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Existence And Uniqueness Result Of Solutions Of A Fractional Differential Boundary Value Problem*

Lamine Guedda[†], Khadidja Iatime[‡]

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Abstract

In this paper, we study the existence and uniqueness of solutions to a class of fractional boundary value problems involving the Caputo fractional derivative. Our methodology relies on reducing the given problem to an equivalent fixed-point problem. Then, by using the topological degree theory for condensing maps via an a priori method, some important results are obtained. An example is also provided to illustrate the practicality of our main results.

1 Introduction

This paper is concerned with the existence and uniqueness of solution to the following fractional boundary value problems (FBVP for short)

$$ {}^cD^r_{0^+}x(t) = g(t, x(t)), \quad t \in I := [0, T], \tag{1}$$

$$ ax(0) - bx(T) = \Psi(x), \tag{2}$$

$$ {}^cD^{r-1}x(T) = \phi(x), \tag{3}$$

where $a, b \in \mathbb{R}$ $(a \neq b)$, $r \in (1, 2)$, ${}^cD^r$ is the Caputo derivative, $g : I \times \mathbb{R} \to \mathbb{R}$, $\phi : C(I) \to \mathbb{R}$ and $\Psi : C(I) \to \mathbb{R}$ are given continuous maps.

Fractional calculus is a branch of mathematics that investigates the properties of integrals and derivatives of non-integer order. The potential of this concept has attracted the interest of many authors. Also in this field, there are different approaches to defining the fractional derivative and the fractional integral, including the approach proposed by Riemann-Liouville, Caputo, ...

In several cases, boundary value problems associated with fractional differential equations have been created to effectively model many phenomena in a variety of applications: astrophysics, acoustic control, chaotic dynamics, chemical engineering, electro-chemistry, economics, optics, medicine, porous media, polymer physics... see [1], [13], [14], [15] and [16]. Recently, the existence and uniqueness of solutions to some classes of fractional boundary value problems were dealt with by several methodologies, including, but not limited to, the fixed point theorems of Schaeffer, Schauder and Banach. For instance, the reader can refer to [2], [3], [4], [5], [6], [7] and the references therein.

The topological degree theory has emerged as the most important tool in investigating many problems encountered in nonlinear analysis. For example, Isaia applied this theory to establish the necessary conditions for the existence of solutions to some nonlinear integral equations see [11] and [18]. A very rich existence theory for fractional differential equations subject to various boundary conditions has been developed for some boundary value problems over the past few years. For instance, the authors of the paper [17] considered the following FBVP:

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)), \quad t \in J = [0, T], \\ au(0) + bu(T) = c, \end{cases}$$

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[†]Department of Mathematics, Faculty of exact sciences, Eloued University, Eloued 39000, Algeria

[‡]Laboratory of "Théorie des opérateurs et EDPs et ses Applications", Eloued University, Eloued 39000,Algeria

where $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and a, b, c are real constants with $a + b \neq 0$. Under some conditions upon f, they prove an existence and uniqueness result by using the degree theory and Banach contraction principle. In [18] Wang et al. apply the topological degree theory to the following non-local Cauchy problems of the form

$$\begin{cases} {}^{c}D^{q}u(t) = f(t, u(t)), \quad t \in J = [0, T], \\ u(0) + g(u) = u_0, \end{cases}$$

where ${}^{c}D^{q}$ is the Caputo fractional derivative of order $q \in (0, 1)$, the function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and $u_0 \in \mathbb{R}$.

R. A. Khan and K. Shah [12] provided sufficient conditions for the existence and uniqueness of solutions to some nonlinear multi-point boundary value problems with nonlinear boundary conditions of the form

$$\begin{cases} -{}^{c}D^{q}u(t) = f(t, u(t)), \quad t \in J = [0, T], \\ u(0) = g(u), u(1) - \sum_{i=1}^{m-2} \lambda_i u(\eta_i), \end{cases}$$

where $1 < q \leq 2$, $\lambda_i, \eta_i \in (0, 1)$ with $\sum_{i=1}^{m-2} \lambda_i \eta_i < 1$, $g, h : C(J, \mathbb{R}) \to \mathbb{R}$ and $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

In [10] Taghareed et al. studied the existence of solutions for the following fractional boundary value problem

$$\begin{cases} {}^{c}D^{q}u(t) = F(t, u(t)), \\ u(0) = \eta(u), u(T) = u_0, \end{cases}$$

where $t \in J := [0, T]$, $q \in (0, 1)$, ${}^{c}D^{q}$ is the Caputo derivative, $F : J \times X \to X$ and $\eta : C(J, X) \to X$ are given continuous maps (X is a Banach space). Also, the study is based on the application of the topological degree approach and fixed point theory. Mainly motivated by these and other works, we will use the coincidence degree theory approach for condensing maps (neither Brouwer degree nor Leray-Schauder degree) to provide sufficient conditions for the existence and uniqueness of solutions to the FBVP (1), (2) and (3). To the best of our knowledge, the boundary conditions (2) and (3) are more general than their counterparts in all problems of this type, and choosing an example in which all conditions are met requires great precision. The rest of this paper is organized as follows: In Section 2, we will state some definitions and lemmas about fractional calculus and the measure of non-compactness. Section three is devoted to recalling the concept of topological degree for condensing perturbations of the identity. In Section 4; we establish sufficient conditions for the existence and uniqueness of solutions for our problem, and we give an example to illustrate the obtained result.

