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An Encoding-Decoding Algorithm Based On Fermat And Mersenne Numbers*

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Abstract

In this study, we present an encoding/decoding algorithm using Fermat and Mersenne numbers. Using Fermat and Mersenne numbers we define Fermat Q-matrices and Mersenne R-matrices. We process these matrices we defined in the same way as in Fibonacci Q-matrix. We can encode any given massage with a special encryption and we can decode the encrypted massage with the method we use. We describe these process step by step. First we define coding process and then we define decoding process. In addition we give a mix model named minesweeper using Fermat Q-and Mersenne R-matrices simultaneously. The purpose of this study is not only to increase the reliability of information security technology, but also to provide the ability to verify information at a high rate.

1 Introduction

Communication for humans has shown its importance throughout history. Encryption depends on the coding/decoding theory. This theory has been studied in different ways. One of the people working on coding/decoding theory is Lester Hill, who created the Hill algorithm. It is included matrix multiplying and matrix inverses. This algorithm is the first step of the modern mathematical theory. However, this method was not safe nowadays. It is relatively broken by using a computer. Alan Turing, a computer scientist, after inventing an encryption machine, broke the encrypted messages made by the Germans Enigma machine during World War II. It helped change the course of World War II.

Many researchers have introduced coding/decoding algorithms using Fibonacci and Lucas numbers [14, 11, 3]. Ucar et al. introduced Fibonacci Q-matrices and R-matrices [14]. Soykan presented matrix sequence of tribonacci and tribonacci-Lucas numbers [11]. Eser et al. presented Narayana numbers in coding theory [3]. Also, Kuloğlu et al. created encryption and decryption algorithm using Pell and Jacobsthal numbers [7, 6]. Prasad introduced a new method for coding/decoding algorithms using Lucas p-numbers [10]. Stakhov presented Cassini formula for Fibonacci matrices [12]. Diskaya et al. presented a new encryption algorithm based on Fibonacci polynomials and matrices [1]. The studies given above encouraged us to study on Mersenne numbers and Fermat numbers. Eser et al. presented Mersenne and Mersenne-Lucas hybrinomial quaternions [2]. Kuloglu et al. presented k-Fermat and k-Mersenne numbers in r-circulant matrices [8]. Uysal et al. presented hyperbolic k-Mersenne numbers k-Mersenne-Lucas octonions [15].

Fermat numbers are defined as follows [13]:

$$F_n = 2^n + 1 \quad , \; n \geq 0.$$

The first few of these sequences are

$$F_n = \{3, 5, 9, 17, 33, , 65, 129, 257, 513, \dots\}$$

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and also, Fermat sequence F_n is defined by the second order recurrence relation [4] as follows:

$$F_n = 3F_{n-1} - 2F_{n-2} n \geq 2.$$

Fermat Q_n -matrices are defined as follows:

$$Q_n = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} . \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+2} & -2F_{n+1} \\ F_{n+1} & -2F_n \end{bmatrix}.$$

Now we introduce Mersenne numbers. Mersenne numbers are defined as

$$M_{n+2} = 3M_{n+1} - 2M_n$$

with the initial conditions $M_0 = 0$ and $M_1 = 1$ in [5].

The first few terms of these sequences are

$$M_n = \{0, 1, 3, 7, 15, 31, 63, 127, 255, 511, \dots\}.$$

Mersenne R_n-matrices are defined by as follows [9]:

$$R_n = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} M_{n+1} & -2M_n \\ M_n & -2M_{n-1} \end{bmatrix}.$$

In this study, we give a new coding-decoding system using Mersenne R-matrices and Fermat Q-matrices. Our study consists of two parts. In the first part, we present a coding-decoding algorithm using Mersenne R-matrices. We explain this model with an example. We divide the message matrix into $2m \times 2m$ block matrices then mapping each letter in the alphabet to a different number, we obtain a codded matrix. In the second part, we present a coding-decoding algorithm Minesweeper model. This model includes both Mersenne Q- and Fermat R-matrices, also we give an example of the algorithm. This model, which is based on the principle of creating a stronger coding system by combining different number sequences of coding, was handled by Uçar and brought a different perspective to the coding algorithm [1].

