$ are also critical points of I thus are weak solutions for the equation (4).

We conclude this section by recalling the following Ricceri-type three critical point theorem which plays a crucial role in the proof of Theorem 1. Before doing that, we recall the following notion: if X is a Banach space, we denote by W_X the class of that functional $E : X \to \mathbb{R}$ having the property that if $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty} E(u_n) \leq E(u)$ then $\{u_n\}$ has a subsequence converging strongly to u.

Theorem 3 *Let X be a separable and reflexive real Banach space, let $E_1 : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional belonging to W_X, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; and $E_2 : X \to \mathbb{R}$ a C^1 functional with a compact derivative. Assume that E_1 has a strict local minimum u_0 with $E_1(u_0) = E_2(u_0) = 0$. Setting the numbers*

$$\tau = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{E_2(u)}{E_1(u)}, \limsup_{u\to u_0} \frac{E_2(u)}{E_1(u)}\right\} \tag{10}$$

and

$$\xi = \sup_{E_1(u)>0} \frac{E_2(u)}{E_1(u)}. \tag{11}$$

Assume that $\tau < \xi$. Then, for each compact interval $[a, b] \subset (1/\xi, 1/\tau)$ (with the conventions $1/0 = \infty$ and $1/\infty = 0$) there exists $\kappa > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $E_3 : X \to \mathbb{R}$ with a compact derivative, there exists $\delta > 0$ such that for each $\theta \in [0, \delta]$, the equation

$$E_1'(u) - \lambda E_2'(u) - \theta E_3'(u) = 0$$

admits at least three solutions in X having norm less than κ.

This theorem is a powerful tool for studying the elliptic problems, see e.g. the references [16, 18, 5, 12, 17, 15].

3 Proofs

In this section, we assume that the assumptions of Theorem 1 are fulfilled.

For every $\lambda \geq 0$ and $\theta \in \mathbb{R}$, let

$$I(u) = E_1(u) - \lambda E_2(u) - \theta E_3(u)$$

where

$$E_1(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\|u\|^4, \quad E_2(u) = \int_{\mathbb{R}^3} \alpha(x)F(u)dx, \quad E_3(u) = \int_{\mathbb{R}^3} \beta(x)G(u)dx. \tag{12}$$

It is clear that E_i are C^1 functionals, $i \in \{1, 2, 3\}$. To complete the proof of Theorem 1, some lemmas need to be proven.

Lemma 1 *The functional E_1 is coercive, sequentially weakly lower semicontinuous which belongs to $W_{H_r^1(\mathbb{R}^3)}$, bounded on each bounded subset of $H_r^1(\mathbb{R}^3)$, and its derivative admits a continuous inverse on $H_r^1(\mathbb{R}^3)^*$.*

Proof. It is clear that E_1 is coercive on $H_r^1(\mathbb{R}^3)$. On account of Brezis [4], the functional E_1 is sequentially weakly lower semicontinuous on $H_r^1(\mathbb{R}^3)$. Now, let $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ which converges weakly to $u \in H_r^1(\mathbb{R}^3)$ and $\liminf_{n\to\infty} E_1(u_n) \leq E_1(u)$. On account of proposition of the norm, we obtain

$$\liminf_{n\to\infty} \|u_n\|_{H^1}^2 \leq \|u\|_{H^1}^2.$$

Thus, standard arguments show that $u_n \to u$ strongly in $H_r^1(\mathbb{R}^3)$, i.e., E_1 belongs to $W_{H_r^1(\mathbb{R}^3)}$. The proposition of the norm implies that E_1 sends bounded sets of $H_r^1(\mathbb{R}^3)$ to bounded sets. It remains to show that the derivative of E_1 has a continuous inverse on $H_r^1(\mathbb{R}^3)^*$. This was asserted in [28]. ∎