2 Preliminaries

For the convenience of the reader, we start with some basic notions from fractional calculus, which are used further in this paper (see [8], [9] and [13]). We denote by $C([0, T])$ the Banach space of all continuous functions on $[0; T]$ with the sup-norm $\|x\|_\infty = \sup_{t \in [0;T]} |x(t)|$ and by $L^1([0, T])$ the Banach space of all measurable functions that are Lebesgue integrable on $[0, T]$ equipped by the norm

$$\|x\|_1 = \int_0^T |x(s)| \, ds.$$

Definition 1 ([13]) *For a given function F in the closed interval $[a, b]$, the r-th fractional order integral of F is defined by*

$$I_{a+}^{r} F(t) = \frac{1}{\Gamma(r)} \int_a^t (t - s)^{r-1} F(s) ds,$$

where Γ is the gamma function.

Definition 2 ([13]) *For a given function F in the closed interval $[a,b]$, the Caputo fractional order derivative of F is defined by*

$$\left({}^{c}D^{r}_{a+}F\right)(t) = \frac{1}{\Gamma(n-r)}\int_a^t (t-s)^{n-r-1}F^{(n)}(s)ds,$$

where $n = [r] + 1$.

Lemma 1 ([17]) *Let $n - 1 < r \leq n$. Then*

$$\begin{aligned} {}^{c}D^{r}_{a+}\left(I^{r}_{a+}h\right)(t) &= h(t), \\ I^{r}_{a+}\left({}^{c}D^{r}_{a+}h\right)(t) &= h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}, \end{aligned}$$

for some $c_i \in \mathbb{R}$, $\quad i = 0, 1, 2, \ldots, n-1, n = [r] + 1$.

In what follows, suppose that X is a Banach space equipped with the norm $\|.\|$, B is the set of all bounded subset of X.

Definition 3 ([9]) *The function $\alpha : B \to \mathbb{R}_+$ defined by*

$$\alpha(M) = \inf\{d > 0 : M \text{ admits a finite cover by sets of diameter } \leq d\},$$

is the Kuratowski measure of non-compactness.

Proposition 1 ([9]) *The following assertions hold:*

1). $\alpha(M) = 0$ if and only if M is relatively compact.

2). α is a seminorm i.e., $\alpha(\lambda M) = |\lambda|\,\alpha(M)$, $\lambda \in \mathbb{R}$ and $\alpha(M_1 + M_2) \leq \alpha(M_1) + \alpha(M_2)$.

3). $M_1 \subset M_2$ implies that $\alpha(M_1) \leq \alpha(M_2)$; $\alpha(M_1 + M_2) = \max\{\alpha(M_1), \alpha(M_2)\}$.

4). $\alpha(conv\,(M)) = \alpha(M)$.

5). $\alpha(\overline{M}) = \alpha(M)$.

Where $conv\,(M)$ is the convex hull of M and $\overline{M}$ designates the closure of M.

In what follows α is the Kuratowski measure of non-compactness.

Definition 4 (see [9], [11]) *Let $\Omega \subset X$ and $F : \Omega \to X$ be a continuous bounded map. One say that F is α-Lipschitz if there exists $k \geq 0$ such that $\alpha(F(M)) \leq k\alpha(M)$ for all $M \subset \Omega$ bounded. In the case $k < 1$, we call F a strict α-contraction. One say that F is α-condensing if $\alpha(F(M)) < \alpha(M)$ for all $M \subset \Omega$ bounded with $\alpha(M) > 0$. Also $F : \Omega \to X$ is Lipschitz if there exists $k > 0$ such that*

$$\|Fx - Fy\| \leq k\|x - y\| \text{ for all } \quad x, y \in \Omega,$$

and if $k < 1$, F is a strict contraction. We denote by $C_\alpha(\bar{\Omega})$ the set of all α-condensing maps on Ω, and by $SC_\alpha(\Omega)$ the class of all strict α-contractions. Note that $SC_\alpha(\Omega) \subset C_\alpha(\Omega)$ and every $F \in C_\alpha(\Omega)$ is α-Lipschitz with constant $k = 1$.

Now, recall the following propositions (the reader can be referred to [8], [9]).

Proposition 2 *If $F, G : \Omega \to X$ are α-Lipschitz maps with constants k, k' respectively, then $F + G : \Omega \to X$ is α-Lipschitz with constant $k + k'$.*

Proposition 3 *If $F : \Omega \to X$ is compact, then F is α-Lipschitz with zero constant.*

Proposition 4 *If $F : \Omega \to X$ is Lipschitz with constant k, then F is α-Lipschitz with the same constant k.*

3 Topological Degree for Condensing Perturbation of the Identity

Firstly, we give an axiomatic definition of a function degree. If Ω is a bounded open subset of X, $H \in C_\alpha\left(\overline{\Omega}\right)$ and $x \in X \diagdown (I-H)(\partial\Omega)$, the triplet $(I-H, \Omega, x)$ is called admissible. Let A be a family of admissible triplets.

Theorem 1 *[17] There exists a one-degree function $D : \mathbb{A} \to \mathbb{Z}$ which satisfies the following properties:*

(D1) Normalization: $D(I, \Omega, y) = 1$ for every $y \in \Omega$.

