Applied Mathematics E-Notes, 24(2024), 379-398 ©
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/ ISSN 1607-2510

Fixed Points Of Cyclic $(\ddot{\sigma}, \ddot{\lambda})$-Admissible Generalized Contraction Type Maps In b-Metric Spaces With Applications*

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Received 20 June 2023

Abstract

This paper explores the existence and uniqueness of fixed points for cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible generalized contraction type maps in complete b-metric spaces. Further, we apply it to a pair of maps and establish the presence of common fixed points. Our results extend/generalize the results of Kumar et al. [21] from the metric space setting to b-metric spaces. We incorporate a few corollaries from our results and present instances to bolster the findings. Several applications are illustrated.

1 Introduction

The cornerstones of fixed point theory are the generalization of contraction conditions in one direction or/and the generalization of the surroundings of the operators being studied in the other direction. One of the most useful findings in fixed point theory is the Banach contraction principle, which is crucial for solving nonlinear equations. Many researchers have established contraction conditions by replacing several general conditions and many fixed point results achieved for contraction type mappings in ambient spaces with their applications in diverse domains throughout the last few decades [7, 8, 10, 11, 12, 14]. Czerwik [13] created the idea of b-metric space or metric type space as a generalization of metric space. Afterwards, many authors studied the existence of fixed points for a single-valued and multi-valued mappings in b-metric spaces under certain contraction conditions [1, 4, 5, 15, 19, 20, 23, 24, 25, 26, 27, 28, 29, 30].

In this paper, $\mathbb{R}$ represents the set of real numbers and $\mathbb{N}$ is the set of all natural numbers.

Definition 1 ([13]) *Suppose that $\tilde{\Theta}$ is a non-empty set and $s \geq 1$ is a given real number. A mapping $\eth : \tilde{\Theta} \times \tilde{\Theta} \to [0, \infty)$ is said to be a b-metric if the following conditions are satisfied: for any $\theta, \varrho, \xi \in \tilde{\Theta}$,*

($\eth_1$) $0 \leq \eth(\theta, \varrho)$ *and* $\eth(\theta, \varrho) = 0$ *if and only if* $\theta = \varrho$,

($\eth_2$) $\eth(\theta, \varrho) = \eth(\varrho, \theta)$,

($\eth_3$) $\eth(\theta, \xi) \leq s[\eth(\theta, \varrho) + \eth(\varrho, \xi)]$.

In this case, the pair $(\tilde{\Theta}, \eth)$ is called a b-metric space with parameter s.

Every metric space is a b-metric space with $s = 1$. In general, b-metric space is not a metric space (see [3]).

Definition 2 ([9]) *Suppose $(\tilde{\Theta}, \eth)$ is a b-metric space. A sequence $\{\theta_n\}$ in $\tilde{\Theta}$ is:*

*Mathematics Subject Classifications: 47H10, 54H25.

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- *b-convergent, if there exists* $\theta \in \tilde{\Theta}$ *such that* $\eth(\theta_n, \theta) \to 0$ *as* $n \to \infty$.
- *b-Cauchy, if* $\eth(\theta_n, \theta_m) \to 0$ *as* $n, m \to \infty$.

In general, b-metric is not necessarily continuous [16].

Definition 3 **([18])** *A pair* $(\digamma, \eta)$ *of self-maps on a set* $\tilde{\Theta}$ *is said to be weakly compatible if* $\digamma\eta\theta = \eta\digamma\theta$ *whenever* $\digamma\theta = \eta\theta$ *for any* $\theta \in \tilde{\Theta}$.

Definition 4 **([2])** *Let* $\tilde{\Theta}$ *be a nonempty set,* $\digamma$ *be a self-map defined on* $\tilde{\Theta}$ *and* $\ddot{\sigma}, \ddot{\lambda} : \tilde{\Theta} \to [0, \infty)$ *be two functions. Then* $\digamma$ *is a cyclic* $(\ddot{\sigma}, \ddot{\lambda})$*-admissible mapping if it has the following properties:*

(i) $\ddot{\sigma}(\theta) \geq 1$ *for some* $\theta \in \tilde{\Theta} \implies \ddot{\lambda}(\digamma\theta) \geq 1$,

(ii) $\ddot{\lambda}(\theta) \geq 1$ *for some* $\theta \in \tilde{\Theta} \implies \ddot{\sigma}(\digamma\theta) \geq 1$.

Definition 5 **([17])** *Let* $\tilde{\Theta}$ *be a nonempty set,* $\ddot{\sigma}, \ddot{\lambda} : \tilde{\Theta} \to [0, \infty)$ *be two mappings and* $\digamma, \eta : \tilde{\Theta} \to \tilde{\Theta}$ *be self-mappings. Then* $\digamma$ *is said to be an* η*-cyclic-*$(\ddot{\sigma}, \ddot{\lambda})$*-admissible mapping if the following conditions are satisfied:*

(i) $\ddot{\sigma}(\eta\theta) \geq 1$ *for some* $\theta \in \tilde{\Theta}$ *implies* $\ddot{\lambda}(\digamma\theta) \geq 1$,

(ii) $\ddot{\lambda}(\eta\theta) \geq 1$ *for some* $\theta \in \tilde{\Theta}$ *implies* $\ddot{\sigma}(\digamma\theta) \geq 1$.

The following lemma is useful in proving our main results.

Lemma 1 **([22])** *Suppose* $(\tilde{\Theta}, \eth)$ *is a b-metric space with coefficient* $s \geq 1$ *and* $\{\theta_n\}$ *is a sequence in* $\tilde{\Theta}$ *such that* $\eth(\theta_n, \theta_{n+1}) \to 0$ *as* $n \to \infty$. *If* $\{\theta_n\}$ *is not a Cauchy sequence then there exist an* $\epsilon > 0$ *and sequences of positive integers* $\{m_k\}$ *and* $\{n_k\}$ *with* $n_k > m_k \geq k$ *such that for every* $k > 0$, *corresponds to* m_k, *we can take* n_k *which is smallest such that* $\eth(\theta_{m_k}, \theta_{n_k}) \geq \epsilon, \eth(\theta_{m_k}, \theta_{n_k - 1}) < \epsilon$ *and*

(i) $\epsilon \leq \underline{\lim}_{k\to\infty} \eth(\theta_{m_k}, \theta_{n_k}) \leq \overline{\lim}_{k\to\infty} \eth(\theta_{m_k}, \theta_{n_k}) \leq s\epsilon$,

(ii) $\frac{\epsilon}{s} \leq \underline{\lim}_{k\to\infty} \eth(\theta_{m_k+1}, \theta_{n_k}) \leq \overline{\lim}_{k\to\infty} \eth(\theta_{m_k+1}, \theta_{n_k}) \leq s^2\epsilon$,

(iii) $\frac{\epsilon}{s} \leq \underline{\lim}_{k\to\infty} \eth(\theta_{m_k}, \theta_{n_k+1}) \leq \overline{\lim}_{k\to\infty} \eth(\theta_{m_k}, \theta_{n_k+1}) \leq s^2\epsilon$,

(iv) $\frac{\epsilon}{s^2} \leq \underline{\lim}_{k\to\infty} \eth(\theta_{m_k+1}, \theta_{n_k+1}) \leq \overline{\lim}_{k\to\infty} \eth(\theta_{m_k+1}, \theta_{n_k+1}) \leq s^3\epsilon$.

In 2019, Zada et al. [6] established the following:

Theorem 1 **([6])** *Let* $(\tilde{\Theta}, \eth)$ *be a complete b-metric space and* $\ddot{\sigma}, \ddot{\lambda} : \tilde{\Theta} \to [0, \infty)$ *be two mappings. If* $\digamma : \tilde{\Theta} \to \tilde{\Theta}$ *and* $\eta : \tilde{\Theta} \to \tilde{\Theta}$ *such that* $\digamma$ *is an* η*-cyclic-*$(\ddot{\sigma}, \ddot{\lambda}) - (\psi, \phi)_s$*-rational contraction map satisfies the following:*

(i) $\digamma\tilde{\Theta} \subseteq \eta\tilde{\Theta}$ *with* $\eta\tilde{\Theta}$ *are closed sub spaces of* $\tilde{\Theta}$;

(ii) there exists $\theta_0 \in \tilde{\Theta}$ *with* $\ddot{\sigma}(\eta\theta_0) \geq 1$ *and* $\ddot{\lambda}(\eta\theta_0) \geq 1$;

(iii) if the sequence $\{\theta_n\}$ *in* $\tilde{\Theta}$ *with* $\ddot{\lambda}(\theta_n) \geq 1$ *for all* n *and* $\theta_n \to \theta$, *then* $\ddot{\lambda}(\theta) \geq 1$;

(iv) $\ddot{\sigma}(\eta a) \geq 1$, $\ddot{\lambda}(\eta b) \geq 1$ *whenever* $\digamma a = \eta a$, $\digamma b = \eta b$.

Then $\digamma$ *and* η *have a unique point of coincidence in* $\tilde{\Theta}$. *Furthermore, if* $\digamma$ *and* η *are weakly compatible, then* $\digamma$ *and* η *have a unique common fixed point in* $\tilde{\Theta}$.

Recently, Kumar et al. [21] proved the following result in complete metric spaces.

Theorem 2 ([21]) *Let $(\tilde{\Theta}, \eth)$ be a complete metric space and $\digamma : \tilde{\Theta} \to \tilde{\Theta}$ satisfies the following condition: for all $\theta, \varrho \in \tilde{\Theta}$ and there exist $\alpha_1 \geq 0$, $\alpha_2, \alpha_3, \alpha_4 > 0$, $0 < h < 1$ with $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 = 1$ such that*

$$\eth(\digamma\theta, \digamma\varrho) \leq \alpha_1 \eth(\theta, \varrho) + \alpha_2[\eth(\theta, \digamma\theta) + \eth(\varrho, \digamma\varrho)] + \alpha_3[\eth(\theta, \digamma\varrho) + \eth(\digamma\theta, \varrho)] + \alpha_4[M(\theta, \varrho) + hm(\theta, \varrho)],$$

where,

$$M(\theta, \varrho) = \max\{\eth(\theta, \digamma\varrho), \eth(\digamma\theta, \varrho)\}, m(\theta, \varrho) = \min\{\eth(\theta, \digamma\varrho), \eth(\digamma\theta, \varrho)\}.$$

Then $\digamma$ has a unique fixed point in $\tilde{\Theta}$.

We introduce 'cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible generalized contraction type maps' in b-metric space and prove the existence of fixed points in complete b-metric spaces. We also extend the same to a pair of maps and prove the existence of common fixed points. We draw some corollaries from our results and provide examples to support our results. We include applications to nonlinear integral and functional equations.

2 Fixed Points of Cyclic $(\ddot{\sigma}, \ddot{\lambda})$-Admissible Generalized Contraction Type Maps

We first introduce cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible generalized contraction type maps in b-metric spaces:

Definition 6 *Suppose $(\tilde{\Theta}, \eth)$ is a b-metric space with $s \geq 1$ and $\ddot{\sigma}, \ddot{\lambda} : \tilde{\Theta} \to [0, \infty)$ are two given mappings. Let $\digamma : \tilde{\Theta} \to \tilde{\Theta}$ be a self-map on $\tilde{\Theta}$ such that $\digamma$ is a cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible mapping. If there exist $\mu_1 \geq 0$, $\mu_2, \mu_3, \mu_4 > 0$, $0 < h < 1$ with $\mu_1 + 2\mu_2 + 2s\mu_3 + 2s\mu_4 = 1$ such that if $\ddot{\sigma}(\theta)\ddot{\lambda}(\varrho) \geq 1$, then*

$$\begin{aligned} s^3\eth(\digamma\theta, \digamma\varrho) \leq\ & \mu_1\eth(\theta, \varrho) + \mu_2[\eth(\theta, \digamma\theta) + \eth(\varrho, \digamma\varrho)] \\ & + \mu_3[\eth(\theta, \digamma\varrho) + \eth(\varrho, \digamma\theta)] + \mu_4[\Delta(\theta, \varrho) + h\delta(\theta, \varrho)] \end{aligned} \tag{1}$$

for all $\theta, \varrho \in \tilde{\Theta}$, where $\Delta(\theta, \varrho) = \max\{\eth(\theta, \digamma\varrho), \eth(\varrho, \digamma\theta)\}$ and $\delta(\theta, \varrho) = \min\{\eth(\theta, \digamma\varrho), \eth(\varrho, \digamma\theta)\}$, then we say that $\digamma$ is a cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible generalized contraction type map.