2 A New Coding-Decoding Algorithm Using Mersenne R-Matrix

In this part first, we give a character table which we use in coding/decoding. The first character is matched with n. This table is according to $mod30$ for n. The size of the message matrix is $2m \times 2m$. We put " •" between each word. The block matrices of M are shown by T_i, $1 \leq i \leq m^2$. The size of each block is 2×2. We use the matrices of the following forms:

A	B	C	D	E	F	G	H	I	J
n	$n+1$	n+2	n+3	n+4	n+5	n+6	n+7	n+8	$n+9$
K	L	M	N	O	P	Q	R	S	T
$n+10$	n+11	$n+12$	$n+13$	$n+14$	$n+15$	$n+16$	$n+17$	$n+18$	$n+19$
U	V	W	X	Y	Z	0	!	?	.
$n+20$	$n+21$	$n+22$	$n+23$	$n+24$	$n+25$	$n+26$	$n+27$	$n+28$	$n+29$

Table 1: Number values corresponding to letters.

$$T_i = \begin{bmatrix} t_1^i & t_2^i \\ t_3^i & t_4^i \end{bmatrix}, \ A_i = \begin{bmatrix} a_1^i & a_2^i \\ a_3^i & a_4^i \end{bmatrix}, \ Q_i = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix}, \ R_n = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}.$$

Now we present the coding-decoding algorithm as follows:

Mersenne Blocking Algorithm

First of all, in case of any minor of $(T_i)_{22}$ is zero we have to add zero sequence of the beginning of the T_i matrix. Our coding algorithm is based on four steps. We give the steps as follows:

1. We denote the number of the block matrices T_i by t. We choose n as follows:

$$n = \begin{cases} t & \text{if } b \leq 3, \\ \left[\left[\frac{t}{2}\right]\right] & \text{otherwise.} \end{cases}$$

2. Using the table as follow, we change matrix M into T_i $\left(1 \leq i \leq m^2\right)$.
3. We compute each $w_i = \det(T_i)$.
4. We find the coded matrix $P = \left[w_i, t_k^i\right]_{k=1,2,4}$.

Decoding Algorithm

To decode the message, we take the coded matrix and apply an inversion process to it and we get the M matrix. Our decoding algorithm is based on eight steps. We give these steps as follows:

First, we know that $\det(T_i) = w_i$.

1. Compute R_n and determine r_j $(1 \leq j \leq 4)$.
2. Compute $r_1 t_1^i + r_3 t_2^i \rightarrow a_1^i$ $\left(1 \leq i \leq m^2\right)$.
3. Compute $r_2 t_1^i + r_4 t_2^i \rightarrow a_2^i$.
4. Solve $-1 \times d_i = a_1^i \left(r_2 x_i + r_4 t_4^i\right) - a_2^i \left(r_1 x_i + r_3 t_4^i\right)$ if i is even.
5. Solve $(-1)2^{n-1} \times d_i = a_1^i \left(r_2 x_i + r_4 t_4^i\right) - a_2^i \left(r_1 x_i + r_3 t_4^i\right)$ if i is odd.
6. Substitute for $x_i = t_3^i$.
7. Construct T_i.
8. and finally, we obtain M.

3 An Example for Coding-Decoding Process

Considering the message "HI! HOW ARE YOU?" we get the following message matrix:

$$M = \begin{bmatrix} H & I & ! & 0 \\ H & O & W & 0 \\ A & R & E & 0 \\ Y & O & U & ? \end{bmatrix}.$$

Using Table 1, we obtain the coded matrix L as follows:

$$L = \begin{bmatrix} 9 & 10 & 29 & 28 \\ 9 & 16 & 24 & 28 \\ 2 & 19 & 6 & 28 \\ 26 & 16 & 22 & 0 \end{bmatrix}.$$

Now we construct the block matrices T_i of L.