(D2) Additivity on the domain: for every disjoint open sets $\Omega_1, \Omega_2 \subset \Omega$ and every $y \notin (I-H)\left(\bar{\Omega}\backslash (\Omega_1 \cup \Omega_2)\right)$ we have

$$D(I-H, \Omega, y) = D(I-H, \Omega_1, y) + D(I-H, \Omega_2, y).$$

(D3) Invariance under homotopy: $D(I - P(t,\cdot), \Omega, y(t))$ is independent of $t \in [0,1]$ for every continuous, bounded map $P : [0,1] \times \bar{\Omega} \to X$ which satisfies

$$\alpha(P([0,1] \times M)) < \alpha(M), \quad \forall M \subset \bar{\Omega} \quad \textit{with} \ \ \alpha(M) > 0$$

and every continuous function $y : [0,1] \to X$ which satisfies

$$y(t) \neq x - P(t,x), \quad \forall t \in [0,1], \quad \forall x \in \partial\Omega.$$

(D4) Existence: $D(I-H, \Omega, y) \neq 0$ implies that $y \in (I-H)(\Omega)$.

(D5) Excision: $D(I-H, \Omega, y) = D(I-H, \Omega_1, y)$ for every open set $\Omega_1 \subset \Omega$ and every y does not belong to $(I-H)\left(\bar{\Omega}\backslash\Omega_1\right)$.

Use this degree to determine the usefulness of the a priori estimation method. In the sequel, we recall the fixed point theorem used to prove the most important existence result. See Isaia [11] for more details.

Theorem 2 *Let $H : X \to X$ be α-condensing operator and*

$$S = \{x \in X : \exists\theta \in [0,1] \ \textit{such that} \ x = \theta H(x)\}.$$

If S is a bounded subset in X, so there exists $\kappa > 0$ such that $S \subset B_\kappa(0)$, then

$$D(I - \theta H, B_\kappa(0), 0) = 1 \quad \forall\theta \in [0,1].$$

Consequently, H has at least one fixed point, and the set of the fixed points of H lies in $B_\kappa(0)$.

4 Existence and Uniqueness Result

In order to discuss the existence and uniqueness of solutions to the FBVP (1), (2), (3), we require the following assumptions:

[H1] There exist two constants $\lambda, \lambda' > 0$ such that

$$|\phi(x) - \phi(v)| \leq \lambda\|x - v\|_\infty \ \text{ and } \ |\Psi(x) - \Psi(v)| \leq \lambda'\|x - v\|_\infty \ \text{ for all } x, v \in C(I).$$

[H2] There exist two functions $\beta', \beta \in L^1([0;T], \mathbb{R}_+)$ and a constant $r_1 \in [0,1)$ such that

$$|g(t,x)| \leq \beta'(t)\,|x|^{r_1} + \beta(t) \ \text{ for any } (t,x) \in I \times \mathbb{R}.$$

[H3] There exist six real constants $\gamma, \delta, \gamma', \delta' > 0$, $r_2, r_3 \in [0,1)$ such that

$$|\phi(x)| \leq \gamma \|x\|_\infty^{r_2} + \delta \ \text{ and } \ |\Psi(x)| \leq \gamma' \|x\|_\infty^{r_3} + \delta' \ \text{ for each } x \in C(I).$$

[H4] There exists a constant $\lambda'' > 0$ such that

$$|g(t,x) - g(t,v)| \leq \lambda'' |x - v| \quad \text{for all } x, v \in \mathbb{R} \text{ and } t \in I,$$

Lemma 2 *Let $1 < r < 2$. The fractional integral equation (FIE for short)*

$$x(t) = \frac{1}{a-b} \int_0^T G(t,s) g(s, x(s)) ds + \frac{1}{a-b} \Psi(x) + (\frac{b}{a-b} T^{r-1} + t T^{r-2}) \Gamma(3-r) \phi(x), \tag{4}$$

where

$$G(t,s) = \begin{cases} \frac{a-b}{\Gamma(r)}(t-s)^{r-1} + \frac{b}{\Gamma(r)}(T-s)^{r-1} - \\ ((a-b)t + bT) T^{r-2} \Gamma(3-r), & 0 \leq s \leq t \leq T, \\ \frac{b}{\Gamma(r)}(T-s)^{r-1} - ((a-b)t + bT) T^{r-2} \Gamma(3-r), & 0 \leq t \leq s \leq T, \end{cases}$$

has a solution $x \in C(I)$ if and only if x is a solution of the FBVP (1), (2), (3).

Proof. Suppose that x is a solution of FBVP (1), (2), (3), then we need to prove that x is also a solution of FIE (4). By the Lemma (1), we have

$$x(t) = I_{0^+}^r g(t, x(t)) + c_0 + c_1 t. \tag{5}$$

And by introducing the Caputo derivative of order $r-1$ on both sides of equation (5) and taking $t = T$, we get

$$\begin{aligned} {}^c D_{0^+}^{r-1} x(t) &= {}^c D_{0^+}^{r-1} (I_{0^+}^r g(t, x(t)) + {}^c D_{0^+}^{r-1}(c_0) + {}^c D_{0^+}^{r-1}(c_1 t) \\ &= I_{0^+}^1 g(t, x(t)) + \frac{c_1}{\Gamma(3-r)} t^{2-r}, \end{aligned}$$

therefore

$${}^c D_{0^+}^{r-1} x(T) = \int_0^T g(s, x(s)) ds + \frac{c_1}{\Gamma(3-r)} T^{2-r} = \phi(x).$$