Example 1 *Let $\tilde{\Theta} = (0, 1) \cup \{1, 2, 3, ...\}$. We define $\eth : \tilde{\Theta} \times \tilde{\Theta} \to [0, \infty)$ by*

$$\eth(\theta, \varrho) = \begin{cases} 0, & \theta = \varrho, \\ (\theta + \varrho)^2, & \theta \neq \varrho. \end{cases}$$

Then $\eth$ is a b-metric with $s = 2$. We define $\digamma : \tilde{\Theta} \to \tilde{\Theta}$ by

$$\digamma(\theta) = \begin{cases} \frac{\theta^2}{100}, & \theta \in (0, 1), \\ 1 + \theta, & \theta \in \{1, 2, 3, ...\}, \end{cases}$$

and $\ddot{\sigma}, \ddot{\lambda} : \tilde{\Theta} \to [0, \infty)$ by

$$\ddot{\sigma}(\theta) = \begin{cases} \frac{3}{1+\theta}, & \theta \in (0, 1), \\ 0, & \textit{otherwise}, \end{cases} \quad \textit{and} \quad \ddot{\lambda}(\theta) = \begin{cases} \frac{4}{1+\theta}, & \theta \in (0, 1), \\ 0, & \textit{otherwise}. \end{cases}$$

It is easy to see that $\digamma$ is a cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible mapping. The next step is to demonstrate that $\digamma$ is a cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible generalized contraction type map. For $\theta, \varrho \in \tilde{\Theta}$ with

$$\ddot{\sigma}(\theta)\ddot{\lambda}(\varrho) \geq 1 \iff \theta, \varrho \in (0, 1).$$

We choose $\mu_1 = \frac{2}{5}$, $\mu_2 = \frac{1}{5}$, $\mu_3 = \frac{1}{40} = \mu_4$, $h = \frac{9}{10}$. *Then we have* $\mu_1 + 2\mu_2 + 2s\mu_3 + 2s\mu_4 = 1$. *For* $\theta \neq \varrho$, *we have*

$$\begin{aligned} s^3 ð(F\theta, F\varrho) &= (2)^3 (\frac{\theta^2}{100} + \frac{\varrho^2}{100})^2 \\ &\leq (\frac{2}{5})(\theta+\varrho)^2 + (\frac{1}{5}) \left[(\theta + \frac{\theta^2}{100})^2 + (\varrho + \frac{\varrho^2}{100})^2 \right] + (\frac{1}{40}) \left[(\theta + \frac{\varrho^2}{100})^2 + (\varrho + \frac{\theta^2}{100})^2 \right] \\ &\quad + (\frac{1}{40}) \left[\max\{(\theta + \frac{\varrho^2}{100})^2, (\varrho + \frac{\theta^2}{100})^2\} + \frac{9}{10} \min\{(\theta + \frac{\varrho^2}{100})^2, (\varrho + \frac{\theta^2}{100})^2\} \right] \\ &\leq \mu_1 ð(\theta, \varrho) + \mu_2 [ð(\theta, F\theta) + ð(\varrho, F\varrho)] + \mu_3 [ð(\theta, F\varrho) + ð(\varrho, F\theta)] \\ &\quad + \mu_4 [\max\{ð(\theta, F\varrho), ð(\varrho, F\theta)\} + h \min\{ð(\theta, F\varrho), ð(\varrho, F\theta)\}]. \end{aligned}$$

Therefore F *is a cyclic* $(\ddot{\sigma}, \ddot{\lambda})$*-admissible generalized contraction type map.*

The following is the main result of this paper.

Theorem 3 *Let* $(\tilde{\Theta}, ð)$ *be a complete b-metric space with* $s \geq 1$ *and* $F : \tilde{\Theta} \to \tilde{\Theta}$ *be a cyclic* $(\ddot{\sigma}, \ddot{\lambda})$*-admissible generalized contraction type map. Further, suppose that there exists* $\theta_0 \in \tilde{\Theta}$ *such that* $\ddot{\sigma}(\theta_0) \geq 1$, $\ddot{\lambda}(\theta_0) \geq 1$ *and either one of the following holds:*

(a_1) F *is b-continuous; or*

(a_2) if $\{\theta_n\} \subseteq \tilde{\Theta}$ *such that* $\theta_n \to \nu$ *and* $\ddot{\lambda}(\theta_n) \geq 1$ *for all* n *then* $\ddot{\lambda}(\nu) \geq 1$.

Then F *has a fixed point and it is unique if* $\ddot{\sigma}(\nu) \geq 1$ *or* $\ddot{\lambda}(\nu) \geq 1$.

Proof. Let $\theta_0 \in \tilde{\Theta}$. Then by the hypotheses, we have $\ddot{\sigma}(\theta_0) \geq 1$ and $\ddot{\lambda}(\theta_0) \geq 1$. We construct a sequence $\{\theta_n\}$ by $\theta_{n+1} = F\theta_n$, $n \in \mathbb{N} \cup \{0\}$. Assume that $\theta_{n_0+1} = \theta_{n_0}$ for some n_0, so that $F\theta_{n_0} = \theta_{n_0+1} = \theta_{n_0}$ and hence the proof.

Without diminishing the generality, we presume that $\theta_n \neq \theta_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\ddot{\sigma}(\theta_0) \geq 1$ and F is a cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible map, we attain $\ddot{\lambda}(\theta_1) = \ddot{\lambda}(F\theta_0) \geq 1$ and that $\ddot{\sigma}(\theta_2) = \ddot{\sigma}(F\theta_1) \geq 1$. Continuing in this manner, we get

$$\ddot{\sigma}(\theta_{2k}) \geq 1 \text{ and } \ddot{\lambda}(\theta_{2k+1}) \geq 1 \text{ for all } k \in \mathbb{N} \cup \{0\}. \tag{2}$$

As $\ddot{\lambda}(\theta_0) \geq 1$ and F is a cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible map, we get $\ddot{\sigma}(\theta_1) = \ddot{\sigma}(F\theta_0) \geq 1$. Therefore it follows that $\ddot{\lambda}(\theta_2) = \ddot{\lambda}(F\theta_1) \geq 1$. Continuing in this way, we get that

$$\ddot{\lambda}(\theta_{2k}) \geq 1 \text{ and } \ddot{\sigma}(\theta_{2k+1}) \geq 1 \text{ for all } k \in \mathbb{N} \cup \{0\}. \tag{3}$$

From the inequalities (2) and (3), we conclude that

$$\ddot{\sigma}(\theta_n) \geq 1 \text{ and } \ddot{\lambda}(\theta_n) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

We now prove that $\lim_{n\to\infty} ð(\theta_n, \theta_{n+1}) = 0$. Since $\ddot{\sigma}(\theta_n)\ddot{\lambda}(\theta_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and from the inequality (1), we get

$$\begin{aligned} s^3 ð(F\theta_n, F\theta_{n+1}) &\leq \mu_1 ð(\theta_n, \theta_{n+1}) + \mu_2 [ð(\theta_n, F\theta_n) + ð(\theta_{n+1}, F\theta_{n+1})] \\ &\quad + \mu_3 [ð(\theta_n, F\theta_{n+1}) + ð(\theta_{n+1}, F\theta_n)] + \mu_4 [\Delta(\theta_n, \theta_{n+1}) + h\delta(\theta_n, \theta_{n+1})], \end{aligned} \tag{4}$$

where

$$\Delta(\theta_n, \theta_{n+1}) = \max\{ð(\theta_n, F\theta_{n+1}), ð(\theta_{n+1}, F\theta_n)\} = ð(\theta_n, \theta_{n+2})$$

and

$$\delta(\theta_n, \theta_{n+1}) = \min\{ð(\theta_n, F\theta_{n+1}), ð(\theta_{n+1}, F\theta_n)\} = 0.$$

From the inequality (4), we have

$$\begin{aligned} s^3\eth(\theta_{n+1},\theta_{n+2}) &\leq \mu_1\eth(\theta_n,\theta_{n+1})+\mu_2[\eth(\theta_n,\theta_{n+1})+\eth(\theta_{n+1},\theta_{n+2})] \\ &\quad +\mu_3[\eth(\theta_n,\theta_{n+2})+\eth(\theta_{n+1},\theta_{n+1})]+\mu_4\eth(\theta_n,\theta_{n+2}). \end{aligned} \tag{5}$$

Now, we suppose that $\eth(\theta_n,\theta_{n+1})<\eth(\theta_{n+1},\theta_{n+2})$ for some $n\in\mathbb{N}\cup\{0\}$ and from the inequality (5),

$$s^3\eth(\theta_{n+1},\theta_{n+2})\leq(\mu_1+2\mu_2+2s\mu_3+2s\mu_4)\eth(\theta_{n+1},\theta_{n+2})=\eth(\theta_{n+1},\theta_{n+2}) \tag{6}$$

which implies that $(s^3-1)\eth(\theta_{n+1},\theta_{n+2})\leq 0$ and hence that $\theta_{n+1}=\theta_{n+2}$, which is a contradiction as $\theta_{n+1}\neq\theta_{n+2}$. Therefore $\eth(\theta_n,\theta_{n+1})\geq\eth(\theta_{n+1},\theta_{n+2})$ for every $n\in\mathbb{N}\cup\{0\}$. Thus, $\{\eth(\theta_n,\theta_{n+1})\}$ is decreasing and there exists $r\geq 0$ such that $\lim_{n\to\infty}\eth(\theta_n,\theta_{n+1})=r$. Let $r>0$. As $n\to\infty$ in (6), we get

$$s^3(r)\leq r \implies (s^3-1)r\leq 0,$$

which a contradiction. Thus, $\lim_{n\to\infty}\eth(\theta_n,\theta_{n+1})=0$. We suppose that $\{\theta_n\}$ is not b-Cauchy. As $\ddot{\sigma}(\theta_{m_k})\geq 1$ and $\ddot{\lambda}(\theta_{n_k})\geq 1$ we obtain $\ddot{\sigma}(\theta_{m_k})\ddot{\lambda}(\theta_{n_k})\geq 1$, using (1) and Lemma 1, we get

$$\begin{aligned} s^3\eth(\theta_{m_k+1},\theta_{n_k+1}) &= s^3\eth(\digamma\theta_{m_k},\digamma\theta_{n_k}) \\ &\leq \mu_1\eth(\theta_{m_k},\theta_{n_k})+\mu_2[\eth(\theta_{m_k},\theta_{m_k+1})+\eth(\theta_{n_k},\theta_{n_k+1})] \\ &\quad +\mu_3[\eth(\theta_{m_k},\theta_{n_k+1})+\eth(\theta_{n_k},\theta_{m_k+1})]+\mu_4[\Delta(\theta_{m_k},\theta_{n_k})+h\delta(\theta_{m_k},\theta_{n_k})], \end{aligned} \tag{7}$$

where

$$\begin{cases} \Delta(\theta_{m_k},\theta_{n_k})=\max\{\eth(\theta_{m_k},\theta_{n_k+1}),\eth(\theta_{n_k},\theta_{m_k+1})\}, \\ \delta(\theta_{m_k},\theta_{n_k})=\min\{\eth(\theta_{m_k},\theta_{n_k+1}),\eth(\theta_{n_k},\theta_{m_k+1})\}. \end{cases} \tag{8}$$

By Lemma 1 and taking limit superior as $k\to\infty$ in (8), we get

$$\begin{cases} \overline{\lim}_{k\to\infty}\Delta(\theta_{m_k},\theta_{n_k})\leq\max\{s^2\epsilon,s^2\epsilon\}=s^2\epsilon, \\ \overline{\lim}_{k\to\infty}\delta(\theta_{m_k},\theta_{n_k})\leq\max\{s^2\epsilon,s^2\epsilon\}=s^2\epsilon. \end{cases} \tag{9}$$

Let $k\to\infty$ in (7) and from (9), we obtain

$$\begin{aligned} s\epsilon &= s^3(\frac{\epsilon}{s^2})\leq s^3[\overline{\lim}_{k\to\infty}\eth(\theta_{m_k+1},\theta_{n_k+1})] \\ &= \overline{\lim}_{k\to\infty}[s^3\eth(\digamma\theta_{m_k},\digamma\theta_{n_k})] \\ &\leq \overline{\lim}_{k\to\infty}[\mu_1\eth(\theta_{m_k},\theta_{n_k})+\mu_2[\eth(\theta_{m_k},\theta_{m_k+1})+\eth(\theta_{n_k},\theta_{n_k+1})] \\ &\quad +\mu_3[\eth(\theta_{m_k},\theta_{n_k+1})+\eth(\theta_{n_k},\theta_{m_k+1})]+\mu_4[\Delta(\theta_{m_k},\theta_{n_k})+h\delta(\theta_{m_k},\theta_{n_k})]] \\ &\leq \mu_1\overline{\lim}_{k\to\infty}\eth(\theta_{m_k},\theta_{n_k})+\mu_2[\overline{\lim}_{k\to\infty}\eth(\theta_{m_k},\theta_{m_k+1})+\overline{\lim}_{k\to\infty}\eth(\theta_{n_k},\theta_{n_k+1})] \\ &\quad +\mu_3[\overline{\lim}_{k\to\infty}\eth(\theta_{m_k},\theta_{n_k+1})+\overline{\lim}_{k\to\infty}\eth(\theta_{n_k},\theta_{m_k+1})] \\ &\quad +\mu_4[\overline{\lim}_{k\to\infty}\Delta(\theta_{m_k},\theta_{n_k})+h\,\overline{\lim}_{k\to\infty}\delta(\theta_{m_k},\theta_{n_k})] \\ &\leq \mu_1(s\epsilon)+2\mu_3(s^2\epsilon)+\mu_4(s^2\epsilon+hs^2\epsilon)\leq(\mu_1+2s\mu_3+2s\mu_4)s\epsilon<s\epsilon, \end{aligned}$$

which is a contradiction. Therefore $\{\theta_n\}$ is a b-Cauchy sequence in $\tilde{\Theta}$. Since $\tilde{\Theta}$ is b-complete, there exists $\nu\in\tilde{\Theta}$ such that $\lim_{n\to\infty}\theta_n=\nu$. Firstly, assume $\digamma$ is b-continuous. Then $\lim_{n\to\infty}\digamma\theta_n=\digamma\nu$ and that $\digamma\nu=\lim_{n\to\infty}\digamma\theta_n=\lim_{n\to\infty}\theta_{n+1}=\nu$. Therefore ν is a fixed point of $\digamma$. We suppose that (a_2) is true, which means, $\ddot{\lambda}(\theta_n)\geq 1$ for all n. Then we have $\ddot{\lambda}(\nu)\geq 1$. Suppose that $\digamma\nu\neq\nu$. The b-triangular inequality gives us

$$\eth(\nu,\digamma\nu)\leq s[\eth(\nu,\digamma\theta_n)+\eth(\digamma\theta_n,\digamma\nu)].$$