Lemma 2 *E_2 and E_3 have compact derivatives.*

Proof. We prove the statement only for E_2; the argument for E_3 is similar. Let $\{u_n\} \subset X$ be a bounded sequence. In particular, for some $c > 0$, one has that $\sup_n \|u_n\|_2 \leq c$ for some $c > 0$. First, we prove that the sequence $\{E_2'(u_n)\} \subset X^*$ is bounded; the latter fact follows from the uniform boundedness principle, i.e., the sequence $\{|\langle E_2'(u_n), v\rangle|\}$ is uniformly bounded for every $v \in X$. Indeed, due to (8), for every $v \in X$ one has

$$|\langle E_2'(u_n), v\rangle| \leq \int_{\mathbb{R}^3} \alpha(x)|f(u_n)||v|dx \leq n_f\|\alpha\|_\infty \int_{\mathbb{R}^3} |u_n||v|dx,$$
$$\leq n_f\|\alpha\|_\infty\|u_n\|_2\|v\|_2 \leq n_f\|\alpha\|_\infty c\|v\|_2 < \infty.$$

Up to a subsequence, $\{E_2'(u_n)\}$ weakly converges to some $h \in X^*$. Arguing by contradiction, we assume that there exists $\delta > 0$ such that

$$\|E_2'(u_n) - h\|_{X^*} > \delta \text{ for all } n \in \mathbb{N}. \tag{13}$$

In particular, for every $n \in \mathbb{N}$, there exists $\{v_n\} \in X$ such that $\|v_n\| = 1$ and

$$\langle E_2'(u_n) - h, v_n\rangle > \delta.$$

Up to a subsequence, we may assume that $\{v_n\}$ weakly converges to some $v \in X$, and $\{v_n\}$ strongly converges to v in $L^3(\mathbb{R}^3)$, since the embedding $X \hookrightarrow L^3(\mathbb{R}^3)$ is compact. Therefore, we obtain

$$\langle E_2'(u_n) - h, v_n\rangle = \langle E_2'(u_n) - h, v\rangle + \langle E_2'(u_n), v_n - v\rangle + \langle h, v - v_n\rangle$$
$$\leq \langle E_2'(u_n) - h, v\rangle + \int_{\mathbb{R}^3} \alpha(x)|f(u_n)||v_n - v|dx + \langle h, v - v_n\rangle,$$

and each term in the above expression tends to be 0. Indeed, the case of the first and last expressions is immediate, while from (f1) and (f2), it follows in particular that for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$|f(s)| \leq \varepsilon|s| + c_\varepsilon s^2 \text{ for all } s \in \mathbb{R}. \tag{14}$$

Therefore,

$$\int_{\mathbb{R}^3} \alpha(x)|f(u_n)||v_n - v|dx \leq \|\alpha\|_\infty(\varepsilon\|u_n\|\|v_n - v\| + c_\varepsilon\|u_n\|_3^2\|v_n - v\|_3).$$

The arbitrariness of ε and the fact that $\{v_n\}$ strongly converges to v in $L^3(\mathbb{R}^3)$ imply that the right-hand side of the above inequality tends to 0. Combining these facts, we arrive at a contradiction with (13), which concludes the proof. ■

Lemma 3 $\limsup_{\|u\|\to\infty} \frac{E_2(u)}{E_1(u)} \leq 0$.

Proof. According to (f1) and (f2), for every $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0, 1)$ such that

$$|f(s)| < \frac{\varepsilon}{2(1 + \|\alpha\|_\infty)}|s| \text{ for all } |s| \leq \delta_\varepsilon \text{ and } |s| \geq \delta_\varepsilon^{-1}.$$

Since $f \in C(\mathbb{R}, \mathbb{R})$, there also exists a number $M_\varepsilon > 0$ such that

$$\frac{|f(s)|}{|s|^q} \leq M_\varepsilon \text{ for all } |s| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}],$$

where $q \in (0, 1)$ is from the hypothesis for $\alpha \in L^{6/(5-q)}(\mathbb{R}^3)$. Combining the above two relations, we obtain that

$$|f(s)| \leq \frac{\varepsilon}{2(1 + \|\alpha\|_\infty)}|s| + M_\varepsilon|s|^q \text{ for all } s \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} E_2(u) &\le \int_{\mathbb{R}^3} \alpha(x)|F(u)|dx \\ &\le \int_{\mathbb{R}^3} \alpha(x) \left[\frac{\varepsilon}{4(1+\|\alpha(x)\|_\infty)} u^2 + \frac{M_\varepsilon}{q+1}|u|^{q+1} \right] \\ &\le \frac{\varepsilon}{4}\|u\|^2 + \frac{M_\varepsilon}{q+1}\|\alpha\|_{6/(5-q)} s_6^{q+1}\|u\|^{q+1}. \end{aligned}$$