So

$$c_1 = T^{r-2} \Gamma(3-r) \left[\phi(x) - \int_0^T g(s, x(s)) ds \right].$$

Then from (2) and (5), we get

$$\begin{aligned} & a c_0 - b(\frac{1}{\Gamma(r)} \int_0^T (T-s)^{r-1} g(s, x(s)) ds + c_0 + c_1 T) \\ = \ & (a-b) c_0 - b c_1 T - \frac{b}{\Gamma(r)} \int_0^T (T-s)^{r-1} g(s, x(s)) ds = \Psi(x). \end{aligned}$$

Substituting c_1 by its value, we get

$$c_0 = \frac{1}{a-b} \left\{ \begin{array}{c} \Psi(x) + b \frac{1}{\Gamma(r)} \int_0^T (T-s)^{r-1} g(s, x(s)) ds + \\ b T^{r-1} \Gamma(3-r) \left[\phi(x) - \int_0^T g(s, x(s)) ds \right] \end{array} \right\}.$$

Consequently

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} g(s,x(s))ds \\ &+\frac{1}{a-b}\left(\Psi(x)+\frac{b}{\Gamma(r)}\int_0^T (T-s)^{r-1} g(s,x(s))ds\right) \\ &+\left(\frac{bT}{a-b}+t\right)T^{r-2}\Gamma(3-r)\left(\phi(x)-\int_0^T g(s,x(s))ds\right). \end{aligned}$$

By simple calculations, we get

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} g(s,x(s))ds+\frac{b}{(a-b)\Gamma(r)}\int_0^T (T-s)^{r-1} g(s,x(s))ds \\ &-(\frac{bT}{a-b}+t)T^{r-2}\Gamma(3-r)\int_0^T g(s,x(s))ds \\ &+\frac{1}{a-b}\Psi(x)+(\frac{bT}{a-b}+t)T^{r-2}\Gamma(3-r)\phi(x). \end{aligned} \tag{6}$$

Finally, we deduce that

$$x(t)=\frac{1}{a-b}\int_0^T G(t,s)g(s,x(s))ds+\frac{1}{a-b}\Psi(x)+(\frac{bT}{a-b}+t)T^{r-2}\Gamma(3-r)\phi(x),$$

where

$$G(t,s)=\begin{cases} \frac{a-b}{\Gamma(r)}(t-s)^{r-1}+\frac{b}{\Gamma(r)}(T-s)^{r-1}- \\ ((a-b)t+bT)T^{r-2}\Gamma(3-r), & 0\le s\le t\le T, \\ \frac{b}{\Gamma(r)}(T-s)^{r-1}-((a-b)t+bT)T^{r-2}\Gamma(3-r), & 0\le t\le s\le T. \end{cases}$$

Now, suppose that x is a solution of FIE (4). By applying ${}^cD^r_{0+}$ on both sides of equation (6) with taking into account the definitions (1), (2) and the properties of Caputo derivatives we get

$${}^cD^r_{0+}x(t)=g(t,x(t))$$

because

$${}^cD^r_{0+}\left[\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}g(s,x(s))ds\right]={}^cD^r_{0+}\left(I^r_{0+}g(t,x(t))\right)=g(t,x(t))$$

and

$${}^cD^r_{0+}(c_2+c_3t)={}^cD^r_{0+}(C)=0,$$

where c_2,c_3 and C are arbitrary constants. On the other hand, we have

$$\begin{aligned} x(0) &= \frac{b}{(a-b)\Gamma(r)}\int_0^T (T-s)^{r-1}g(s,x(s))ds-\frac{bT^{r-1}}{a-b}\Gamma(3-r)\int_0^T g(s,x(s))ds \\ &+\frac{1}{a-b}\Psi(x)+\frac{bT^{r-1}}{a-b}\Gamma(3-r)\phi(x) \end{aligned}$$

and

$$\begin{aligned} x(T) &= \frac{a}{a-b}\int_0^T (T-s)^{r-1}g(s,x(s))ds-\frac{a}{a-b}T^{r-1}\Gamma(3-r)\int_0^T g(s,x(s))ds \\ &+\frac{1}{a-b}\Psi(x)+\frac{a}{a-b}T^{r-1}\Gamma(3-r)\phi(x). \end{aligned}$$

Then

$$ax(0) - bx(T) = \frac{a-b}{a-b}\Psi(x) = \Psi(x)$$

and

$${}^{c}D_{0^+}^{r-1}x(t) = \int_0^t g(s,x(s))ds - \frac{t^{2-r}}{\Gamma(3-r)}T^{r-2}\Gamma(3-r)\int_0^T g(s,x(s))ds + \frac{t^{2-r}}{\Gamma(3-r)}T^{r-2}\Gamma(3-r)\phi(x).$$

Hence,

$${}^{c}D_{0^+}^{r-1}x(T) = \int_0^T g(s,x(s))ds - \int_0^T g(s,x(s))ds + \phi(x) = \phi(x),$$

which completes the proof. ■

From expression (4), we can define the following operators:

$$\begin{aligned} F_1 &: C(I) \to C(I) \\ (F_1x)(t) &= \frac{1}{a-b}\int_0^T G(t,s)g(s,x(s))ds, \qquad t \in I, \\ F_2 &: C(I) \to C(I) \\ (F_2x)(t) &= \frac{1}{a-b}\Psi(x) + (\frac{b}{a-b}T^{r-1} + tT^{r-2})\Gamma(3-r)\phi(x), \qquad t \in I, \\ P &: C(I) \to C(I), \quad Px = F_1x + F_2x. \end{aligned}$$

By definition of operators ϕ, Ψ and the continuity of g, the operator P is well defined and the fractional integral equation (FIE for short) (4) can be written as the following operator equation:

$$x = Px = F_1x + F_2x. \tag{7}$$

Therefore, each solution of the fractional FBVP (1), (2), (3) is a fixed point of the operator P.

