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An Exactly Solvable Planar Differential System Of Degree Six With An Explicit Algebraic Limit Cycle*

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Abstract

A multi-parameter and planar first-order autonomous differential system of degree six is shown to be integrable. More specifically, we show that it possesses a time-dependent first integral and an autonomous one. This enables us to analytically obtain the solutions. Moreover, under some appropriate conditions, we prove that this system admits an explicit hyperbolic algebraic limit cycle. Finally, we draw the possible phase portraits on the Poincaré disk and illustrate the geometric behavior of the trajectories of the system.

1 Introduction and Statement of the Main Result

Recent years have seen rapid progress in the study of second-order nonlinear ordinary differential equations. Some of these equations are of particular interest due to their frequent occurrence in other scientific areas. As examples, we can cite the Liénard equation [17, 35], the Rayleigh equation [37] and autonomous systems leading to these types of equations such as Kukles' system [5, 19] and Kolmogorov system [36]. However, one of the significant challenges in the study of these nonlinear differential equations and systems is to identify which ones are integrable. This can be achieved through the investigation of integrability, which enables the gathering of all necessary data explicitly from solutions or implicitly from invariants such as first integrals, inverse integrating factors, and invariant algebraic curves, among others.

We recall that an autonomous differential system of dimension n is completely integrable if it has $n-1$ independent first integrals, and therefore the exact solutions of the system can be obtained by intersecting the level sets of these first integrals (for more details, see [9, 28]).

For a planar differential system, the knowledge of a first integral is of great importance in the study of its dynamical behavior. Several analytical methods have been proposed to tackle integrability, each with its own advantages and disadvantages. These methods include Noether symmetries [34], Lie symmetries [32, 6], the Darbouxian theory of integrability [14], direct methods [21, 22] and Painlevé analysis [7, 13]. As a particularly interesting example of the last approach, Nucci and Leach [31] proposed a model for an infectious disease expressed by

$$\begin{aligned} \dot{x} &= -\beta xy - \mu x + \gamma y + \mu K, \\ \dot{y} &= \beta xy - (\mu + \gamma)\, y, \end{aligned} \tag{1}$$

where the dot stands for differentiation with respect to time, $x(t)$ and $y(t)$ represent the susceptibles and the infectives of the population, respectively, β is the infectivity coefficient of the typical Lotka-Volterra interaction term, μK is the birth rate, μ is the proportionate death rate and γ represents the recovery

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coefficient. Using the Painlevé analysis method, the authors showed that system (1) is integrable under certain assumptions. After that, they obtained the solution analytically with the help of the Lie theory of transformation groups.

In [12], Chandrasekar et al. studied the second-order nonlinear ordinary differential equations of the form

$$\ddot{x} = \frac{P}{Q}, \quad P, Q \in \mathbb{C}\left[t, x, \dot{x}\right], \tag{2}$$

where $\dot{x} = dx/dt$ and P, Q are polynomials in t, x and $\dot{x}$ with coefficients in the field of complex numbers. The authors used the extended Prelle-Singer method (for more details, see [15]) to find the first integrals and general solutions, as well as proposed a new technique to identify linearizing transformations.

Continuing the study on the integrability and exact solutions of nonlinear ordinary differential equations, Chandrasekar et al. [11] employed the same method to investigate the integrability of the following class of nonlinear oscillators described by the planar differential system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\left(k_1 x^q + k_2\right) y - k_3 x^{2q+1} - k_4 x^{q+1} - \lambda_1 x, \end{aligned} \tag{3}$$

where the parameters λ_1, q and $k_i, i = 1, 2, 3, 4$ are real. Moreover, the authors obtained integrating factors and general solutions for the integrable cases.