For every $u \neq 0$, we have that

$$\begin{aligned} \frac{E_2(u)}{E_1(u)} &\le \frac{\frac{\varepsilon}{4}\|u\|^2 + \frac{M_\varepsilon}{q+1}\|\alpha\|_{6/(5-q)} s_6^{q+1}\|u\|^{q+1}}{\frac{1}{2}\|u\|^2 + \frac{1}{4}\|u\|^4} \\ &\le \frac{\varepsilon}{2} + 2\frac{M_\varepsilon}{q+1}\|\alpha\|_{6/(5-q)} s_6^{q+1}\|u\|^{q-1}. \end{aligned}$$

Taking the "lim sup" the above estimation when $\|u\| \to \infty$, the arbitrariness of $\varepsilon > 0$ gives the required inequality. ∎

Lemma 4 $\limsup_{u \to 0} \frac{E_2(u)}{E_1(u)} \le 0$.

Proof. A similar argument as in (14) shows that for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|F(s)| \le \frac{\varepsilon}{4(1+\|\alpha\|_\infty)} s^2 + c_\varepsilon |s|^3 \text{ for all } s \in \mathbb{R}.$$

This inequality implies that for every $u \in X$, we have

$$\begin{aligned} E_2(u) &\le \int_{\mathbb{R}^3} \alpha(x)|F(u)|dx \\ &\le \int_{\mathbb{R}^3} \alpha(x) \left[\frac{\varepsilon}{4(1+\|\alpha(x)\|_\infty)} u^2 + c_\varepsilon |u|^3 \right] dx \\ &\le \frac{\varepsilon}{4}\|u\|^2 + c_\varepsilon s_3^3 \|\alpha\|_\infty \|u\|^3. \end{aligned}$$

Thus, for every $u \neq 0$.

$$\frac{E_2(u)}{E_1(u)} \le \frac{\frac{\varepsilon}{4}\|u\|^2 + c_\varepsilon s_3^3\|\alpha\|_\infty \|u\|^3}{\frac{1}{2}\|u\|^2 + \frac{1}{4}\|u\|^4} \le \frac{\varepsilon}{2} + 2c_\varepsilon s_3^3 \|\alpha\|_\infty \|u\|,$$

and the argument is similar to the previous lemma. ∎

For any $0 \le r_1 \le r_2$, let $A[r_1, r_2] = \{x \in \mathbb{R}^3 : r_1 \le |x| \le r_2\}$ be the closed annulus (perhaps degenerate) with radials r_1 and r_2.

By assumption, since $\alpha \in L^\infty(\mathbb{R}^3)$ is a radially symmetric function with $\alpha \ge 0$ and $\alpha \not\equiv 0$, there are real numbers $R > r \ge 0$ and $\alpha_0 > 0$ such that

$$\operatorname{essinf}_{x \in A[r,R]} \alpha(x) \ge \alpha_0. \tag{15}$$

Let $s_0 \in \mathbb{R}$ from (f3). For a fixed element $\sigma \in (0,1)$, define the function $u_\sigma \in X$ such that

(a) $\operatorname{supp} u_\sigma \subseteq A[(r - (1-\sigma)(R-r)_+, R)]$;

(b) $u_\sigma = s_0$ for every $x \in A[r, r + \sigma(R-r)]$;

(c) $\|u_\sigma\|_\infty \le |s_0|$.

where we use the notation $t_+ = \max(0,t)$ for $t \in \mathbb{R}$. A simple calculation shows that

$$E_1(u_\sigma) \geq \frac{1}{2}\|u_\sigma\|^2 \geq \frac{2\pi s_0^2}{3}[(r+\sigma(R-r))^3 - r^3], \tag{16}$$

and

$$\begin{aligned} E_2(u_\sigma) \geq & \frac{4\pi}{3}[\alpha_0 F(s_0)(r+\sigma(R-r))^3 - r^3 - \|\alpha\|_\infty \max_{|t|\leq|s_0|}|F(t)| \\ & \times \left(r^3 - (r-(1-\sigma)(R-r))_+^3 + R^3 - (r+\sigma(R-r))^3\right)] \end{aligned} \tag{17}$$