Remark 1 *For all* $(t,s) \in I \times I$;

$$|G(t,s)| \leq (|a-b| + |b|)\left(\frac{1}{\Gamma(r)} + \Gamma(3-r)\right)T^{r-1} := R_1. \tag{8}$$

Lemma 3 *Under the assumption* $[H2]$, *the operator* F_1 *is completely continuous and satisfies the following growth condition*

$$\|F_1x\|_\infty \leq \frac{R_1}{|a-b|}\left\|\beta'\right\|_1 \|x\|_\infty^{r_1} + \frac{R_1}{|a-b|}\|\beta\|_1. \tag{9}$$

Proof. Let $\{x_n\}$ be a sequence of the bounded set $B(\kappa) = \{u \in C(I); \|u\|_\infty \leq \kappa\}$ such that $||x_n - x||_\infty \to 0$ as $n \to \infty$. we have to show that $||F_1x_n - F_1x||_\infty \to 0$ as $n \to \infty$. Due to the continuity of g, it is obvious that $||g(s,x_n(s)) - g(s,x(s))|| \to 0$ as $n \to \infty$. Using the triangular inequality and condition $[H2]$, we get for each $t \in I$,

$$\begin{aligned} |g(t,x_n(t)) - g(t,x(t))| &\leq |g(t,x_n(t)|) + |g(t,x(t))| \\ &\leq \beta'(t)\left[|x_n(t)|^{r_1} + |x(t)|^{r_1}\right] + 2\beta(t) \\ &\leq 2\left(\kappa^{r_1}\beta'(t) + \beta(t)\right). \end{aligned}$$

Then

$$|G(t,s)|\,|g(s,x_n(s)) - g(s,x(s))| \leq 2R_1\left(\kappa^{r_1}\beta'(s) + \beta(s)\right).$$

As $t \to 2R_1 \left(\kappa^{r_1}\beta'(t) + \beta(t)\right)$ is Lebesgue integrable function on $[0,T]$. by means of the Lebesgue-dominated convergence theorem

$$\int_0^T |G(t,s)|\,|g(s,x_n(s)) - g(s,x(s))|\,ds \ \text{ tends to } 0 \ \text{ as } \ n \to \infty,$$

then,

$$||F_1x_n - F_1x||_\infty \le \frac{1}{|a-b|}\int_0^T |G(t,s)|\,|g(s,x_n(s)) - g(s,x(s))| \ \text{ tends to } 0 \ \text{ as } \ n \to \infty.$$

Which means that F_1 is continuous.

Let B be a bounded subset of $C(I)$, i.e., there exists $\kappa > 0$ such that $B \subseteq B_\kappa(0)$. As function g is continuous function, then there exists $M > 0$ such that for all $(t,x) \in [0,T] \times B$

$$|g(s,x(s))| \le M,$$

because in this case $(s,x(s)) \in [0,T] \times [-\kappa,\kappa]$. Therefore, for every $x \in B$ and $0 \le t \le T$, we have

$$\begin{aligned} |(F_1x)(t)| &= \left|\frac{1}{a-b}\int_0^T G(t,s)g(s,x(s))ds\right| \\ &\le \frac{R_1}{|a-b|}\int_0^T |g(s,x(s))|\,ds \\ &\le \frac{R_1}{|a-b|}\int_0^T M ds = \frac{R_1MT}{|a-b|}. \end{aligned}$$

Consequently, $F_1(B)$ is uniformly bounded in $C(I)$. For $0 \le t_1 \le t_2 \le T$, we have

$$\begin{aligned} |(F_1x)(t_2) - (F_1x)(t_1)| &= \left| \begin{array}{c} \frac{1}{\Gamma(r)}\left(\int_0^{t_2}(t_2-s)^{r-1}g(s,x(s))ds - \int_0^{t_1}(t_1-s)^{r-1}g(s,x(s))ds\right) \\ -T^{r-2}\Gamma(3-r)(t_2-t_1)\int_0^T g(s,x(s))ds \end{array} \right| \\ &= \left| \begin{array}{c} \frac{1}{\Gamma(r)}\int_0^{t_1}\left((t_2-s)^{r-1} - (t_1-s)^{r-1}\right)g(s,x(s))ds + \\ \frac{1}{\Gamma(r)}\int_{t_1}^{t_2}(t_2-s)^{r-1}g(s,x(s))ds - T^{r-2}\Gamma(3-r)(t_2-t_1)\int_0^T g(s,x(s))ds \end{array} \right| \\ &\le \frac{M}{\Gamma(r)}\left[\begin{array}{c} \int_0^{t_1}\left((t_2-s)^{r-1} - (t_1-s)^{r-1}\right)ds + \int_{t_1}^{t_2}(t_2-s)^{r-1}ds \\ +T^{r-1}\Gamma(r)\Gamma(3-r)(t_2-t_1) \end{array} \right] \\ &= \frac{M}{\Gamma(r+1)}\left[(t_2^r - t_1^r)\right] + T^{r-1}M\Gamma(3-r)(t_2-t_1) \to 0 \text{ uniformly as } t_1 \to t_2. \end{aligned}$$

Therefore, $F_1(B)$ is equicontinuous. Hence, as a result of Arzela Ascoli theorem, the operator F_1 is completely continuous.