If we consider the upper limit to be $n\to\infty$, we obtain

$$\frac{1}{s}\eth(\nu,F\nu)\le\overline{\lim_{n\to\infty}}\,\eth(F\theta_n,F\nu). \tag{10}$$

Also, we have $\eth(F\theta_n,F\nu)\le s[\eth(F\theta_n,\nu)+\eth(\nu,F\nu)]$. Taking upper limit as $n\to\infty$, we get

$$\overline{\lim_{n\to\infty}}\,\eth(F\theta_n,F\nu)\le s\eth(\nu,F\nu). \tag{11}$$

From the inequalities (10) and (11), we have

$$\frac{1}{s}\eth(\nu,F\nu)\le\overline{\lim_{n\to\infty}}\,\eth(F\theta_n,F\nu)\le s\eth(\nu,F\nu). \tag{12}$$

Since $\ddot{\sigma}(\theta_n)\ddot{\lambda}(\nu)\ge 1$, from the inequalities (1) and (12),

$$\begin{aligned}\eth(\nu,F\nu) &\le s^2\eth(\nu,F\nu)\\ &= s^3(\frac{1}{s}\eth(\nu,F\nu))\\ &\le \overline{\lim_{n\to\infty}}\,s^3\eth(F\theta_n,F\nu)\\ &\le \overline{\lim_{n\to\infty}}\,[\mu_1\eth(\theta_n,\nu)+\mu_2[\eth(\theta_n,F\theta_n)+\eth(\nu,F\nu)]\\ &\quad+\mu_3[\eth(\theta_n,F\nu)+\eth(\nu,F\theta_n)]+\mu_4[\Delta(\theta_n,\nu)]+h\delta(\theta_n,\nu)],\end{aligned} \tag{13}$$

where

$$\Delta(\theta_n,\nu)=\max\{\eth(\theta_n,F\nu),\eth(\nu,F\theta_n)\}\ \text{ and }\ \delta(\theta_n,\nu)=\min\{\eth(\theta_n,F\nu),\eth(\nu,F\theta_n)\}.$$

If we take the limit superior as $n\to\infty$ then we get

$$\overline{\lim_{n\to\infty}}\,\Delta(\theta_n,\nu)\le s\eth(\nu,F\nu)\ \text{ and }\ \overline{\lim_{n\to\infty}}\,\delta(\theta_n,\nu)=0.$$

From the inequality (13), we get

$$\eth(\nu,F\nu)\le(\mu_2+s\mu_3+s\mu_4)\eth(\nu,F\nu)<\eth(\nu,F\nu),$$

which is a contradiction. Thus, $F\nu=\nu$. Therefore ν is a fixed point of F in $\tilde{\Theta}$.

Assume that $\nu\ne\nu'$ be two fixed points of F. Then $\ddot{\sigma}(\nu)\ge 1$ or $\ddot{\lambda}(\nu)\ge 1$ and $\ddot{\sigma}(\nu')\ge 1$ or $\ddot{\lambda}(\nu')\ge 1$. Since F is a cyclic $(\ddot{\sigma},\ddot{\lambda})$-admissible mapping, we obtain that

$$\ddot{\sigma}(\nu)\ge 1\implies\ddot{\lambda}(F(\nu))=\ddot{\lambda}(\nu)\ge 1$$

and

$$\ddot{\lambda}(\nu)\ge 1\implies\ddot{\sigma}(F(\nu))=\ddot{\sigma}(\nu)\ge 1.$$

Hence $\ddot{\sigma}(\nu)\ge 1$ and $\ddot{\lambda}(\nu)\ge 1$. Now,

$$\ddot{\sigma}(\nu')\ge 1\implies\ddot{\lambda}(F(\nu'))=\ddot{\lambda}(\nu')\ge 1$$

and

$$\ddot{\lambda}(\nu')\ge 1\implies\ddot{\sigma}(F(\nu'))=\ddot{\sigma}(\nu')\ge 1.$$

Thus, $\ddot{\sigma}(\nu)\ge 1$, $\ddot{\lambda}(\nu)\ge 1$, $\ddot{\sigma}(\nu')\ge 1$ and $\ddot{\lambda}(\nu')\ge 1\implies\ddot{\sigma}(\nu)\ddot{\lambda}(\nu')\ge 1$.

From the inequality (1), we get

$$\begin{aligned}s^3\eth(F\nu,F\nu') &\le \mu_1\eth(\nu,\nu')+\mu_2[\eth(\nu,F\nu)+\eth(\nu',F\nu')]+\mu_3[\eth(\nu',F\nu)+\eth(\nu,F\nu')]\\ &\quad+\mu_4[\Delta(\nu,\nu')+h\delta(\nu,\nu')],\end{aligned} \tag{14}$$

where

$$\Delta(\nu,\nu') = \max\{\eth(\nu', F\nu), \eth(\nu, F\nu')\} = \eth(\nu,\nu')$$

and

$$\delta(\nu,\nu') = \min\{\eth(\nu', F\nu), \eth(\nu, F\nu')\} = \eth(\nu,\nu').$$

From (14), we have

$$s^3\eth(\nu,\nu') \leq (\mu_1 + 2\mu_2 + 2\mu_3 + \mu_4(1+h))\eth(\nu,\nu') < \eth(\nu,\nu'),$$

which is a contradiction. Hence $\nu = \nu'$. This completes the proof. ■

Definition 7 *Suppose $(\tilde{\Theta}, \eth)$ is a b-metric space with parameter $s \geq 1$, $\ddot{\sigma}, \ddot{\lambda} : \tilde{\Theta} \to [0,\infty)$ are two given mappings and $F, \eta : \tilde{\Theta} \to \tilde{\Theta}$ are two self-mappings on $\tilde{\Theta}$ with F is an η-cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible mapping. If there exist $\mu_1 \geq 0$, $\mu_2, \mu_3, \mu_4 > 0$, $0 < h < 1$ with $\mu_1 + 2\mu_2 + 2s\mu_3 + 2s\mu_4 = 1$ such that if $\ddot{\sigma}(\eta\theta)\ddot{\lambda}(\eta\varrho) \geq 1$, then*

$$\begin{aligned} s^3\eth(F\theta, F\varrho) \leq\ & \mu_1\eth(\eta\theta,\eta\varrho) + \mu_2[\eth(\eta\theta, F\theta) + \eth(\eta\varrho, F\varrho)] \\ & + \mu_3[\eth(\eta\theta, F\varrho) + \eth(\eta\varrho, F\theta)] + \mu_4[\Delta_s(\theta,\varrho) + h\delta_s(\theta,\varrho)] \end{aligned} \tag{15}$$

for all $\theta, \varrho \in \tilde{\Theta}$, where $\Delta_s(\theta,\varrho) = \max\{\eth(\eta\theta, F\varrho), \eth(\eta\varrho, F\theta)\}$ and $\delta_s(\theta,\varrho) = \min\{\eth(\eta\theta, F\varrho), \eth(\eta\varrho, F\theta)\}$. Then we say that F is an η-cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible generalized contraction type map.

Example 2 Let $\tilde{\Theta} = (0,1) \cup \{1,2,3,...\}$. We define $\eth : \tilde{\Theta} \times \tilde{\Theta} \to [0,\infty)$ by

$$\eth(\theta,\varrho) = \begin{cases} 0, & \theta = \varrho, \\ (\theta+\varrho)^2, & \theta \neq \varrho. \end{cases}$$

Then, $\eth$ is a b-metric with $s = 2$. We define $F, \eta : \Theta \to \Theta$ by

$$F(\theta) = \begin{cases} \frac{\theta^2}{100}, & \theta \in (0,1), \\ 1 + \frac{2}{3(\theta+1)}, & \theta \in \{1,2,3,...\}, \end{cases}$$

$$\eta(\theta) = \begin{cases} \theta, & \theta \in (0,1), \\ 1+\theta, & \theta \in \{1,2,3,...\}, \end{cases}$$

and $\ddot{\sigma}, \ddot{\lambda} : \tilde{\Theta} \to [0,\infty)$ by

$$\ddot{\sigma}(\theta) = \begin{cases} \frac{4}{\theta}, & \theta \in (0,1), \\ 0, & \textit{otherwise}, \end{cases} \quad \textit{and} \quad \ddot{\lambda}(\theta) = \begin{cases} \frac{3}{\theta}, & \theta \in (0,1), \\ 0, & \textit{otherwise}. \end{cases}$$

Since

$$\ddot{\sigma}(\eta\theta) = \frac{4}{\eta\theta} = \frac{4}{\theta} \geq 1 \iff \theta \in (0,1),$$

we have $\ddot{\lambda}(F\theta) = \frac{3}{F\theta} = \frac{300}{\theta^2} \geq 1$ and also $\theta \in \tilde{\Theta}$, $\ddot{\lambda}(\eta\theta) = \frac{3}{\eta\theta} = \frac{3}{\theta} \geq 1 \iff \theta \in (0,1)$, we have $\ddot{\sigma}(F\theta) = \frac{4}{F\theta} = \frac{400}{\theta^2} \geq 1$. Therefore F is an η-cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible mapping. Now, we show that F is an η-cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible generalized contraction type map.

For $\theta, \varrho \in \tilde{\Theta}$ with $\ddot{\sigma}(\eta\theta)\ddot{\lambda}(\eta\varrho) \geq 1 \iff \theta, \varrho \in (0,1)$. Hence, for $\theta, \varrho \in (0,1)$ with $\theta \neq \varrho$, we choose

$$\mu_1 = \frac{2}{5}, \ \ \mu_2 = \frac{1}{5}, \ \ \mu_3 = \frac{1}{40} = \mu_4, \ \ h = \frac{9}{10}.$$

Then we have $\mu_1 + 2\mu_2 + 2s\mu_3 + 2s\mu_4 = 1$. We now consider

$$s^3\eth(F\theta, F\varrho) = (2)^3(\frac{\theta^2}{100} + \frac{\theta^2}{100})^2$$

$$\begin{aligned}
&\leq \frac{2}{5}(\theta+\varrho)^2+\frac{1}{5}[(\theta+\frac{\theta^2}{100})^2+(\varrho+\frac{\varrho^2}{100})^2]+\frac{1}{40}[(\theta+\frac{\varrho^2}{100})^2+(\varrho+\frac{\theta^2}{100})^2] \\
&\quad +\mu_4\left[\max\{(\theta+\frac{\varrho^2}{100})^2,(\varrho+\frac{\theta^2}{100})^2\}+h\min\{(\theta+\frac{\varrho^2}{100})^2,(\varrho+\frac{\theta^2}{100})^2\}\right] \\
&= \mu_1\eth(\eta\theta,\eta\varrho)+\mu_2[\eth(\eta\theta,\digamma\theta)+\eth(\eta\varrho,\digamma\varrho)]+\mu_3[\eth(\eta\theta,\digamma\varrho)+\eth(\eta\varrho,\digamma\theta)] \\
&\quad +\mu_4[\max\{\eth(\eta\theta,\digamma\varrho),\eth(\eta\varrho,\digamma\theta)\}+h\min\{\eth(\eta\theta,\digamma\varrho),\eth(\eta\varrho,\digamma\theta)\}].
\end{aligned}$$

Thus, $\digamma$ is an η-cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible generalized contraction type map.

Theorem 4 *Let $(\tilde{\Theta}, \eth)$ be a complete b-metric space with coefficient $s \geq 1$ and $\digamma$ be an η-cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible generalized contraction type map. Further, if $\theta_0 \in \tilde{\Theta}$ with $\ddot{\sigma}(\eta\theta_0) \geq 1, \ddot{\lambda}(\eta\theta_0) \geq 1$ and the following conditions hold:*

(b_1) $\digamma(\tilde{\Theta}) \subseteq \eta(\tilde{\Theta})$ with $\eta(\tilde{\Theta})$ is closed subspace of $\tilde{\Theta}$;

(b_2) if $\{\theta_n\} \subseteq \tilde{\Theta}$ with $\ddot{\lambda}(\theta_n) \geq 1$ for all n and $\theta_n \to \nu$, then $\ddot{\lambda}(\nu) \geq 1$;

(b_3) $\ddot{\sigma}(\eta u) \geq 1$ and $\ddot{\lambda}(\eta v) \geq 1$ whenever $\digamma u = \eta u$ and $\digamma v = \eta v$.