1. We can divide the L matrix of size 4×4 by the matrices T_i of size 2×2 since $m = 2$, we have four blocks:

$$T_1 = \begin{bmatrix} H & I \\ H & O \end{bmatrix}, \quad T_2 = \begin{bmatrix} ! & 0 \\ W & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} A & R \\ Y & O \end{bmatrix}, \quad T_4 = \begin{bmatrix} E & 0 \\ U & ? \end{bmatrix}.$$

H	I	!	0	H	O	W	0
9	10	29	28	9	16	24	28
A	R	E	0	Y	O	U	?
2	19	6	28	26	16	22	0

2. Since $t = 4 \geq 3$, we calculate $n = \left[\left[\frac{t}{2}\right]\right] = 2$. For $n = 2$, using the character Table 1., we get:
3. We have the elements of the blocks T_i $(1 \leq i \leq 4)$ matrix as follows:

$t_1^1 = 9$	$t_2^1 = 10$	$t_3^1 = 9$	$t_4^1 = 16$
$t_1^2 =29$	$t_2^2 = 28$	$t_3^2 = 24$	$t_4^2 = 28$
$t_1^3 = 2$	$t_2^3 = 19$	$t_3^3 = 26$	$t_4^3 = 16$
$t_1^4 = 6$	$t_2^4 = 28$	$t_3^4 = 22$	$t_4^4 = 0$

$$w_1 = 54, w_2 = 140, \quad w_3 = -462, \quad w_4 = -616.$$

4. At the end we have encoded matrix P as follows:

$$P = \begin{bmatrix} 54 & 9 & 10 & 16 \\ 140 & 29 & 28 & 28 \\ -462 & 2 & 19 & 16 \\ -616 & 6 & 28 & 0 \end{bmatrix}.$$

Decoding algorithm:

1. It is known that

$$R_2 = \begin{bmatrix} 7 & -2.3 \\ 3 & -2.1 \end{bmatrix}.$$

It should be noted here that the numbers 7, 3, and 1 are Mersenne numbers and the number -2 specified in the matrix is not included in the decoding process.

2. The elements of R_2 are denoted by

$$r_1 = 7, \ r_2 = 3, \ r_3 = 3 \text{ and } r_4 = 1.$$

3. We calculate the elements a_1^i and we composite the matrix E_i:

$$a_1^1 \ = 93, \ a_1^2 = 287, \ a_1^3 = 71 \text{ and } a_1^4 = 126.$$

4. Now we calculate the elements a_2^i and we composite the matrix E_i:

$$a_2^1 \ = 37, \ a_2^2 \ = 115, \ a_2^3 \ = 25 \text{ and } a_2^4 \ = 46.$$

5. We find the x_i values of the equations as follows:

$$(-1)\, 2^{n-1} w_i = a_1^i \left(r_2 x_i + r_4 t_4^i\right) - a_2^i \left(r_1 x_i + r_3 t_4^i\right).$$

For $n = 2$,

$$(-1)\, 2\, (54) = 93\, (3x_1 + 16) - 37\, (7x_1 + 48) \Rightarrow \ x_1 = 9,$$

$$(-1)\, 2\, (140) = 287\, (3x_2 + 28) - 115\, (7x_2 + 84) \Rightarrow \ x_2 = 24,$$

$$(-1)\, 2\, (-462) = 71\, (3x_3 + 16) - 25\, (7x_3 + 48) \Rightarrow \ x_1 = 26,$$

$$(-1)\, 2\, (-616) = 126\, (3x_4 + 0) - 46\, (7x_4 + 0) \Rightarrow \ x_4 = 22.$$