Thereafter, Lin and Han [23] proved that for specific parameter values of q, k_3, k_4, the integrable system (3) possesses a stable limit cycle. They also determined its explicit parametric representation. This limit cycle was obtained a long time ago in [1] and [4] using the method of invariant curves. In the same direction, the present paper aims to study the class of first-order autonomous differential systems

$$\begin{aligned} \dot{x} &= \tfrac{dx}{dt} = P(x, y), \\ \dot{y} &= \tfrac{dy}{dt} = Q(x, y), \end{aligned} \tag{4}$$

where

$$\begin{aligned} P(x, y) &= a(-hwx - 2hay - 2hbx^2 + wx^3 + wa^2xy^2 + 2wabx^3y + wb^2x^5), \\ Q(x, y) &= 2hx - hway + hwbx^2 + 4habxy + 4hb^2x^3 + wax^2y + wa^3y^3 \\ &\quad -wbx^4 + wa^2bx^2y^2 - wab^2x^4y - wb^3x^6, \end{aligned} \tag{5}$$

and a, b, w and h are real parameters. We prove that this class of differential systems is integrable, allowing us to obtain exact analytical solutions. Furthermore, we show that under certain conditions on the parameters, the system possesses a hyperbolic algebraic limit cycle. As far as we know, there are very few examples of planar differential systems that exhibit simultaneously first integral, explicit solutions, and algebraic limit cycles.

Our main result is the following.

Theorem 1 *Let $X = (P, Q)$ be the vector field given by equations (4) and (5). Let also*

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 \colon (bx^2 + ay)^2 + x^2 - h = 0\right\}.$$

When a, b, $w \in \mathbb{R}^*$ and $h \in \mathbb{R}$, the following statements hold:

1. *X has an autonomous first integral H, given by*

$$H(x, y) = \frac{\left(bx^2 + ay\right)^2 + x^2}{\left(bx^2 + ay\right)^2 + x^2 - h} \exp\left(w \arctan\left(a\frac{y}{x} + bx\right)\right), \quad \forall (x, y) \in \mathbb{R}^2.$$

2. *X has a non-autonomous first integral I, given by*

$$I(x, y, t) = \frac{\left(bx^2 + ay\right)^2 + x^2}{\left(bx^2 + ay\right)^2 + x^2 - h} \exp(2ahwt), \quad \forall (x, y, t) \in \mathbb{R}^3.$$

3. If $h > 0$, then X has two exact solutions $(x_i(t), y_i(t))$, $i \in \{1, 2\}$, $t \in \mathbb{R}$, given by

$$x_i(t) = (-1)^i \sqrt{\frac{h\cos^2(2aht)}{1 + hC\exp(2ahwt)}},$$

$$y_i(t) = -\frac{\sqrt{h}\cos^2(2aht)\left(b\sqrt{h} + (-1)^{i+1}\tan(2aht)\sqrt{\frac{1+hC\exp(2ahwt)}{\cos^2(2aht)}}\right)}{a(1+hC\exp(2ahwt))},$$

where $C \geq 0$;

4. If $h > 0$, then Γ is a hyperbolic algebraic limit cycle for X and the following statements hold:

(a) If $aw < 0$ (resp. $aw > 0$), then Γ is stable (resp. unstable);

(b) If $aw < 0$ (resp. $aw > 0$), then Γ is the global sink (resp. source) of X;

(c) Γ is the unique limit cycle of X.

Moreover, the phase portraits of X in the Poincaré disk are topologically equivalent to those shown in Figure 1.

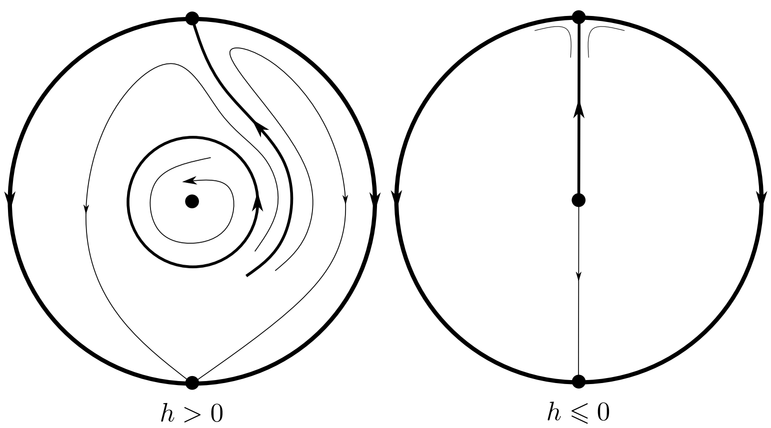


Figure 1: The topological distinct phase portraits of X.