On the other hand, in view of the remark (1) and the condition $[H1]$, we have for each $x \in C(I)$ and all $t \in I$

$$\begin{aligned} |F_1x(t)| &= \left|\frac{1}{a-b}\int_0^T G(t,s)g(s,x(s))ds\right| \\ &\le \frac{1}{|a-b|}\int_0^T |G(t,s)|\,|g(s,x(s))|\,ds \\ &\le \frac{R_1}{|a-b|}\int_0^T \left(\beta'(t)\,|x(s)|^{r_1} + \beta(t)\right)ds \\ &\le \frac{R_1}{|a-b|}\left(\|\beta'\|_1\,\|x\|_\infty^{r_1} + \|\beta\|_1\right) = \frac{R_1}{|a-b|}\|\beta'\|_1\,\|x\|_\infty^{r_1} + \frac{R_1}{|a-b|}\|\beta\|_1. \end{aligned}$$

Then,

$$\|F_1 x\|_\infty \le \frac{R_1}{|a-b|}\left\|\beta'\right\|_1 \|x\|_\infty^{r_1} + \frac{R_1}{|a-b|}\|\beta\|_1 \,,$$

which completes the proof. ■

Lemma 4 *Under the assumption* $[H1]$ *and* $[H3]$, *the operator* F_2 *satisfies the Lipschitz condition with constant*

$$l = \frac{\lambda'}{|a-b|} + \left(\frac{|b|}{|a-b|}+1\right) T^{r-1}\Gamma(3-r)\lambda,$$

consequently F_2 *is* α*-lipschitz with the same constant and fulfills the following growth condition:*

$$\begin{aligned} \|F_2 x\|_\infty \quad &\le \quad \frac{\gamma'}{|a-b|}\|x\|_\infty^{r_3} + \left(\frac{|b|}{|a-b|}+1\right) T^{r-1}\Gamma(3-r)\gamma\|x\|_\infty^{r_2} \\ &\quad + \left(\frac{|b|}{|a-b|}+1\right) T^{r-1}\Gamma(3-r)\delta + \frac{\delta'}{|a-b|}. \end{aligned} \tag{10}$$

Proof. By the triangular inequality and the assumption $[H1]$, we have for all $u, v \in C(I)$,

$$\begin{aligned} |F_2 u(t) - F_2 v(t)| &= \left|\frac{1}{a-b}(\Psi(u) - \Psi(v)) + (\frac{b}{a-b}T^{r-1} + tT^{r-2})\Gamma(3-r)(\phi(u)-\phi(v))\right| \\ &\le \frac{1}{|a-b|}|\Psi(u) - \Psi(v)| + \left(\frac{|b|}{|a-b|}+1\right) T^{r-1}\Gamma(3-r)|\phi(u)-\phi(v)| \\ &\le \frac{\lambda'}{|a-b|}\|u-v\|_\infty + \left(\frac{|b|}{|a-b|}+1\right) T^{r-1}\Gamma(3-r)\lambda\|u-v\|_\infty \\ &= \left[\frac{\lambda'}{|a-b|} + \left(\frac{|b|}{|a-b|}+1\right) T^{r-1}\Gamma(3-r)\lambda\right]\|u-v\|_\infty \,. \end{aligned}$$

In view of proposition (4), we conclude that F_2 is α-lipschitz with constant l.

Further, using condition $[H3]$, we get for all $x \in C(I)$, $t \in I$,

$$\begin{aligned} |F_2 x(t)| &= \left|\frac{1}{a-b}\Psi(x) + (\frac{b}{a-b}T^{r-1} + tT^{r-2})\Gamma(3-r)\phi(x)\right| \\ &\le \frac{1}{|a-b|}|\Psi(x)| + \left|(\frac{b}{a-b}T^{r-1} + tT^{r-2})\Gamma(3-r)\right||\phi(x)| \\ &\le \frac{1}{|a-b|}\left(\gamma'\|x\|_\infty^{r_3} + \delta'\right) + \left|(\frac{b}{a-b}T^{r-1} + tT^{r-2})\Gamma(3-r)\right|\left(\gamma\|x\|_\infty^{r_2} + \delta\right). \end{aligned}$$

Then

$$\begin{aligned} \|F_2 x\|_\infty \quad &\le \quad \frac{\gamma'}{|a-b|}\|x\|_\infty^{r_3} + \left(\frac{|b|}{|a-b|}+1\right) T^{r-1}\Gamma(3-r)\gamma\|x\|_\infty^{r_2} \\ &\quad + \left(\frac{|b|}{|a-b|}+1\right) T^{r-1}\Gamma(3-r)\delta + \frac{\delta'}{|a-b|}, \end{aligned}$$

which completes the proof. ■

Theorem 3 *If the conditions* $[H1]$–$[H3]$ *hold, the fractional integral equation (4) and then the FBVP (1), (2), (3) admits at least one solution in* $C(I)$ *provided that*

$$l = \frac{\lambda'}{|a-b|} + \left(\frac{|b|}{|a-b|}+1\right) T^{r-1}\Gamma(3-r)\lambda < 1.$$

Moreover, in this case, the set of solutions of this problem is bounded.