6. We rename x_i as follows:

$$x_i = t_3^1 = 9, \ x_2 = t_3^2 = 24, \ x_3 = 26 \text{ and } x_4 = 22.$$

7. We construct the block matrices T_i:

$$T_1 = \begin{bmatrix} 9 & 10 \\ 9 & 16 \end{bmatrix},\ T_2 = \begin{bmatrix} 29 & 28 \\ 24 & 28 \end{bmatrix},\ T_3 = \begin{bmatrix} 2 & 19 \\ 26 & 16 \end{bmatrix},\ T_4 = \begin{bmatrix} 6 & 28 \\ 22 & 0 \end{bmatrix}.$$

8. and finally we obtain the message matrix L as follows:

$$L = \begin{bmatrix} 9 & 10 & 29 & 28 \\ 9 & 16 & 24 & 28 \\ 2 & 19 & 6 & 28 \\ 26 & 16 & 22 & 0 \end{bmatrix}.$$

4 A Mixed Model: Minesweeper

In this part, we introduce Q_n-Fermat matrices and R_n-Mersenne matrices. We use these matrices simultaneously. The purpose of this model is to obtain a stronger encryption method. For this we use Fermat matrices for odd indices and Mersenne matrices for even indices.

Coding Algorithm

We give the steps for coding as follows:

1. Divide the matrix L into blocks T_i $\left(1 \leq i \leq m^2\right)$.
2. Choose n.
3. Determine t_j^i $\left(1 \leq j \leq m^2\right)$.
4. Compute w_i.
5. Construct matrix F.
6. End of algorithm.

Decoding Algorithm

We give the steps for decoding as follows:

1. Compute Q_n.
2. Compute R_n.
3. Compute $q_1 t_1^i + q_3 t_2^i \to a_1^i,\ i = 2l+1$ for $0 \leq l \leq 2m$.
   Compute $r_1 t_1^i + r_3 t_2^i \to a_1^i,\ i = 2l$ for $0 \leq l \leq 2m$.
   Compute $q_2 t_1^i + q_4 t_2^i \to a_2^i,\ i = 2l+1$ for $0 \leq l \leq 2m$.
   Compute $r_2 t_1^i + r_4 t_2^i \to a_2^i,\ i = 2l$ for $0 \leq l \leq 2m$.
4. Solve $(-1)^{i+1} 2^{n-1} \times w_i = a_1^i \left(r_2 t_3^i + r_4 x_i\right) - a_2^i \left(r_1 t_3^i + r_3 x_i\right),\ i = 2p,\ 0 \leq p \leq 2m$.
   Solve $(-1)^{i+1} 2^{n-1} \times w_i = a_1^i \left(q_2 t_3^i + q_4 x_i\right) - a_2^i \left(q_1 t_3^i + q_3 x_i\right),\ i = 2p+1,\ 0 \leq p \leq 2m$.
5. Substitute for $x_i = t_4^i$.
6. Construct T_i.
7. Construct L.
8. End of the algorithm.

4.1 An Example for Coding-Decoding Process

Considering the massage "MIXED MODELLING FOR CRYPTOGRAPHY" we get the following message matrix:

$$M=\begin{bmatrix} M & I & X & E & D & 0 \\ M & O & D & E & L & L \\ I & N & G & 0 & F & O \\ R & 0 & C & R & Y & P \\ T & O & G & R & A & P \\ H & Y & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6}.$$

Coding Algorithm

We give the steps for coding as follows:

1. Divide the matrix M into blocks T_i $\left(1 \leq i \leq m^2\right)$:

$$T_1 = \begin{bmatrix} M & I \\ M & O \end{bmatrix},\ T_2 = \begin{bmatrix} X & E \\ D & E \end{bmatrix},\ T_3 = \begin{bmatrix} D & 0 \\ L & L \end{bmatrix},$$

$$T_4 = \begin{bmatrix} I & N \\ R & 0 \end{bmatrix},\ T_5 = \begin{bmatrix} G & 0 \\ C & R \end{bmatrix},\ T_6 = \begin{bmatrix} F & O \\ Y & P \end{bmatrix},\ T_7 = \begin{bmatrix} T & O \\ H & Y \end{bmatrix},\ T_8 = \begin{bmatrix} G & R \\ 0 & 0 \end{bmatrix},\ T_9 = \begin{bmatrix} A & P \\ 0 & 0 \end{bmatrix}.$$