As many different techniques are necessary to prove our main result, it is useful to include a section that summarizes these techniques, see for instance [25, 26, 27, 3, 10]. For the sake of self-containment, we have introduced some preliminary results in Section 2. Theorem 1 is proved in Section 3.

2 Preliminary Results

2.1 First Integrals and Invariant Algebraic Curves

Consider a vector field denoted by $X = (P, Q)$, where P and Q are polynomials. We say that X is integrable if there exists a non-constant analytic function $H : \mathbb{R}^2 \to \mathbb{R}$, referred to as a *first integral* of X, such that the orbits of X are contained in the level sets of H. More specifically, if an orbit of X is given by $(x(t), y(t))$ for $t \in I$, where $I \subset \mathbb{R}$, then there exists a constant $c \in \mathbb{R}$ such that $H(x(t), y(t)) = c$, for all $t \in I$. Note that H is a first integral of X if and only if

$$P(x, y)\frac{\partial H}{\partial x}(x, y) + Q(x, y)\frac{\partial H}{\partial y}(x, y) = 0, \tag{6}$$

for all $(x, y) \in \mathbb{R}^2$. If the first integral depends on the time t, i.e., $H = H(x, y, t)$, then we say that H is a *non-autonomous first integral* of X if

$$P(x,y)\frac{\partial H}{\partial x}(x,y,t) + Q(x,y)\frac{\partial H}{\partial y}(x,y,t) + \frac{\partial H}{\partial t}(x,y,t) = 0, \tag{7}$$

for all $(x, y, t) \in \mathbb{R}^2 \times I$. Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be a real polynomial. We say that F is an *invariant* for X if there exists a real polynomial $K\colon \mathbb{R}^2 \to \mathbb{R}$, called the *cofactor* associated with the set $F(x, y) = 0$, such that

$$P(x,y)\frac{\partial F}{\partial x}(x,y) + Q(x,y)\frac{\partial F}{\partial y}(x,y) = K(x,y)F(x,y), \tag{8}$$

for all $(x, y) \in \mathbb{R}^2$. Note that if n is the maximum of the degrees of P and Q, then $\deg K \leqslant n - 1$. Also, note that the set $F(x, y) = 0$ is invariant under the flow of X. Particularly, if the set $F(x, y) = 0$ contains an oval, it is called an *algebraic limit cycle*. For more details about first integrals, invariant algebraic curves and algebraic limit cycles, refer to Chapter 8 of [16] and the references cited therein.

2.2 Singular Points

Consider X as a polynomial vector field represented by (P, Q). A point $q \in \mathbb{R}^2$ is said to be a *singularity* of X if $P(q) = Q(q) = 0$. The Jacobian matrix J of the vector field X at q is given by

$$J(q) = \begin{pmatrix} \frac{\partial P}{\partial x}(q) & \frac{\partial P}{\partial y}(q) \\ \frac{\partial Q}{\partial x}(q) & \frac{\partial Q}{\partial y}(q) \end{pmatrix}. \tag{9}$$

Let λ_1 and λ_2 be the eigenvalues of the Jacobian matrix $J(q)$, which are the roots of the characteristic polynomial equation

$$\lambda^2 - T(q)\lambda + D(q) = 0,$$

with $D(q) = \lambda_1\lambda_2$ and $T(q) = \lambda_1 + \lambda_2$ are the determinant and trace of $J(q)$, respectively. The singularity q is said to be:

1. *Hyperbolic* if both eigenvalues have non-zero real parts. Here, we distinguish:
   (a) If $D(q) < 0$, then q is a saddle.
   (b) If $D(q) > 0$ and $T(q) > 0$, then q is an unstable focus/node.
   (c) If $D(q) > 0$ and $T(q) < 0$, then q is a stable focus/node.
2. *Non-degenerate monodromic* if $D(q) > 0$ and $T(q) = 0$. In this case, q is a weak focus or a center.
3. *Semi-hyperbolic* if $D(q) = 0$ and $T(q) \neq 0$.
4. *Nilpotent* if $D(q) = T(q) = 0$ and $J(q)$ is not identically zero.
5. *Linearly zero* if $D(q) = T(q) = 0$ and $J(q)$ is identically zero.