Then $\digamma$ and η have a unique common fixed point whenever the pair $(\digamma, \eta)$ is weakly compatible.

Proof. Let $\theta_0 \in \tilde{\Theta}$. Then by the hypotheses, we have $\ddot{\sigma}(\theta_0) \geq 1$ and $\ddot{\lambda}(\theta_0) \geq 1$. We can define two sequences $\{\theta_n\}$ and $\{\varrho_n\}$ in $\tilde{\Theta}$ by $\varrho_n = \digamma\theta_n = \eta\theta_{n+1}$ for all n. Consider $\varrho_{n_0+1} = \varrho_{n_0}$ for some $n_0 \in \mathbb{N} \cup \{0\}$ i.e., θ_{n_0+1} is a coincidence point of $\digamma$ and η. In order to maintain generality, we therefore suppose that $\varrho_{n+1} \neq \varrho_n$ for all $n \in \mathbb{N} \cup \{0\}$. As $\ddot{\sigma}(\eta\theta_0) \geq 1$ and $\digamma$ is an η-cyclic-$(\ddot{\sigma}, \ddot{\lambda})$-admissible map, we obtain $\ddot{\lambda}(\eta\theta_1) = \ddot{\lambda}(\digamma\theta_0) \geq 1$ and that $\ddot{\sigma}(\eta\theta_2) = \ddot{\sigma}(\digamma\theta_1) \geq 1$. Continuing in this manner, we achieve

$$\ddot{\sigma}(\eta\theta_{2k}) \geq 1 \quad \text{and} \quad \ddot{\lambda}(\eta\theta_{2k+1}) \geq 1 \text{ for all } k \in \mathbb{N} \cup \{0\}. \tag{16}$$

Also, we have $\ddot{\lambda}(\eta\theta_0) \geq 1$ and $\digamma$ is an η-cyclic-$(\ddot{\sigma}, \ddot{\lambda})$-admissible map, we get $\ddot{\sigma}(\eta\theta_1) = \ddot{\sigma}(\digamma\theta_0) \geq 1$. Therefore it follows that $\ddot{\lambda}(\eta\theta_2) = \ddot{\lambda}(\digamma\theta_1) \geq 1$. Generally, we continue this process, we can find that

$$\ddot{\lambda}(\eta\theta_{2k}) \geq 1 \text{ and } \ddot{\sigma}(\eta\theta_{2k+1}) \geq 1 \text{ for all } k \in \mathbb{N} \cup \{0\}. \tag{17}$$

Hence, from these inequalities (16) and (17), we conclude that

$$\ddot{\sigma}(\eta\theta_n) \geq 1, \ddot{\lambda}(\eta\theta_n) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\},$$

it suggests that $\ddot{\sigma}(\eta\theta_n)\ddot{\lambda}(\eta\theta_{n+1}) \geq 1$.

From this inequality (15), we have

$$\begin{aligned}
s^3\eth(\digamma\theta_n,\digamma\theta_{n+1}) &\leq \mu_1\eth(\eta\theta_n,\eta\theta_{n+1})+\mu_2[\eth(\eta\theta_n,\digamma\theta_n)+\eth(\eta\theta_{n+1},\digamma\theta_{n+1})] \\
&\quad +\mu_3[\eth(\eta\theta_n,\digamma\theta_{n+1})+\eth(\eta\theta_{n+1},\digamma\theta_n)] \\
&\quad +\mu_4[\Delta_s(\theta_n,\theta_{n+1})+h\delta_s(\theta_n,\theta_{n+1})],
\end{aligned} \tag{18}$$

where

$$\Delta_s(\theta_n,\theta_{n+1}) = \max\{\eth(\eta\theta_n,\digamma\theta_{n+1}),\eth(\eta\theta_{n+1},\digamma\theta_n)\} = \eth(\varrho_{n-1},\varrho_{n+1})$$

and

$$\delta_s(\theta_n,\theta_{n+1}) = \min\{\eth(\eta\theta_n,\digamma\theta_{n+1}),\eth(\eta\theta_{n+1},\digamma\theta_n)\} = 0.$$

From this inequality (18), we have

$$\begin{aligned}
s^3\eth(\varrho_n,\varrho_{n+1}) &\leq \mu_1\eth(\varrho_{n-1},\varrho_n)+\mu_2[\eth(\varrho_{n-1},\varrho_n)+\eth(\varrho_n,\varrho_{n+1})] \\
&\quad +\mu_3[\eth(\varrho_{n-1},\varrho_{n+1})+\eth(\varrho_n,\varrho_n)]+\mu_4\eth(\varrho_{n-1},\varrho_{n+1}).
\end{aligned} \tag{19}$$

If $\eth(\varrho_{n-1}, \varrho_n) < \eth(\varrho_n, \varrho_{n+1})$ for some $n \in \mathbb{N}$ then from (19), we get that

$$s^3\eth(\varrho_n, \varrho_{n+1}) \leq (\mu_1 + 2\mu_2 + 2s\mu_3 + 2s\mu_4)\eth(\varrho_n, \varrho_{n+1}) = \eth(\varrho_n, \varrho_{n+1})$$

which implies that $(s^3-1)\eth(\varrho_n, \varrho_{n+1}) \leq 0$ and that $\varrho_n = \varrho_{n+1}$, which a contradiction. Therefore $\eth(\varrho_{n-1}, \varrho_n) \geq \eth(\varrho_n, \varrho_{n+1})$ for all $n \in \mathbb{N}$. Hence, the sequence $\{\eth(\varrho_n, \varrho_{n+1})\}$ is decreasing in $\tilde{\Theta}$. So there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} \eth(\varrho_n, \varrho_{n+1}) = r$.

Assume that $r > 0$ and from the inequality (19), we obtain

$$s^3\eth(\varrho_n, \varrho_{n+1}) \leq (\mu_1 + 2\mu_2 + 2s\mu_3 + 2s\mu_4)\eth(\varrho_{n-1}, \varrho_n) = \eth(\varrho_{n-1}, \varrho_n). \tag{20}$$

On taking as $n \to \infty$ in (20), we have $s^3(r) \leq r$ implies that $(s^3 - 1)r \leq 0$, which is a contradiction. Thus, $\lim\limits_{n\to\infty} \eth(\varrho_n, \varrho_{n+1}) = 0$. Next, we demonstrate that $\{\varrho_n\}$ is b-Cauchy on the contrary way. Assume that $\{\varrho_n\}$ is not b-Cauchy. We have $\ddot{\sigma}(\varrho_{m_k+1}) \geq 1$ and $\ddot{\lambda}(\varrho_{n_k+1}) \geq 1$ and that $\ddot{\sigma}(\varrho_{m_k+1})\ddot{\lambda}(\varrho_{n_k+1}) \geq 1$. From the inequality (15) and using Lemma 1,

$$\begin{aligned}
s^3\eth(\varrho_{m_k+1}, \varrho_{n_k+1}) &= s^3\eth(F\theta_{m_k+1}, F\theta_{n_k+1}) \\
&\leq \mu_1\eth(\eta\theta_{m_k+1}, \eta\theta_{n_k+1}) + \mu_2[\eth(\eta\theta_{m_k+1}, F\theta_{m_k+1}) + \eth(\eta\theta_{n_k+1}, F\theta_{n_k+1})] \\
&\quad +\mu_3[\eth(\eta\theta_{m_k+1}, F\theta_{n_k+1}) + \eth(\eta\theta_{n_k+1}, F\theta_{m_k+1})] \\
&\quad +\mu_4[\Delta_s(\theta_{m_k+1}, \theta_{n_k+1}) + h\delta_s(\theta_{m_k+1}, \theta_{n_k+1})] \\
&= \mu_1\eth(\varrho_{m_k}, \varrho_{n_k}) + \mu_2[\eth(\varrho_{m_k}, \varrho_{m_k+1}) + \eth(\varrho_{n_k}, \varrho_{n_k+1})] \\
&\quad +\mu_3[\eth(\varrho_{m_k}, \varrho_{n_k+1}) + \eth(\varrho_{n_k}, \varrho_{m_k+1})] \\
&\quad +\mu_4[\Delta_s(\theta_{m_k+1}, \theta_{n_k+1}) + h\delta_s(\theta_{m_k+1}, \theta_{n_k+1})],
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
\varDelta_s(\theta_{m_k+1}, \theta_{n_k+1}) &= \max\{\eth(\eta\theta_{m_k+1}, F\theta_{n_k+1}), \eth(\eta\theta_{n_k+1}, F\theta_{m_k+1})\} \\
&= \max\{\eth(\varrho_{m_k}, \varrho_{n_k+1}), \eth(\varrho_{n_k}, \varrho_{m_k+1})\}
\end{aligned}$$

and

$$\delta_s(\theta_{m_k}, \theta_{n_k}) = \min\{\eth(\varrho_{m_k}, \varrho_{n_k+1}), \eth(\varrho_{n_k}, \varrho_{m_k+1})\}.$$

On taking limit superior as $k \to \infty$ and by Lemma 1, we get

$$\begin{cases} \overline{\lim\limits_{k\to\infty}} \Delta_s(\theta_{m_k+1}, \theta_{n_k+1}) \leq \max\{s^2\epsilon, s^2\epsilon\} = s^2\epsilon, \\ \overline{\lim\limits_{k\to\infty}} \delta_s(\theta_{m_k+1}, \theta_{n_k+1}) \leq \max\{s^2\epsilon, s^2\epsilon\} = s^2\epsilon. \end{cases} \tag{22}$$

Taking limit superior now as $k \to \infty$ in (21), using (22) and (i)–(iv) of Lemma 1, we have

$$\begin{aligned}
s\epsilon &= s^3(\frac{\epsilon}{s^2}) \leq s^3[\overline{\lim\limits_{k\to\infty}} \eth(\varrho_{m_k+1}, \varrho_{n_k+1})] \\
&= \overline{\lim\limits_{k\to\infty}}[s^3\eth(F\theta_{m_k+1}, F\theta_{n_k+1})] \\
&\leq \overline{\lim\limits_{k\to\infty}}[\mu_1\eth(\eta\theta_{m_k+1}, \eta\theta_{n_k+1}) + \mu_2[\eth(\eta\theta_{m_k+1}, F\theta_{m_k+1}) + \eth(\eta\theta_{n_k+1}, F\theta_{n_k+1})] \\
&\quad +\mu_3[\eth(\eta\theta_{m_k+1}, F\theta_{n_k+1}) + \eth(\eta\theta_{n_k+1}, F\theta_{m_k+1})] \\
&\quad +\mu_4[\Delta_s(\theta_{m_k+1}, \theta_{n_k+1}) + h\delta_s(\theta_{m_k+1}, \theta_{n_k+1})]] \\
&\leq \mu_1 \overline{\lim\limits_{k\to\infty}} \eth(\varrho_{m_k}, \varrho_{n_k}) + \mu_2[\overline{\lim\limits_{k\to\infty}} \eth(\varrho_{m_k}, \varrho_{m_k+1}) + \overline{\lim\limits_{k\to\infty}} \eth(\varrho_{n_k}, \varrho_{n_k+1})] \\
&\quad +\mu_3[\overline{\lim\limits_{k\to\infty}} \eth(\varrho_{m_k}, \varrho_{n_k+1}) + \overline{\lim\limits_{k\to\infty}} \eth(\varrho_{n_k}, \varrho_{m_k+1})] \\
&\quad +\mu_4[\overline{\lim\limits_{k\to\infty}} \Delta_s(\theta_{m_k+1}, \theta_{n_k+1}) + h \overline{\lim\limits_{k\to\infty}} \delta_s(\theta_{m_k+1}, \theta_{n_k+1})]
\end{aligned}$$

$$\leq \quad \mu_1(s\epsilon) + 2\mu_3(s^2\epsilon) + \mu_4(s^2\epsilon + hs^2\epsilon) \leq (\mu_1 + 2s\mu_3 + 2s\mu_4)s\epsilon < s\epsilon,$$

which is a contradiction. Thus, $\{\varrho_n\}$ is b-Cauchy in $(\tilde{\Theta}, \eth)$. Since $\tilde{\Theta}$ is b-complete, there exists ν in $\tilde{\Theta}$ such that $\lim_{n\to\infty} \varrho_n = \nu$. Then we have $\lim_{n\to\infty} F\theta_n = \lim_{n\to\infty} \eta\theta_{n+1} = \nu$. Since $\eta(\tilde{\Theta})$ is closed, we can find $u \in \tilde{\Theta}$ such that $\eta u = \nu$. Thus, we prove that $F(u) = \nu$. Suppose $F(u) \neq \nu$. The b-triangular inequality leads to the following:

$$\eth(\nu, Fu) \leq s[\eth(\nu, F\theta_n) + \eth(F\theta_n, Fu)] \implies \frac{1}{s}\eth(\nu, Fu) \leq \overline{\lim_{n\to\infty}}\, \eth(F\theta_n, Fu).$$

Also, $\eth(F\theta_n, Fu) \leq s[\eth(F\theta_n, \nu) + \eth(\nu, Fu)]$ implies

$$\overline{\lim_{n\to\infty}}\, \eth(F\theta_n, Fu) \leq s\eth(\nu, Fu).$$

Thus,

$$\frac{1}{s}\eth(\nu, Fu) \leq \overline{\lim_{n\to\infty}}\, \eth(F\theta_n, Fu) \leq s\eth(\nu, Fu).$$