2. Since $t = 9 > 3$, we calculate $n = \left[\left[\frac{t}{2}\right]\right] = 4$. For $n = 4$, we use the character table as follows for matrix

M :

M	*I*	*X*	*E*	*D*	0	*M*	*O*	*D*	*E*	*L*	*L*
16	12	27	8	7	0	16	18	7	8	15	15
I	*N*	*G*	*0*	*F*	*O*	*R*	*0*	*C*	*R*	*Y*	*P*
12	17	10	0	9	18	21	0	6	21	28	19
T	O	G	R	A	P	H	Y	0	0	0	0
23	18	10	21	4	19	11	28	0	0	0	0

3. We get the elements of the blocks T_i $(1 \leq i \leq 9)$ as follows:

$t_1^1 = 16$	$t_2^1 = 12$	$t_3^1 = 16$	$t_4^1 = 18$
$t_1^2 = 27$	$t_2^2 = 8$	$t_3^2 = 7$	$t_4^2 = 8$
$t_1^3 = 7$	$t_2^3 = 0$	$t_3^3 = 15$	$t_4^3 = 15$
$t_1^4 = 12$	$t_2^4 = 17$	$t_3^4 = 21$	$t_4^4 = 0$
$t_1^5 = 10$	$t_2^5 = 0$	$t_3^5 = 6$	$t_4^5 = 21$
$t_1^6 = 9$	$t_2^6 = 18$	$t_3^6 = 28$	$t_4^6 = 19$
$t_1^7 = 23$	$t_2^7 = 18$	$t_3^7 = 11$	$t_4^7 = 28$
$t_1^8 = 10$	$t_2^8 = 21$	$t_3^8 = 0$	$t_4^8 = 0$
$t_1^9 = 4$	$t_2^9 = 19$	$t_3^9 = 0$	$t_4^9 = 0$

4. Calculating the determinants w_i of blocks of T_i we find:

$w_1 = 96$
$w_2 = 160$
$w_3 = 105$
$w_4 = -357$
$w_5 = 210$
$w_6 = -333$
$w_7 = 446$
$w_8 = 0$
$w_9 = 0$

5. Now we can construct the matrix $F_i = \left[w_i, t_k^i\right]$ as follows for $k \in \{1,2,3\}$

$$\begin{bmatrix} 96 & 16 & 12 & 16 \\ 160 & 27 & 8 & 7 \\ 105 & 7 & 0 & 15 \\ -357 & 12 & 17 & 21 \\ 210 & 10 & 0 & 6 \\ -333 & 9 & 18 & 28 \\ 446 & 23 & 18 & 11 \\ 0 & 10 & 21 & 0 \\ 0 & 4 & 19 & 0 \end{bmatrix}.$$

6. End of the algorithm.

Decoding Algorithm

1. Let's find Q_4 :

$$Q_4 = \begin{bmatrix} F_6 & -2F_5 \\ F_5 & -2F_4 \end{bmatrix} = \begin{bmatrix} 65 & -2.33 \\ 33 & -2.17 \end{bmatrix}.$$

It should be noted here that the numbers 65, 33, and 17 are Fermat numbers and the number -2 specified in the matrix is not included in the decoding process.