Proof. In view of propositions (2), (3) the operator F_1 is α-Lipschitz with constant 0 and F_2 is α-Lipschitz with constant

$$l = \frac{\lambda'}{|a-b|} + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\lambda.$$

Then $P = F_1 + F_2$ is α-Lipschitz with constant $0 + l = l < 1$. Therefore, the operator P is a strict α-contraction i.e P is α-condensing operator.

Now, by using the theorem (2), we will prove that $S = \{x \in C(I); \exists\theta \in [0,1] \text{ such that } x = \theta P(x)\}$ is a bounded set in $C(I)$.

Consider $x \in S$ (arbitrary), in view of results (9) and (10), we get

$$\begin{aligned} \|x\|_\infty &= \|\theta P(x)\|_\infty = \theta \|Px\|_\infty = \theta \|F_1 x + F_2 x\|_\infty \\ &\le \|F_1 x\|_\infty + \|F_2 x\|_\infty \\ &\le \frac{R_1}{|a-b|} \|\beta'\|_1 \|x\|_\infty^{r_1} + \frac{\gamma'}{|a-b|} \|x\|_\infty^{r_3} + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\gamma \|x\|_\infty^{r_2} \\ &\quad + \frac{R_1}{|a-b|} \|\beta\|_1 + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\delta + \frac{\delta'}{|a-b|}. \end{aligned}$$

Then,

$$\|x\|_\infty \le A_1 \|x\|_\infty^{r_1} + A_2 \|x\|_\infty^{r_2} + A_3 \|x\|_\infty^{r_3} + A, \tag{11}$$

where

$$A_1 := \frac{R_1}{|a-b|} \|\beta'\|_1,$$

$$A_2 := \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\gamma,$$

$$A_3 := \frac{\gamma'}{|a-b|},$$

$$A := \frac{R_1}{|a-b|} \|\beta\|_1 + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\delta + \frac{\delta'}{|a-b|}.$$

The inequality (11), together with $0 < r_i < 1$ and $A_i > 0$ for $i = 1, 2, 3$ implies that there is a constant $C > 0$ such that $\|x\|_\infty \le C$ i.e. S is bounded in $C(I)$. If it is not the case, assume by contradiction that $\|x\|_\infty \to +\infty$ and dividing both sides of inequality (11) by $\|x\|_\infty$, we find

$$1 \le \frac{A_1}{\|x\|_\infty^{1-r_1}} + \frac{A_2}{\|x\|_\infty^{1-r_2}} + \frac{A_3}{\|x\|_\infty^{1-r_3}} + \frac{A}{\|x\|_\infty} \to 0.$$

Thus, the operator P has at least a solution in $C(I)$, and the set of fixed points of P is bounded. ■

Remark 2 *In the conditions $[H2]$, $[H3]$, if there exists $i \in \{1,2,3\}$ such that $r_i = 1$, the above result remains true provided that $A_i < 1$. As example (partcular case), for $\Psi(x) = \int_0^T x(s)\,ds$ with $0 < T < 1$, we have $|\Psi(x) - \Psi(v)| \le T \|x - v\|_\infty$ and $|\Psi(x)| \le T \|x\|_\infty^1 + \delta'$ for all $x, v \in C(I)$ and arbitrary $\delta' > 0$.*

Theorem 4 *In addition of condtions of theorem (3), if the conditions $[H4]$ hold, the BVP (1), (2), (3) admits a unique solution in $C(I)$ provided that*

$$\frac{\lambda'' R_1 T}{|a-b|} + l < 1.$$

Proof. For $u, v \in C(I)$, by $[H1]$ we have $|(F_2 u)(t) - (F_2 v)(t)| \leq l \|u - v\|_\infty$ and by $[H4]$,

$$\begin{aligned} |(F_1 u)(t) - (F_1 v)(t)| &= \frac{1}{a-b} \int_0^T G(t,s)\left(g(s,u(s)) - g(s,v(s))\right) ds \\ &\leq \frac{\lambda'' \|x - v\|_\infty}{|a-b|} \int_0^T |G(t,s)|\, ds \leq \frac{\lambda'' R_1 T}{|a-b|} \|x - v\|_\infty . \end{aligned}$$

Consequently,

$$\begin{aligned} |(Pu)(t) - (Pv)(t)| &\leq |(F_1 u)(t) - (F_1 v)(t)| + (F_2 u)(t) - (F_2 v)(t) \\ &\leq \left[\frac{\lambda'' R_1 T}{|a-b|} + l\right] \|x - v\|_\infty . \end{aligned}$$

As $\frac{\lambda'' R_1 T}{|a-b|} + l < 1$, the operator $P : C(I) \to C(I)$ is a contraction, then by Banach contraction principle it admits a unique fixed point in $C(I)$. ■