Since $\varrho_n \to \nu$, it follows that $\ddot{\lambda}(\varrho_n) = \ddot{\lambda}(\eta\theta_{n+1}) \geq 1$. By the hypotheses, we obtain $\ddot{\lambda}(\nu) = \ddot{\lambda}(\eta u) \geq 1$ and that $\ddot{\sigma}(\eta\theta_n)\ddot{\lambda}(\eta u) \geq 1$ for all $n \in \mathbb{N}$. From the inequality (15), we get

$$\begin{aligned} s^3\eth(F\theta_n, Fu) \quad &\leq \quad \mu_1\eth(\eta\theta_n, \eta u) + \mu_2[\eth(\eta\theta_n, F\theta_n) + \eth(\eta u, Fu)] \\ &\quad +\mu_3[\eth(\eta\theta_n, Fu) + \eth(\eta u, F\theta_n)] + \mu_4[\Delta_s(\theta_n, u) + h\delta_s(\theta_n, u)], \end{aligned} \tag{23}$$

where

$$\Delta_s(\theta_n, u) = \max\{\eth(\eta\theta_n, Fu), \eth(\eta u, F\theta_n)\} = \max\{\eth(\varrho_{n-1}, Fu), \eth(\eta u, \varrho_n)\}$$

and

$$\delta_s(\theta_n, u) = \min\{\eth(\varrho_{n-1}, Fu), \eth(\eta u, \varrho_n)\}.$$

On letting $n \to \infty$ in (23), we get

$$\begin{aligned} s^2\eth(\nu, Fu) \quad &= \quad s^3(\frac{1}{s}\eth(\nu, Fu)) \\ &\leq \quad \overline{\lim_{n\to\infty}}\, s^3\eth(F\theta_n, Fu) \\ &\leq \quad \overline{\lim_{n\to\infty}}\, [\mu_1\eth(\eta\theta_n, \eta u) + \mu_2[\eth(\eta\theta_n, F\theta_n) + \eth(\eta u, Fu)] \\ &\quad +\mu_3[\eth(\eta\theta_n, Fu) + \eth(\eta u, F\theta_n)] + \mu_4[\Delta_s(\theta_n, u) + h\delta_s(\theta_n, u)]], \end{aligned} \tag{24}$$

where

$$\overline{\lim_{n\to\infty}}\, \Delta_s(\theta_n, u) \leq \max\{s\eth(\nu, Fu), 0\} = s\eth(\nu, Fu)$$

and

$$\overline{\lim_{n\to\infty}}\, \delta_s(\theta_n, u) \leq \min\{s\eth(\nu, Fu), 0\} = 0.$$

From the inequality (24), we have

$$s^2\eth(\nu, Fu) \leq (\mu_2 + s\mu_3 + s\mu_4)\eth(\nu, Fu) < d(\nu, Fu),$$

which is a contradiction. Thus, $Fu = \eta u = \nu$. We show that F and η has a unique point of coincidence ν. Let $\nu \neq \nu'$ be a point of coincidence. After it, there is $v \in \tilde{\Theta}$ with $Fv = \eta v = \nu'$. By the hypotheses, we have $\ddot{\sigma}(\eta u)\ddot{\lambda}(\eta v) \geq 1$. Thus, from the inequality (15), we get that

$$\begin{aligned} s^3\eth(\nu, v') \quad &= \quad s^3\eth(Fu, Fv) \\ &\leq \quad \mu_1\eth(\eta u, \eta v) + \mu_2[\eth(\eta u, Fu) + \eth(\eta v, Fv)] \\ &\quad +\mu_3[\eth(\eta u, Fv) + \eth(\eta v, Fu)] + \mu_4[\Delta_s(u, v) + h\delta_s(u, v)], \end{aligned}$$

where

$$\Delta_s(u,v) = \max\{\eth(\eta u, \digamma v), \eth(\eta v, \digamma u)\} = \eth(\nu, \nu')$$

and

$$\delta_s(u,v) = \min\{\eth(\eta u, \digamma v), \eth(\eta v, \digamma u)\} = \eth(\nu, \nu').$$

Therefore

$$s^3\eth(\nu,\nu') \le [\mu_1 + 2\mu_3 + \mu_4(1+h)]\eth(\nu,\nu') < \eth(\nu,\nu'),$$

which is a contradiction. Thus, $\nu = \nu'$. Since the pair $(\digamma, \eta)$ is weakly compatible and $\digamma u = \eta u$, we have

$$\digamma\nu = \digamma\eta u = \eta\digamma u = \eta\nu = w.$$

So w is a point of coincidence of $\digamma$ and η. But ν is unique point of coincidence of $\digamma$ and η so $w = \nu$ and that $\nu = \digamma\nu = \eta\nu$. Further, if $\xi = \digamma\xi = \eta\xi$ then ξ is a point of coincidence of $\digamma$ and η. By the uniqueness of coincidence point, we get that $\nu = \xi$. Thus ν is a unique common fixed point of $\digamma$ and η. ■

3 Examples and Corollaries

We start the section with an illustration of Theorem 3.

Example 3 Suppose $\tilde{\Theta} = [1,2] \cup \{3,4,5,...\}$. We define $\eth : \tilde{\Theta} \times \tilde{\Theta} \to [0,\infty)$ by

$$\eth(\theta,\varrho) = \begin{cases} 0, & \theta = \varrho, \\ \frac{1}{\theta+\varrho}, & \theta,\varrho \in [1,2], \\ 4 + \frac{1}{\theta+\varrho}, & \theta,\varrho \in \{3,4,5,...\}, \\ 2, & \textit{otherwise.} \end{cases}$$

It is easy to see that $\eth$ is a b-metric with $s = \frac{27}{26}$. We define $\digamma : \tilde{\Theta} \to \tilde{\Theta}$ by

$$\digamma(\theta) = \begin{cases} 2 - \frac{\theta}{3}, & \theta \in [1,2], \\ 1 + \frac{2}{3\theta}, & \theta \in \{3,4,5,...\}, \end{cases}$$

and $\ddot{\sigma}, \ddot{\lambda} : \tilde{\Theta} \to [0,\infty)$ by

$$\ddot{\sigma}(\theta) = \begin{cases} \frac{3}{1+\theta}, & \theta \in [1,2], \\ 0, & \textit{otherwise,} \end{cases}$$

$$\ddot{\lambda}(\theta) = \begin{cases} \frac{4}{1+\theta}, & \theta \in [1,2], \\ 0, & \textit{otherwise.} \end{cases}$$

Since $\ddot{\sigma}(\theta) \ge 1$ if and only if $\theta \in [1,2]$, we have $\ddot{\lambda}(\digamma(\theta)) = \frac{4}{1+\digamma\theta} = \frac{4}{3-\frac{\theta}{3}} \ge 1$ and also $\theta \in \tilde{\Theta}$, $\ddot{\lambda}(\theta) \ge 1$ if and only if $\theta \in [1,2]$, we have

$$\ddot{\sigma}(\digamma(\theta)) = \frac{3}{1+\digamma\theta} = \frac{3}{3-\frac{\theta}{3}} \ge 1.$$

As a result $\digamma$ is a cyclic $(\ddot{\sigma},\ddot{\lambda})$-admissible mapping. Now we prove that the mapping $\digamma$ is a cyclic $(\ddot{\sigma},\ddot{\lambda})$-admissible generalized contraction type map. For $\theta,\varrho \in \tilde{\Theta}$ with $\ddot{\sigma}(\theta)\ddot{\lambda}(\varrho) \ge 1$ if and only if $\theta,\varrho \in [1,2]$. Hence, for $\theta,\varrho \in [1,2]$, we have $\digamma\theta = 2 - \frac{\theta}{3}$, $\digamma\varrho = 2 - \frac{\varrho}{3}$. We choose

$$\mu_1 = \frac{1}{10}, \;\; \mu_2 = \frac{1}{4}, \;\; \mu_3 = \frac{13}{135} = \mu_4, \;\; h = \frac{9}{10}.$$

Then we have $\mu_1 + 2\mu_2 + 2s\mu_3 + 2s\mu_4 = 1$. For $\theta \ne \varrho$,

$$\eth(\digamma\theta,\digamma\varrho) = \frac{3}{12-(\theta+\varrho)}, \;\; \eth(\theta,\varrho) = \frac{1}{\theta+\varrho}, \;\; \eth(\theta,\digamma\theta) = \frac{3}{2(\theta+1)}, \;\; \eth(\varrho,\digamma\varrho) = \frac{3}{2(1+\varrho)},$$

$$\eth(\theta,\digamma\varrho)=\frac{3}{3\theta-\varrho+2},\quad \eth(\varrho,\digamma\theta)=\frac{3}{3\varrho-\theta+2},$$
$$\max\{\eth(\theta,\digamma\varrho),\eth(\varrho,\digamma\theta)\}=\{\frac{3}{3\theta-\varrho+2},\frac{3}{3\varrho-\theta+2}\}\geq\frac{3}{7}$$

and

$$\min\{\eth(\theta,\digamma\varrho),\eth(\varrho,\digamma\theta)\}=\{\frac{3}{3\theta-\varrho+2},\frac{3}{3\varrho-\theta+2}\}\geq\frac{3}{7}.$$

Now we consider

$$\begin{aligned}
s^3\eth(\digamma\theta,\digamma\varrho) &= (\frac{27}{26})^3(\frac{3}{12-(\theta+\varrho)})\\
&\leq (\frac{27}{26})^3(\frac{3}{8})\\
&\leq (\frac{1}{10})(\frac{1}{4})+(\frac{1}{4})[\frac{1}{2}+\frac{1}{2}]+(\frac{13}{135})[\frac{3}{7}+\frac{3}{7}]+(\frac{13}{135})[\frac{3}{7}+\frac{27}{70}]\\
&\leq \mu_1(\frac{1}{\theta+\varrho})+\mu_2[\frac{3}{2(\theta+1)}+\frac{3}{2(1+\varrho)}]+\mu_3[\frac{3}{3\theta-\varrho+2}+\frac{3}{3\varrho-\theta+2}]\\
&\quad+\mu_4[\max\{\frac{3}{3\theta-\varrho+2},\frac{3}{3\varrho-\theta+2}\}+h\min\{\frac{3}{3\theta-\varrho+2},\frac{3}{3\varrho-\theta+2}\}]\\
&\leq \mu_1\eth(\eta\theta,\eta\varrho)+\mu_2[\eth(\eta\theta,\digamma\theta)+\eth(\eta\varrho,\digamma\varrho)]\\
&\quad+\mu_3[\eth(\eta\theta,\digamma\varrho)+\eth(\eta\varrho,\digamma\theta)]+\mu_4[\Delta(\theta,\varrho)+h\delta(\theta,\varrho)].
\end{aligned}$$

Therefore $\digamma$ satisfies all the hypotheses of Theorem 3 and $\frac{3}{2}$ is the unique fixed point.

The following is an example in support of Theorem 4.

Example 4 Let $\tilde{\Theta}=[1,2]\cup\{3,4,5,...\}$. We define $\eth:\tilde{\Theta}\times\tilde{\Theta}\to[0,\infty)$ by

$$\eth(\theta,\varrho)=\begin{cases}0, & \theta=\varrho,\\ \frac{1}{\theta}+\frac{1}{\varrho}, & \theta,\varrho\in[1,2],\\ 5+\frac{1}{\theta}+\frac{1}{\varrho}, & \theta,\varrho\in\{3,4,5,...\},\\ \frac{5}{2}, & \textit{otherwise.}\end{cases}$$

It is obvious that $\eth$ is a b-metric with $s=\frac{8}{7}$. We define $\digamma,\eta:\tilde{\Theta}\to\tilde{\Theta}$ by

$$\digamma(\theta)=\begin{cases}\theta(3-\theta), & \theta\in[1,2],\\ 1+\frac{2}{3(\theta+1)}, & \theta\in\{3,4,5,...\},\end{cases}$$

$$\eta(\theta)=\begin{cases}\frac{\theta^2+2}{3}, & \theta\in[1,2],\\ 1+\frac{2}{3(\theta+1)}, & \theta\in\{3,4,5,...\},\end{cases}$$

and $\ddot{\sigma},\ddot{\lambda}:\tilde{\Theta}\to[0,\infty)$ by

$$\ddot{\sigma}(\theta)=\begin{cases}\frac{4}{\theta}, & \theta\in[1,2],\\ 0, & \textit{otherwise,}\end{cases}$$

$$\ddot{\lambda}(\theta)=\begin{cases}\frac{3}{\theta} & \theta\in[1,2],\\ 0 & \textit{otherwise.}\end{cases}$$