2. Now let's find: $R_4 = \begin{bmatrix} M_5 & -2M_4 \\ M_4 & -2M_3 \end{bmatrix} = \begin{bmatrix} 31 & 15 \\ 15 & 7 \end{bmatrix}.$

$a_1^1 = 732$	$a_2^1 = 380$
$a_1^2 = 957$	$a_2^2 = 461$
$a_1^3 = 231$	$a_2^3 = 119$
$a_1^4 = 657$	$a_2^4 = 299$
$a_1^5 = 330$	$a_2^5 = 170$
$a_1^6 = 549$	$a_2^6 = 261$
$a_1^7 = 1065$	$a_2^7 = 553$
$a_1^8 = 625$	$a_2^8 = 346$
$a_1^9 = 355$	$a_2^9 = 239$

3. If i is odd we use Fermat Q-matrices otherwise we use Mersenne R-matrices and we get as follows:

$$768 = 732\,(272 + 9x_1) - 380\,(528 + 17x_1) \Rightarrow x_1 = 18,$$

$$-1280 = 957\,(105 + 7x_2) - 461\,(217 + 15x_2) \Rightarrow x_2 = 8,$$

$$840 = 231\,(255 + 9x_3) - 119\,(495 + 17x_3) \Rightarrow x_3 = 15,$$

$$-2856 = 627\,(315 + 7x_4) - 299\,(651 + 15x_4) \Rightarrow x_4 = 0,$$

$$1680 = 330\,(102 + 9x_5) - 170\,(198 + 17x_5) \Rightarrow x_5 = 21,$$

$$2664 = 549\,(420 + 7x_6) - 261\,(868 + 15x_6) \Rightarrow x_6 = 19,$$

$$3568 = 1065\,(187 + 9x_7) - 553\,(363 + 17x_7) \Rightarrow x_7 = 28,$$

$$0 = 625\,(0 + 7x_8) - 346\,(0 + 15x_8) \Rightarrow x_8 = 0,$$

$$0 = 355\,(0 + 9x_9) - 239\,(0 + 17x_9) \Rightarrow x_9 = 0.$$

4. We rename x_i as follows:

$$x_1 = t_4^1,\ \ x_2 = t_4^2,\ x_3 = t_4^3,\ x_4 = t_4^4,\ x_5 = t_4^5,\ x_6 = t_4^6,\ x_7 = t_4^7,\ x_8 = t_4^8,\ x_9 = t_4^9.$$

5. Now we can construct the block matrices T_i.

$$T_1 = \begin{bmatrix} 16 & 12 \\ 16 & 18 \end{bmatrix},\ T_2 = \begin{bmatrix} 28 & 8 \\ 7 & 8 \end{bmatrix},\ T_3 = \begin{bmatrix} 7 & 0 \\ 15 & 15 \end{bmatrix},$$

$$T_4 = \begin{bmatrix} 12 & 17 \\ 4 & 0 \end{bmatrix},\ T_5 = \begin{bmatrix} 10 & 0 \\ 6 & 21 \end{bmatrix},\ T_6 = \begin{bmatrix} 9 & 18 \\ 28 & 19 \end{bmatrix},\ T_7 = \begin{bmatrix} 23 & 18 \\ 11 & 28 \end{bmatrix},\ T_8 = \begin{bmatrix} 10 & 21 \\ 0 & 0 \end{bmatrix},\ T_9 \ = \begin{bmatrix} 4 & 9 \\ 0 & 0 \end{bmatrix}.$$

6. At the end of the process we get L matrix combining the matrices T_i.

7.

$$M = \begin{bmatrix} 16 & 12 & 27 & 8 & 7 & 0 \\ 16 & 18 & 7 & 8 & 15 & 15 \\ 12 & 17 & 10 & 0 & 9 & 18 \\ 4 & 0 & 6 & 21 & 28 & 19 \\ 23 & 18 & 10 & 21 & 4 & 9 \\ 11 & 28 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6}.$$

Thus, the algorithm ends.

5 Conclusions

In this study, after giving a coding-decoding algorithm for Mersenne R-matrices and Fermat Q- matrices, we used Fermat and Mersenne matrices at the same time in Minesweeper model. Using this model, we ensure to massage security and increase reliability and transfer information correctly. Different security coding\decoding can be created by applying this work to all number sequences.

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