5 Example

Consider the boundary value problem

$$\begin{cases} {}^c D_{0^+}^{\frac{3}{2}} x(t) = \frac{1}{90}\left(2 + t + 3\left(t^2 + 1\right)\sin^2 x(t)\right), \\ 2x(0) - x(\frac{\pi}{2}) = \frac{1}{\pi^2} \int_0^{\frac{\pi}{4}} x(s)\, ds, \\ {}^c D_{0^+}^{\frac{1}{2}} x(\frac{\pi}{2}) = \frac{1}{8}\cos^2 x(0). \end{cases} \tag{12}$$

In this example, $r = \frac{3}{2}$, $T = \frac{\pi}{4}$, $a = 2$, $b = 1$, $g(t,x) = \frac{1}{90}\left(2 + t + 3\left(t^2+1\right)\sin^2 x\right)$, $\phi(x) = \frac{1}{8}\cos^2 x(0)$, $\Psi(x) = \frac{1}{\pi^2}\int_0^{\frac{\pi}{4}} x(s)\, ds$. Then

$$R_1 = 2\left(\frac{1}{\Gamma\left(\frac{3}{2}\right)} + \Gamma\left(\frac{3}{2}\right)\right)\sqrt{\frac{\pi}{4}} = 2 + \frac{\pi}{2} \simeq 3.57.$$

Also, we have for all $x, v \in C\left(\left[0, \frac{\pi}{4}\right]\right)$;

$$\begin{aligned} |\phi(x) - \phi(v)| &= \frac{1}{8}\left|\cos^2(x(0)) - \cos^2(v(0))\right| \\ &= \frac{1}{8}\left|\cos(x(0)) + \cos(v(0))\right| \left|\cos(x(0)) - \cos(v(0))\right| \\ &\leq \frac{2}{8}.\left|2\sin\left(\frac{x(0) + v(0)}{2}\right)\right| \left|\sin\frac{x(0) - v(0)}{2}\right| \\ &\leq \frac{2}{8}.2.1\left|\frac{x(0) - v(0)}{2}\right| \\ &\leq \frac{1}{4}|x(0) - v(0)| \leq \frac{1}{4}\|x - v\|_\infty , \end{aligned}$$

and

$$\begin{aligned} |\phi(x)| &= \frac{1}{8}\left|1 - \sin^2 x(0)\right| \leq \frac{1 + \sin^2 x(0)}{8} \leq \frac{1 + \sqrt{|\sin x(0)|}}{8} \\ &\leq \frac{\sqrt{|x(0)|} + 1}{8} \leq \frac{1}{8}\|x\|_\infty^{\frac{1}{2}} + \frac{1}{8}. \end{aligned}$$

For each $t \in \left[0, \frac{\pi}{4}\right]$ and all $x, v \in \mathbb{R}$, we have

$$\begin{aligned} |g(t,x) - g(t,v)| &= \left|\frac{1}{30}\left(t^2+1\right)\right| \left|\sin^2 x - \sin^2 v\right| \\ &= \left|\frac{1}{30}\left(t^2+1\right)\right| \left|\sin x + \sin v\right| \left|\sin x - \sin v\right| \\ &\leq \frac{\left(t^2+1\right)}{15} \left|2\cos\left(\frac{x+v}{2}\right)\right| \left|\sin\left(\frac{x-v}{2}\right)\right| \\ &\leq \frac{2}{15}|x-v|. \end{aligned}$$

It′s easy to see that

$$|\Psi(x) - \Psi(v)| \leq \frac{1}{4\pi}\|x-v\|_\infty, \quad |\Psi(x)| \leq \frac{1}{4\pi}\|x\|_\infty$$

with

$$\frac{1}{4\pi} \simeq 7.9577 \times 10^{-2} < 1 \quad \text{and} \quad g(t,x) \leq \frac{\left(t^2+1\right)}{30}|x|^{\frac{1}{2}} + \frac{2+t}{90}.$$

Then , we conclude that $\lambda = \frac{1}{4}$, $\lambda' = \frac{1}{4\pi}$, $\gamma = \frac{1}{8} = \delta$, $r_2 = \frac{1}{2}$, $\gamma' = \frac{1}{4\pi}$, δ'(arbitrary) $r_3 = 1$, $\beta'(t) = \frac{\left(t^2+1\right)}{30}$, $\beta(t) = \frac{2+t}{90}$, $r_1 = \frac{1}{2}$, $\lambda'' = \frac{2}{15}$,

$$l = \frac{\lambda'}{|a-b|} + \left(\frac{|b|}{|a-b|} + 1\right) T^{r-1}\Gamma(3-r)\lambda = \frac{1}{4\pi} + 2\sqrt{\frac{\pi}{4}}\Gamma(\frac{3}{2})\frac{1}{4} \simeq 0.39270 < 1,$$

$$\frac{\lambda'' R_1 T}{|a-b|} + l \simeq 0.37385 + 0.39270 = 0.76655 < 1.$$

Consequently, all conditions of theorem (4) hold, which prove that the problem (12) has a unique solution in $C\left(\left[0, \frac{\pi}{4}\right]\right)$.

6 Conclusion

Throughout this paper, sufficient conditions for the existence and uniqueness of solutions to FBVP (1), (2), (3) have been determined, at first by transforming it into a fixed-point problem involving two operators, one of which is compact, then using topological degree theory and Banach's contraction principle. Moreover, a particular case has been studied. Finally, a good example is provided to illustrate the applicability of our results. In the future, we look forward to solving problems of this kind, but with weaker assumptions.

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