Since $\ddot{\sigma}(\eta\theta)=\frac{4}{\eta\theta}=\frac{12}{\theta^2+2}\geq1$ if and only if $\theta\in[1,2]$, we have $\ddot{\lambda}(\digamma(\theta))=\frac{3}{\digamma\theta}=\frac{3}{\theta(3-\theta)}\geq1$ and also $\theta\in\tilde{\Theta}$, $\ddot{\lambda}(\eta\theta)=\frac{3}{\eta\theta}=\frac{9}{\theta^2+2}\geq1$ if and only if $\theta\in[1,2]$, we have $\ddot{\sigma}(\digamma(\theta))=\frac{4}{\digamma\theta}=\frac{4}{\theta(3-\theta)}\geq1$. Therefore $\digamma$ is an η-cyclic $(\ddot{\sigma},\ddot{\lambda})$-admissible mapping. Clearly $\digamma(\tilde{\Theta})\subseteq\eta(\tilde{\Theta})$ and $\eta(\tilde{\Theta})$ is a closed subspace of $\tilde{\Theta}$. Now we show that $\digamma$ is an η-cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible generalized contraction type map. For $\theta, \varrho \in \tilde{\Theta}$ with $\ddot{\sigma}(\eta\theta)\ddot{\lambda}(\eta\varrho) \geq 1$ if and only if $\theta, \varrho \in [1,2]$. Hence, for $\theta, \varrho \in [1,2]$ with $\theta \neq \varrho$, we choose

$$\mu_1 = \frac{1}{4},\ \ \mu_2 = \frac{1}{2},\ \ \mu_3 = \frac{7}{128} = \mu_4,\ \ h = \frac{9}{10}.$$

Then we have $\mu_1 + 2\mu_2 + 2s\mu_3 + 2s\mu_4 = 1$. Now

$$\eth(\digamma\theta, \digamma\varrho) = \frac{1}{\theta(3-\theta)} + \frac{1}{\varrho(3-\varrho)}, \eth(\eta\theta, \eta\varrho) = \frac{3}{\theta^2+2} + \frac{3}{\varrho^2+2}, \eth(\eta\theta, \digamma\theta) = \frac{3}{\theta^2+2} + \frac{1}{\theta(3-\theta)},$$

$$\eth(\eta\varrho, \digamma\varrho) = \frac{3}{\varrho^2+2} + \frac{1}{\varrho(3-\varrho)}, \eth(\eta\theta, \digamma\varrho) = \frac{3}{\theta^2+2} + \frac{1}{\varrho(3-\varrho)}, \eth(\eta\varrho, \digamma\theta) = \frac{3}{\varrho^2+2} + \frac{1}{\theta(3-\theta)},$$

$$\max\{\eth(\eta\theta, \digamma\varrho), \eth(\eta\varrho, \digamma\theta)\} = \{\frac{3}{\theta^2+2} + \frac{1}{\varrho(3-\varrho)}, \frac{3}{\varrho^2+2} + \frac{1}{\theta(3-\theta)}\} \geq \frac{1}{2}$$

and

$$\min\{\eth(\eta\theta, \digamma\varrho), \eth(\eta\varrho, \digamma\theta)\} = \{\frac{3}{\theta^2+2} + \frac{1}{\varrho(3-\varrho)}, \frac{3}{\varrho^2+2} + \frac{1}{\theta(3-\theta)}\} \geq \frac{1}{2}.$$

Now we consider

$$\begin{aligned}
s^3\eth(\digamma\theta, \digamma\varrho) &= (\frac{27}{26})^3(\frac{1}{\theta(3-\theta)} + \frac{1}{\varrho(3-\varrho)}) \\
&\leq (\frac{8}{7})^3(\frac{1}{2} + \frac{1}{2}) \\
&\leq (\frac{1}{4})(\frac{1}{2} + \frac{1}{2}) + (\frac{1}{2})[\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}] + (\frac{7}{128})[\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}] \\
&\quad + (\frac{7}{128})[\frac{1}{2} + \frac{1}{2} + \frac{9}{10}(\frac{1}{2} + \frac{1}{2})] \\
&\leq \mu_1(\frac{3}{\theta^2+2} + \frac{3}{\varrho^2+2}) + \mu_2[\frac{3}{\theta^2+2} + \frac{1}{\theta(3-\theta)} + \frac{3}{\varrho^2+2} + \frac{1}{\varrho(3-\varrho)}] \\
&\quad + \mu_3[\frac{3}{\theta^2+2} + \frac{1}{\varrho(3-\varrho)} + \frac{3}{\varrho^2+2} + \frac{1}{\theta(3-\theta)}] + \mu_4[\frac{1}{2} + \frac{9}{10}\frac{1}{2}] \\
&\leq \mu_1\eth(\eta\theta, \eta\varrho) + \mu_2[\eth(\eta\theta, \digamma\theta) + \eth(\eta\varrho, \digamma\varrho)] + \mu_3[\eth(\eta\theta, \digamma\varrho) + \eth(\eta\theta, \digamma\varrho)] \\
&\quad + \mu_4[\max\{\eth(\eta\theta, \digamma\varrho), \eth(\eta\varrho, \digamma\theta)\} + h\min\{\eth(\eta\theta, \digamma\varrho), \eth(\eta\varrho, \digamma\theta)\}].
\end{aligned}$$

Therefore $\digamma$ and η satisfy all the hypotheses of Theorem 4 and 2 is the unique common fixed point.

If we take $\ddot{\sigma}(\theta) = \ddot{\lambda}(\theta) = 1$ in Theorem 3 and Theorem 4, we get Corollary 1 and Corollary 2 respectively.

Corollary 1 *Let $(\tilde{\Theta}, \eth)$ be a complete b-metric space with parameter $s \geq 1$. Suppose $\digamma : \tilde{\Theta} \to \tilde{\Theta}$ be a continuous self-map satisfies the following:*

$$\begin{aligned}
s^3\eth(\digamma\theta, \digamma\varrho) &\leq \mu_1\eth(\theta, \varrho) + \mu_2[\eth(\theta, \digamma\theta) + \eth(\varrho, \digamma\varrho)] \\
&\quad + \mu_3[\eth(\theta, \digamma\varrho) + \eth(\varrho, \digamma\theta)] + \mu_4[\Delta(\theta, \varrho) + h\delta(\theta, \varrho)],
\end{aligned}$$

where $\Delta(\theta, \varrho) = \max\{\eth(\theta, \digamma\varrho), \eth(\varrho, \digamma\theta)\}$, $\delta(\theta, \varrho) = \min\{\eth(\theta, \digamma\varrho), \eth(\varrho, \digamma\theta)\}$, for all $\theta, \varrho \in \tilde{\Theta}$ and the exist $\mu_1 \geq 0$, $\mu_2, \mu_3, \mu_4 > 0$, $0 < h < 1$ with $\mu_1 + 2\mu_2 + 2s\mu_3 + 2s\mu_4 = 1$. Then $\digamma$ has a unique fixed point.

Remark 1 *Corollary 1 extends and generalizes Theorem 2 to b-metric space.*

Corollary 2 *Suppose $(\tilde{\Theta}, \eth)$ is a complete b-metric space with parameter $s \geq 1$ and $\digamma, \eta : \tilde{\Theta} \to \tilde{\Theta}$ are two self-maps with $\digamma(\tilde{\Theta}) \subseteq \eta(\tilde{\Theta}), \eta(\tilde{\Theta})$ is closed subspace of $\tilde{\Theta}$ and $\digamma, \eta$ satisfy the following:*

$$s^3\eth(\digamma\theta, \digamma\varrho) \leq \mu_1\eth(\eta\theta, \eta\varrho) + \mu_2[\eth(\eta\theta, \digamma\theta) + \eth(\eta\varrho, \digamma\varrho)]$$

$$+\mu_3[\eth(\eta\theta, F\varrho) + \eth(\eta\varrho, F\theta)] + \mu_4[\Delta_s(\theta, \varrho) + h\delta_s(\theta, \varrho)],$$

where

$$\Delta_s(\theta, \varrho) = \max\{\eth(\eta\theta, F\varrho), \eth(\eta\varrho, F\theta)\}, \quad \delta_s(\theta, \varrho) = \min\{\eth(\eta\theta, F\varrho), \eth(\eta\varrho, F\theta)\},$$

for all $\theta, \varrho \in \tilde{\Theta}$ *and there exist* $\mu_1 \geq 0$, $\mu_2, \mu_3, \mu_4 > 0$, $0 < h < 1$ *with* $\mu_1 + 2\mu_2 + 2s\mu_3 + 2s\mu_4 = 1$. *If the pair* (F, η) *is weakly compatible, then there is a unique common fixed point for* F *and* η *in* $\tilde{\Theta}$.

4 Application to Nonlinear Integral Equations

Suppose $\tilde{\Omega} = \tilde{C}[a_l, b_u]$ is a set of real valued continuous functions on $[a_l, b_u]$. We define $\eth : \tilde{\Omega} \times \tilde{\Omega} \to \mathbb{R}^+$ by

$$\eth(\zeta, \xi) = \max_{\varsigma \in [a_l, b_u]} |\zeta(\tau) - \xi(\tau)|^{\wp},$$

for all $\zeta, \xi \in \tilde{\Omega}$ with $s = 2^{\wp - 1}$, $\wp > 1$ a real number. In this section, we present unique solution to nonlinear integral equations of the Fredholm type defined by

$$\zeta(\varsigma) = \aleph(\varsigma) + \mu \int_{a_l}^{b_u} \tilde{\mathcal{D}}(\varsigma, \tau, \zeta(\varsigma)) d\tau \tag{25}$$

where $\zeta \in \tilde{C}[a_l, b_u]$, $\mu \in \mathbb{R}$, $\varsigma, \tau \in [a_l, b_u]$, $\tilde{\mathcal{D}} : [a_l, b_u] \times [a_l, b_u] \times \mathbb{R} \to \mathbb{R}$ and $\aleph : [a_l, b_u] \to \mathbb{R}$ are continuous.

Let $\Im : \tilde{\Omega} \to \tilde{\Omega}$ be a mappings defined as

$$\Im(\zeta(\varsigma)) = \aleph(\varsigma) + \mu \int_{a_l}^{b_u} \tilde{\mathcal{D}}(\varsigma, \tau, \zeta(\varsigma)) d\tau. \tag{26}$$

Considering the following:

($\Im_1$) $\gamma : [a_l, b_u] \times [a_l, b_u] \to \mathbb{R}^+$ is continuous with

$$\max_{\tau \in [a_l, b_u]} \int_{a_l}^{b_u} \gamma(\varsigma, \tau) d\tau \leq \frac{1}{(b_u - a_l)^{\wp - 1}} \quad \text{and} \quad |\mu| \leq 1;$$

($\Im_2$) there exists $\zeta_0 \in \tilde{\Omega}$ such that $\eta_1(\zeta_0) \geq 0$ and $\eta_2(\zeta_0) \geq 0$;

($\Im_3$) if $\{\zeta_n\} \subseteq \tilde{\Omega}$ such that $\lim_{n \to \infty} \zeta_n = \zeta$ and $\eta_2(\zeta_n) \geq 0$ for all n, then $\eta_2(\zeta) \geq 0$;

($\Im_4$) $\eta_1(\zeta) \geq 0$ for some $\zeta \in \tilde{\Omega} \implies \eta_2(\Im\zeta) \geq 0$ and $\eta_2(\zeta) \geq 0$ for some $\zeta \in \tilde{\Omega} \implies \eta_1(\Im\zeta) \geq 0$;

($\Im_5$) $\eta_1(u) \geq 0$ and $\eta_2(v) \geq 0$ whenever $\Im u = u$ and $\Im v = v$;

($\Im_6$) if $\eta_1(\Im\zeta) \geq 0$, $\eta_2(\Im\xi) \geq 0$, for all $\zeta, \xi \in \tilde{\Omega}$ such that for $\varsigma, \tau \in [a_l, b_u]$, the following holds:

$$\left|\tilde{\mathcal{D}}(\varsigma, \tau, \zeta(\varsigma)) - \tilde{\mathcal{D}}(\varsigma, \tau, \xi(\varsigma)\right|^{\wp} \leq \frac{1}{2^{3\wp - 3}} \gamma(\varsigma, \tau) \nabla_s(\zeta, \xi)$$

where

$$\begin{aligned} \nabla_s(\zeta, \xi) \quad = \quad & \mu_1 |\zeta - \xi|^{\wp} + \mu_2 [|\zeta - \Im\zeta|^{\wp} + |\xi - \Im\xi|^{\wp}] \\ & + \mu_3[|\zeta - \Im\xi|^{\wp} + |\xi - \Im\zeta|^{\wp}] \\ & + \mu_4[\max\{|\zeta - \Im\xi|^{\wp}, |\xi - \Im\zeta|^{\wp}\} + h \min\{|\zeta - \Im\xi|^{\wp}, |\xi - \Im\zeta|^{\wp}\}]. \end{aligned}$$

Theorem 5 *Let $\Im : \tilde{\Omega} \to \tilde{\Omega}$ be defined by (26) for which the conditions $(\Im_1)$–$(\Im_6)$ hold. Then (25) has a unique solution in $\tilde{\Omega}$.*

Proof. Now we prove that $\Im$ is a cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible generalized contraction type map. We define $\ddot{\sigma}, \ddot{\lambda} : \tilde{\Omega} \to \tilde{\Omega}$ as

$$\ddot{\sigma}(\zeta) = \begin{cases} 1, & \eta_1(\zeta) > 0 \text{ where } \zeta \in \tilde{\Omega}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\ddot{\lambda}(\zeta) = \begin{cases} 1, & \eta_2(\zeta) > 0 \text{ where } \zeta \in \tilde{\Omega}, \\ 0, & \text{otherwise.} \end{cases}$$