We observe that for σ close enough to 1, the right-hand sides of both inequalities become strictly positive; therefore, we can define the number

$$\lambda^* = \inf_{E_2(u)>0} \frac{E_1(u)}{E_2(u)}. \tag{18}$$

Proof of Theorem 1. We apply Theorem 3, by choosing $X = H_r^1(\mathbb{R}^3)$, as well as E_1, E_2 and E_3 from (12). On account of Lemmas 1 and 2, the functionals E_1 and E_2 fulfill the hypotheses of Theorem 3. Moreover, E_1 has a strict global minimum $u_0 = 0$, and $E_1(0) = E_2(0) = 0$. The definition of the number τ in Theorem 3, see (10), and Lemmas 3 and 4 give that $\tau = 0$. Therefore, we may apply Theorem 3: for every compact interval $[a,b] \subset (\lambda^*, \infty)$ there exists $\kappa > 0$ such that for each $\lambda \in [a,b]$ there exists $\delta > 0$ with the property that for every $\theta \in [0,\delta]$, the equation $I(u) = E_1'(u) - \lambda E_2'(u) - \theta E_3'(u) = 0$ admits at least three solutions $u_i \in X$, $i \in \{1,2,3\}$, having X-norms less than κ. Note that we may repeat the above argument with $-E_3$ instead of the function E_3, by obtaining an interval of the form $[-\delta, \delta]$ for the parameter θ.

On account of (f2) and (g1), one has $f(0) = g(0) = 0$, thus 0 is a solution to (4); consequently, there exist at least two nontrivial solutions $u_i \in X$ to problem (4), ($i \in \{1,2\}$) with the required properties, which concludes the proof. ■

Remark 4 *Since the expression of λ^* is involved (see (18)), we give in the sequel an upper estimate of it which can be easily calculated. This fact can be done in terms of α_0, s_0, σ_0, R and r, see (15), where $\sigma_0 \in (0,1)$ is such a number for which the right hand side of (17) becomes positive, i.e., $M(\alpha_0, s_0, \sigma_0, R, r) > 0$. In order to avoid technicalities, we assume that $r = 0$ which slightly restricts our arguments, imposing that α does not vanish near the origin; see (15). The truncation function $u_{\sigma_0} \in H_r^1(\mathbb{R}^3)$ defined by*

$$u_{\sigma_0} = \begin{cases} 0 & \text{if } |x| > R, \\ s_0 & \text{if } |x| < \sigma_0 R, \\ \frac{s_0}{R(1-\sigma_0)}(R-|x|) & \text{if } \sigma_0 R \leq |x| \leq R, \end{cases}$$

verifies the properties (a)–(c) from above. Moreover, we have

$$E_1(u_{\sigma_0}) \leq \frac{2\pi}{3}Rs_0^2\left[R^2 + \frac{1+\sigma_0+\sigma_0^2}{1-\sigma_0}\right] + \frac{4b\pi^2}{9}R^2 s_0^4\left[R^2 + \frac{1+\sigma_0+\sigma_0^2}{1-\sigma_0}\right]^2.$$

Combining the above estimation with relation (17), we obtain

$$\lambda^* \leq \frac{N}{M} = \lambda_0.$$

Now, the conclusions of Theorem 1 are valid for every $\lambda \geq \lambda_0$.

The proof of Theorem 2 is based on a direct calculation.

Proof of Theorem 2. Assume that $u \in H^1(\mathbb{R}^3)$ is a solution of (3). Multiplying (3) by the test function $u \in H^1(\mathbb{R}^3)$, we obtain

$$\begin{aligned} \|u\|^2 + \|u\|^4 &= \lambda \int_{\mathbb{R}^N} \alpha(x) f(u) u dx \\ &\leq \lambda \int_{\mathbb{R}^N} |\alpha(x)||f(u)||u| dx \\ &\leq \lambda \|\alpha\|_{L^\infty} c_f \|u\|^2. \end{aligned}$$

Now, fix $0 \leq \lambda < c_f^{-1}\|\alpha\|_{L^\infty}$ arbitrarily and note that $b > 0$, the above estimate implies $u = 0$, which concludes the proof. ∎

4 Final Remarks

The reader can observe that we considered only symmetric-based compactness arguments together with the principle of symmetric criticality result. The compactness result can be replaced by a Bartsch-type (coercivity-based) compactness (see reference [30, 25, 9, 32, 6, 22, 14, 19, 10, 23, 31] for a closely related approach). In this situation, the proof of the result is almost the same as Theorem 1, so we omit the details.

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