From the condition $(\Im_6)$, $\eta_1(\Im\zeta) \geq 0$ and $\eta_2(\Im\xi) \geq 0$, for all $\zeta, \xi \in \tilde{\Omega}$ so that $\ddot{\sigma}(\Im\zeta)\ddot{\lambda}(\Im\xi) \geq 1$. Let $q \in \mathbb{R}$ be such that $\frac{1}{\wp} + \frac{1}{q} = 1$ using the Hölder's inequality and $\zeta, \xi \in \tilde{\Omega}$ and from $(\Im_1)$ and $(\Im_6)$, for all ς. So we have

$$\begin{aligned}
\eth(\Im\zeta, \Im\xi) &= \max_{\varsigma\in[a_l,b_u]} |\Im\zeta(\varsigma) - \Im\xi(\varsigma)|^{\wp} \\
&= \max_{\varsigma\in[a_l,b_u]} |\mu|^{\wp} \left| \int_{a_l}^{b_u} \tilde{\mathcal{D}}(\varsigma,\tau,\zeta(\varsigma))d\tau - \int_{a_l}^{b_u} \tilde{\mathcal{D}}(\varsigma,\tau,\xi(\varsigma))d\tau \right|^{\wp} \\
&= \max_{\varsigma\in[a_l,b_u]} |\mu|^{\wp} \left| \int_{a_l}^{b_u} (\tilde{\mathcal{D}}(\varsigma,\tau,\zeta(\varsigma)) - \tilde{\mathcal{D}}(\varsigma,\tau,\xi(\varsigma)))d\tau \right|^{\wp} \\
&\leq \left[\max_{\varsigma\in[a_l,b_u]} |\mu|^{\wp} \left(\int_{a_l}^{b_u} 1^{\wp} d\tau \right)^{\frac{1}{q}} \left(\int_{a_l}^{b_u} \left| (\tilde{\mathcal{D}}(\varsigma,\tau,\zeta(\varsigma)) - \tilde{\mathcal{D}}(\varsigma,\tau,\xi(\varsigma)))d\tau \right|^{\wp} \right)^{\frac{1}{\wp}} \right]^{\wp} \\
&\leq (b_u - a_l)^{\frac{\wp}{q}} \max_{\varsigma\in[a_l,b_u]} \left(\int_{a_l}^{b_u} \left| \tilde{\mathcal{D}}(\varsigma,\tau,\zeta(\varsigma) - \tilde{\mathcal{D}}(\varsigma,\tau,\xi(\varsigma) \right|^{\wp} d\tau \right) \\
&= (b_u - a_l)^{\wp-1} \max_{\varsigma\in[a_l,b_u]} \left(\int_{a_l}^{b_u} \left| \tilde{\mathcal{D}}(\varsigma,\tau,\zeta(\varsigma) - \tilde{\mathcal{D}}(\varsigma,\tau,\xi(\varsigma) \right|^{\wp} d\tau \right) \\
&\leq (b_u - a_l)^{\wp-1} \max_{\varsigma\in[a_l,b_u]} \int_{a_l}^{b_u} \frac{1}{2^{3\wp-3}} \gamma(\varsigma,\tau) \nabla_s(\zeta,\xi) \\
&= \max_{\varsigma\in[a_l,b_u]} \int_{a_l}^{b_u} \frac{(b_u - a_l)^{\wp-1}}{2^{3\wp-3}} \gamma(\varsigma,\tau)[\mu_1 |\zeta - \xi|^{\wp} + \mu_2[|\zeta - \Im\zeta|^{\wp} + |\xi - \Im\xi|^{\wp}] \\
&\quad + \mu_3[|\zeta - \Im\xi|^{\wp} + |\xi - \Im\zeta|^{\wp}] + \mu_4[\max\{|\zeta - \Im\xi|^{\wp}, |\xi - \Im\zeta|^{\wp}\} \\
&\quad + h \min\{|\zeta - \Im\xi|^{\wp}, |\xi - \Im\zeta|^{\wp}\}]]
\end{aligned}$$

which implies that

$$\begin{aligned}
s^3 \eth(\Im\zeta, \Im\xi) \leq\ & \mu_1 \eth(\zeta,\xi) + \mu_2[\eth(\zeta,\Im\zeta) + \eth(\xi,\Im\xi)] + \mu_3[\eth(\zeta,\Im\xi) + \eth(\xi,\Im\zeta)] \\
& + \mu_4 \left[\max\{\eth(\zeta,\Im\xi), \eth(\xi,\Im\zeta)\} + h \min\{\eth(\zeta,\Im\xi), \eth(\xi,\Im\zeta)\} \right].
\end{aligned}$$

The hypotheses of Theorem 3 is satisfied. Hence $\Im$ defined in (26) has a unique solution. ■

5 Application to Dynamic Programming

We discuss the following existence of bounded solution for functional equations that arises in dynamic programming. Let 'opt' represents inf or sup, $\tilde{\Theta}_1$ and $\tilde{\Theta}_2$ are two Banach spaces; $\tilde{\mathcal{D}} \subseteq \tilde{\Theta}_1$ is the decision space; $\tilde{\mathcal{S}} \subseteq \tilde{\Theta}_2$ is the state space; $\mho(\tilde{\mathcal{S}})$, the set of all bounded real valued functions on $\tilde{\mathcal{S}}$ with b-metric is defined by:

$$\eth(p_x, p_y) = \sup_{t \in \tilde{\mathcal{S}}} |p_x(t) - p_y(t)|^r, \text{ for all } p_x, p_y \in \mho(\tilde{\mathcal{S}}) \text{ with parameter } s = 2^{r-1}.$$

Now we consider the following functional equations:

$$\begin{cases} F(v_s) = \text{opt}_{v_d \in \tilde{\mathcal{D}}}\{\zeta_1(v_s, v_d) + \mathcal{C}_1(v_s, v_d, F(\omega_1(v_s, v_d)))\} \text{ for all } v_s \in \tilde{\mathcal{S}}, \\ \eta(v_s) = \text{opt}_{v_d \in \tilde{\mathcal{D}}}\{\zeta_2(v_s, v_d) + \mathcal{C}_2(v_s, v_d, \eta(\omega_2(v_s, v_d)))\} \text{ for all } v_s \in \tilde{\mathcal{S}}, \end{cases} \tag{27}$$

where v_d is a decision vector, v_s is a state vector where as ω_1, ω_2 denotes the transformations of the process, and $F(v_s), \eta(v_s)$ indicates the optimal return functions.

Let $\Im, \aleph : \mho(\tilde{\mathcal{S}}) \to \mho(\tilde{\mathcal{S}})$ be two mappings defined by:

$$\begin{cases} \Im f(v_s) = \text{opt}_{v_d \in \tilde{\mathcal{D}}}\{\zeta_1(v_s, v_d) + \mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d)))\} \text{ for all } v_s \in \tilde{\mathcal{S}}, \\ \aleph f(v_s) = \text{opt}_{v_d \in \tilde{\mathcal{D}}}\{\zeta_2(v_s, v_d) + \mathcal{C}_2(v_s, v_d, f(\omega_2(v_s, v_d)))\} \text{ for all } v_s \in \tilde{\mathcal{S}}, \end{cases} \tag{28}$$

where $(v_s, f) \in \tilde{\mathcal{S}} \times \mho(\tilde{\mathcal{S}})$.

Let $\xi_1, \xi_2 : \mho(\tilde{\mathcal{S}}) \times \mathbb{R}$. Assume the following:

(DP_1) $\Im(\mho(\tilde{\mathcal{S}})) \subseteq \aleph(\mho(\tilde{\mathcal{S}}))$ and $\aleph(\mho(\tilde{\mathcal{S}}))$ is closed subspace of $\mho(\tilde{\mathcal{S}})$,

(DP_2) if $f_0 \in \mho(\tilde{\mathcal{S}})$ such that $\xi_1(\aleph f_0) \geq 0$ and $\xi_2(\aleph f_0) \geq 0$,

(DP_3) if $\{f_n\}$ is a sequence in $\mho(\tilde{\mathcal{S}})$ such that $\lim_{n\to\infty} f_n = f$ and $\xi_2(f_n) \geq 0$ for all n then $\xi_2(f) \geq 0$,

(DP_4) if $\xi_1(\Im f) \geq 0$, $\xi_2(\aleph g) \geq 0$, for all $f, g \in \mho(\tilde{\mathcal{S}})$ and there exist $\mu_1 \geq 0$, $\mu_2, \mu_3, \mu_4 > 0$, $0 < h < 1$ such that $\mu_1 + 2\mu_2 + 2s\mu_3 + 2s\mu_4 = 1$ then for all $(v_s, v_d, f, g) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{D}} \times \mho(\tilde{\mathcal{S}}) \times \mho(\tilde{\mathcal{S}})$, we have

$$|\mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d)))| + |\zeta_1(v_s, v_d) - \zeta_2(v_s, v_d)| \leq (2^{3-3r}\Delta_s(f, g))^{\frac{1}{r}},$$

where

$$\begin{aligned} \Delta_s(f, g) \quad = \quad & \mu_1 |\aleph f - \aleph g|^r + \mu_2[|\aleph f - \Im f|^r + |\aleph g - \Im g|^r] \\ & +\mu_3[|\aleph f - \Im g|^r + |\aleph g - \Im f|^r] \\ & +\mu_4[\max\{|\aleph f - \Im g|^r, |\aleph g - \Im f|^r\} + h \min\{|\aleph f - \Im g|^r, |\aleph g - \Im f|^r\}]. \end{aligned}$$

(DP_5) $\xi_1(\aleph f) \geq 0$ for some $f \in \mho(\tilde{\mathcal{S}}) \implies \xi_2(\Im f) \geq 0$ and $\xi_2(\aleph f) \geq 0$ for some $f \in \mho(\tilde{\mathcal{S}}) \implies \xi_1(\Im f) \geq 0$,

(DP_6) $\xi_1(\aleph u) \geq 0$ and $\xi_2(\aleph v) \geq 0$ whenever $\Im u = \aleph u$ and $\Im v = \aleph v$,

(DP_7) for some $f \in \mho(\tilde{\mathcal{S}}), \Im\aleph f = \aleph\Im f$ whenever $\Im f = \aleph f$,

(DP_8) $\omega_i, \mathcal{C}_i$ are bounded for $i = 1, 2$.

Theorem 6 *Suppose $\Im, \aleph : \mho(\tilde{\mathcal{S}}) \to \mho(\tilde{\mathcal{S}})$ are defined by (28) for which the conditions (DP_1)–(DP_8) hold. Then (27) has a unique bounded common solution in $\mho(\tilde{\mathcal{S}})$.*

Proof. Take $\epsilon > 0$. Let $p \in \tilde{\mathcal{S}}$, $f, g \in \mho(\tilde{\mathcal{S}})$ with $\zeta_1(\aleph f) \geq 0$ and $\zeta_2(\aleph g) \geq 0$.Since $\omega_i, \mathcal{C}_i$ are bounded for $i = 1, 2$, there exists $L \geq 0$ such that

$$\sup\{\|\omega_1(v_s, v_d)\|, \|\omega_2(v_s, v_d)\|, \|\mathcal{C}_1(v_s, v_d, t)\|, \|\mathcal{C}_2(v_s, v_d, t)\| : (v_s, v_d, t) \in \tilde{\mathcal{S}} \times \tilde{\mathcal{D}} \times \mathbb{R}\} \leq L. \tag{29}$$

Now we demonstrate $\Im$ is an $\aleph$-cyclic $(\ddot{\sigma}, \ddot{\lambda})$-admissible generalized contraction type map. We define $\ddot{\sigma}, \ddot{\lambda} : \mho(\tilde{\mathcal{S}}) \to \mho(\tilde{\mathcal{S}})$ as

$$\ddot{\sigma}(f) = \begin{cases} 1, & \xi_1(f) > 0 \textit{ where } f \in \mho(\tilde{\mathcal{S}}), \\ 0, & \textit{otherwise}, \end{cases}$$

and

$$\ddot{\lambda}(f) = \begin{cases} 1, & \xi_2(f) > 0 \textit{ where } f \in \mho(\tilde{\mathcal{S}}), \\ 0, & \textit{otherwise}. \end{cases}$$

From the condition (DP_4), $\xi_1(\Im f) \geq 0$ and $\xi_2(\aleph g) \geq 0$, for all $f, g \in \mho(\tilde{\mathcal{S}})$ so that$\ddot{\sigma}(\aleph f)\ddot{\lambda}(\aleph g) \geq 1$. First, we assume that $opt_{v_d \in \tilde{\mathcal{D}}} = \inf\limits_{v_d \in \tilde{\mathcal{D}}}$.By using (28), we can find $v_d \in \tilde{\mathcal{D}}$ and $(v_s, f, g) \in \tilde{\mathcal{S}} \times \mho(\tilde{\mathcal{S}}) \times \mho(\tilde{\mathcal{S}})$ such that

$$\Im f(v_s) > \mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) + \zeta_1(v_s, v_d) - \epsilon, \tag{30}$$

$$\Im g(v_s) > \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d))) + \zeta_2(v_s, v_d) - \epsilon, \tag{31}$$

$$\Im f(v_s) \leq \mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) + \zeta_1(v_s, v_d), \tag{32}$$

$$\Im g(v_s) \leq \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d))) + \zeta_2(v_s, v_d). \tag{33}$$

From (30) and (33), we get

$$\begin{aligned} & \Im f(v_s) - \Im g(v_s) \\ > \; & \mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d))) + \zeta_1(v_s, v_d) - \zeta_2(v_s, v_d) - \epsilon \\ \geq \; & -\{|\mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d)))| + |\zeta_1(v_s, v_d) - \zeta_2(v_s, v_d)| + \epsilon\}. \end{aligned} \tag{34}$$

Also by using (31) and (32), we have

$$\begin{aligned} & \Im f(v_s) - \Im g(v_s) \\ \leq \; & \mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d))) + \zeta_1(v_s, v_d) - \zeta_2(v_s, v_d) + \epsilon \\ \leq \; & |\mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d)))| + |\zeta_1(v_s, v_d) - \zeta_2(v_s, v_d)| + \epsilon. \end{aligned} \tag{35}$$

From (34) and (35), we have

$$\begin{aligned} |\Im f(v_s) - \Im g(v_s)| &< \mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d))) + \zeta_1(v_s, v_d) - \zeta_2(p, q) + \epsilon \\ &\leq |\mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d)))| + |\zeta_1(v_s, v_d) - \zeta_2(p, q)| + \epsilon. \end{aligned}$$

Suppose that $opt_{v_d \in \tilde{\mathcal{D}}} = \sup\limits_{v_d \in \tilde{\mathcal{D}}}$.Again by using the inequality (28), we can find $v_d \in \tilde{\mathcal{D}}$ and $(v_s, f, g) \in \tilde{\mathcal{S}} \times \mho(\tilde{\mathcal{S}}) \times \mho(\tilde{\mathcal{S}})$ such that

$$\Im f(v_s) < \mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) + \zeta_1(v_s, v_d) + \epsilon, \tag{36}$$

$$\Im g(v_s) < \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d))) + \zeta_2(v_s, v_d) + \epsilon, \tag{37}$$

$$\Im f(v_s) \geq \mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) + \zeta_1(v_s, v_d), \tag{38}$$

$$\Im g(v_s) \geq \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d))) + \zeta_2(v_s, v_d). \tag{39}$$

From (36) and (39), we get

$$\begin{aligned} & \Im f(v_s) - \Im g(v_s) \\ < \; & \mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d))) + \zeta_1(v_s, v_d) - \zeta_2(v_s, v_d) + \epsilon \\ \leq \; & |\mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d)))| + |\zeta_1(v_s, v_d) - \zeta_2(v_s, v_d)| + \epsilon. \end{aligned} \tag{40}$$

Also by using (37) and (38), we have

$$\Im f(v_s) - \Im g(v_s)$$

$$
\begin{aligned}
&\geq \mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d))) + \zeta_1(v_s, v_d) - \zeta_2(v_s, v_d) - \epsilon \\
&\geq -\{|\mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d)))| + |\zeta_1(v_s, v_d) - \zeta_2(v_s, v_d)| + \epsilon\}. \quad (41)
\end{aligned}
$$

From (40) and (41), we have

$$
\begin{aligned}
& |\Im f(v_s) - \Im g(v_s)| \\
< \;& \mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d))) + \zeta_1(v_s, v_d) - \zeta_2(v_s, v_d) + \epsilon \\
\leq \;& |\mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d)))| + |\zeta_1(v_s, v_d) - \zeta_2()| + \epsilon. \quad (42)
\end{aligned}
$$

On letting $\epsilon \to 0$ in (42), we obtain

$$
|\Im f(v_s) - \Im g(v_s)| \leq |\mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d)))| + |\zeta_1(v_s, v_d) - \zeta_2(v_s, v_d)|.
$$

By using the condition (DP_4), we have

$$
\begin{aligned}
|\Im f(v_s) - \Im g(v_s)| &\leq |\mathcal{C}_1(v_s, v_d, f(\omega_1(v_s, v_d))) - \mathcal{C}_2(v_s, v_d, g(\omega_2(v_s, v_d)))| + |\zeta_1(v_s, v_d) - \zeta_2(v_s, v_d)| \\
&\leq (2^{3-3r} \Delta_s(f, g))^{\frac{1}{r}} \\
&= (2^{3-3r}(\mu_1 |\aleph f - \aleph g|^r + \mu_2[|\aleph f - \Im f|^r + |\aleph g - \Im g|^r] \\
&\quad + \mu_3[|\aleph f - \Im g|^r + |\aleph g - \Im f|^r] \\
&\quad + \mu_4[\max\{|\aleph f - \Im g|^r, |\aleph g - \Im f|^r\} + h \min\{|\aleph f - \Im g|^r, |\aleph g - \Im f|^r\}]))^{\frac{1}{r}} \\
&\leq (2^{3-3r} \sup_{v_s \in \tilde{\mathcal{S}}} (\mu_1 |\aleph f - \aleph g|^r + \mu_2[|\aleph f - \Im f|^r + |\aleph g - \Im g|^r] \\
&\quad + \mu_3[|\aleph f - \Im g|^r + |\aleph g - \Im f|^r] \\
&\quad + \mu_4[\max\{|\aleph f - \Im g|^r, |\aleph g - \Im f|^r\} + h \min\{|\aleph f - \Im g|^r, |\aleph g - \Im f|^r\}]))^{\frac{1}{r}} \\
&= (2^{3-3r}(\mu_1 \eth(\aleph f, \aleph g) + \mu_2[\eth(\aleph f, \Im f) + \eth(\aleph g, \Im g)] \\
&\quad + \mu_3[\eth(\aleph f, \Im g) + \eth(\aleph g, \Im f)] \\
&\quad + \mu_4[\max\{\eth(\aleph f, \Im g), \eth(\aleph g, \Im f)\} + h \min\{\eth(\aleph f, \Im g), \eth(\aleph g, \Im f)\}]))^{\frac{1}{r}}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|\Im f(v_s) - \Im g(v_s)|^r &\leq 2^{3-3r}(\mu_1 \eth(\aleph f, \aleph g) + \mu_2[\eth(\aleph f, \Im f) + \eth(\aleph g, \Im g)] \\
&\quad + \mu_3[\eth(\aleph f, \Im g) + \eth(\aleph g, \Im f)] \\
&\quad + \mu_4[\max\{\eth(\aleph f, \Im g), \eth(\aleph g, \Im f)\} + h \min\{\eth(\aleph f, \Im g), \eth(\aleph g, \Im f)\}]).
\end{aligned}
$$

Now, for all $f, g \in \mho(\tilde{\mathcal{S}})$, we have

$$
\begin{aligned}
s^3 \eth(\Im f(v_s), \Im g(v_s)) &= 2^{3r-3} \sup_{p \in \tilde{\mathcal{S}}} |\Im f(v_s) - \Im g(v_s)|^r \\
&\leq \mu_1 \eth(\aleph f, \aleph g) + \mu_2[\eth(\aleph f, \Im f) + \eth(\aleph g, \Im g)] + \mu_3[\eth(\aleph f, \Im g) + \eth(\aleph g, \Im f)] \\
&\quad + \mu_4[\max\{\eth(\aleph f, \Im g), \eth(\aleph g, \Im f)\} + h \min\{\eth(\aleph f, \Im g), \eth(\aleph g, \Im f)\}].
\end{aligned}
$$

It is clear that Theorem 6 satisfies all the hypotheses of Theorem 4. According to Theorem 4, a unique common fixed point for $\Im$ and $\aleph$ exists in $\mho(\tilde{\mathcal{S}})$, implies that the functional equations which are defined in (27) has a unique bounded common solution. ■

Acknowledgment: The authors are sincerely thankful to the anonymous referee for the valuable suggestions which helped us to improve the quality of the paper.

References

[1] T. Abdeljawad, Meir-Keeler α-contractive fixed and common fixed point theorems, Fixed Point Theory Appl., 2013, 2013:110, 3 pp.

[2] S. Alizadeh, F. Moradlou and P. Salimi, Some fixed point results for (α, β)-(ψ, φ)-contractive mappings, Filomat, 28(2014), 635–647.

[3] G. V. R. Babu and D. R. Babu, Fixed points of Suzuki $\mathcal{Z}$-contraction type maps in b-metric spaces, Advances in the Theory of Nonlinear Anal. Appl., 4(2020), 14–28.

[4] G. V. R. Babu and D. R. Babu, Fixed points of almost Geraghty contraction type maps/generalized contraction maps with rational expressions in b-metric spaces, Commun. Nonlinear Anal., 6(2019), 40–59.

[5] V. A. Babu, D. R. Babu and N. S. Prasad, Coupled fixed points of generalized rational type $\mathcal{Z}$-contraction maps in b-metric spaces, Int. J. Nonlinear Anal. Appl., 13(2022), 789–802.

[6] B. Zada, M. Sarwar, F. Jarad and T. Abdeljawad, Common fixed point theorem via cyclic (α, β)–$(\psi, \phi)_s$-contraction with applications, Symmetry, 11(2019), 15 pages.

[7] R. Bellman and E. S. Lee, Functional equations arising in dynamic programming, Aequationes Math., 17(1978), 1–18.

[8] J. Bogin, A generalization of a fixed point theorem of Goebel, Kirk and Shimi, Canad. Math. Bull., 19(1976), 7–12.

[9] M. Boriceanu, M-F. Bota and A. Petrusel, Multivalued fractals in b-metric spaces, Cent. Eur. J. Math., 8(2010), 367–377.

[10] L. B. Ćirić, Fixed point for generalized multi-valued contractions, Mat. Vesnik, 9(1972), 265–272.

[11] L. B. Ćirić, On some non-expansive type mappings and fixed points, Indian J. Pure Appl. Math., 3(1993), 145–149.

[12] L. B. Ćirić, A new class of non-expansive type mappings and fixed points, Czechoslovak Math. J., 49(1999), 891–899.

[13] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostraviensis, 1(1993), 5–11.

[14] M. Gregus, A fixed point theorem in Banach spaces, Boll. Unione Mat. Ital. Sez. A, 17(1980), 193–198.

[15] G. Nallaselli, A. J. Gnanaprakasam, G. Mani, Z. D. Mitrovic, A. Aloqaily and N. Mlaiki, Integral equation via fixed point theorems on a new type of convex contraction in b-metric and 2-metric spaces, Mathematics, 11(2023), 344.

[16] N. Hussain, V. Paraneh, J. R. Roshan and Z. Kadelburg, Fixed points of cycle weakly (ψ, φ, L, A, B)-contractive mappings in ordered b-metric spaces with applications, Fixed Point Theory Appl., 2013(2013), 256, 18 pages.

[17] H. Isik, B. Samet and C. Vetro, Cyclic admissible contraction and applications to functional equations in dynamic programming, Fixed Point Theor. Appl., 2015, 19 pages.

[18] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci., 9(1986), 771–779.

[19] A. Latif, H. Isik and A. H. Ansari, Fixed points and functional equation problems via cyclic admissible generalized contractive type mapping, J. Nonlinear Sci. Appl., 9(2016), 22–34.

[20] D. K. Patel, T. Abdeljawad and D. Gopal, Common fixed points of generalized Meir-Keeler α-contractions, Fixed Point Theory Appl., 2013, 2013:260, 16 pp.

[21] R. Kumar, N. Chandra, N. Garakoti and M. C. Joshi, Some results on non-expansive type mappings, Appl. Math. E-Notes, 22(2022), 287–298.

[22] J. R. Roshan, V. Parvaneh and Z. Kadelburg, Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces, J. Nonlinear Sci. Appl., 7(2014), 229–245.

[23] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for $\alpha - \psi$-contractive type mappings, Nonlinear Anal., 75(2012), 2154–2165.

[24] W. Sintunavarat. Fixed point results in b-metric spaces approach to the existence of a solution for nonlinear integral equations, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, 110(2016), 585–600.

[25] W. Sintunavarat, S. Plubtieng and P. Katchang, Fixed point result and applications on b-metric space endowed with an arbitrary binary relation, Fixed Point Theory Appl., 2013, 2013:296, 13 pp.

[26] W. Sintunavarat, A new approach to $(\alpha - \psi)$-contractive mappings and generalized Ulam-Hyers stability, well-posedness and limit shadowing results, Carpathian J. Math., 31(2015), 395–601.

[27] W. Sintunavarat, Nonlinear integral equations with new admissibility types in b-metric spaces, J. Fixed Point Theory Appl., 18(2016), 397–416.

[28] O. Yamaod, W. Sintunavarat and Y. J. Cho, Common fixed point theorems for generalized cyclic contraction pairs in b-metric spaces with applications, Fixed Point Theory Appl., 2015, 2015:164, 18 pp.

[29] O. Yamaod, W. Sintunavarat and Y. J. Cho, Existence of a common solution for a system of nonlinear integral equations via fixed point methods in b-metric spaces, Open Math. 14(2016), 128–245.

[30] O. Yamaod and W. Sintunavarat, Fixed point theorems for (α, β)–(ψ, ϕ)-contractive mapping in b-metric spaces with some numerical results and applications, J. Nonlinear Sci. Appl., 9(2016), 22–34.