We study the local phase portraits at hyperbolic, semi-hyperbolic and nilpotent singular points using Theorems 2.15, 2.19 and 3.5 of [16], respectively. For the linearly zero singularities, we determine their local phase portraits by applying the blow-up technique.

2.3 The Blow-up Technique

Let the origin be an isolated singularity of a polynomial vector field X, thus we can perform the change of coordinates $\phi : \mathbb{S}^1 \times \mathbb{R}_+ \to \mathbb{R}^2$ given by $\phi(\theta, r) = (r\cos\theta, r\sin\theta) = (x, y)$, where $\mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$. Consequently, it is possible to obtain the vector field X_0 in $\mathbb{S}^1 \times \mathbb{R}_+$ by pullback, i.e., $X_0 = D\phi^{-1}X$. When the k-jet of X (which is the k-th order Taylor expansion of X, denoted as j_k) is zero at the origin, the k-jet of X_0 is also zero at every point in $\mathbb{S}^1 \times \{0\}$. As a result, one can define the vector $\hat{X} = \frac{1}{r^k}X_0$ by taking the first $k \in \mathbb{N}$ satisfying $j_k(X(0,0)) = 0$ and $j_{k+1}(X(0,0)) \neq 0$. Therefore, understanding the behavior of $\hat{X}$ near $\mathbb{S}^1$ is equivalent to understanding the behavior of X near the origin. One can also see that $\mathbb{S}^1$ is invariant under the flow of $\hat{X}$. For more details on this technique, see [2] or Chapter 3 of [16]. The vector field $\hat{X}$ is also given by

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r^{k+1}}, \qquad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^{k+2}}.$$

There exists a generalization of the blow-up technique, known as the *quasihomogeneous blow-up*. This time, we consider the change of coordinates $\psi(\theta, r) = (r^\alpha\cos\theta, r^\beta\sin\theta) = (x, y)$ for $(\alpha, \beta) \in \mathbb{N}^2$. In a way analogous to the previous technique, there exists a vector field X_0 in $\mathbb{R}_+ \times \mathbb{S}^1$. For some $k \in \mathbb{N}$ maximal, one can define $X_{\alpha,\beta} = \frac{1}{r^k}X_0$ and observe that this vector field is expressed as follows

$$\dot{r} = \xi(\theta)\frac{\cos\theta\, r^\beta\dot{x} + \sin\theta\, r^\alpha\dot{y}}{r^{\alpha+\beta+k-1}}, \qquad \dot{\theta} = \xi(\theta)\frac{\alpha\cos\theta\, r^\alpha\dot{y} - \beta\sin\theta\, r^\beta\dot{x}}{r^{\alpha+\beta+k}},$$

where $\xi(\theta) = (\beta\sin^2\theta + \alpha\cos^2\theta)^{-1}$. Since $\xi(\theta) > 0$ for all $\theta \in \mathbb{S}^1$, hence, it can be removed through a change in the time variable. Thus, we have

$$\dot{r} = \frac{\cos\theta\, r^\beta\dot{x} + \sin\theta\, r^\alpha\dot{y}}{r^{\alpha+\beta+k-1}}, \qquad \dot{\theta} = \frac{\alpha\cos\theta\, r^\alpha\dot{y} - \beta\sin\theta\, r^\beta\dot{x}}{r^{\alpha+\beta+k}}.$$

Similarly to the previous technique, for studying the behavior of $X_{\alpha,\beta}$ near $\mathbb{S}^1$, we study the behavior of X near the origin.

2.4 The Poincaré Compactification

To investigate the singularities at infinity of the planar vector field $X = (P, Q)$ of degree $n \in \mathbb{N}$, we employ the *Poincaré compactification*. The *Poincaré compactified vector field* $p(X)$ is an analytic vector field generated on the sphere $\mathbb{S}^2$, as follows (for more details on this technique, see [20] or Chapter 5 of [16]). The *Poincaré sphere* is denoted by

$$\mathbb{S}^2 = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$$

and the *equator* is represented by $\mathbb{S}^1 = \{(y_1, y_2, y_3) \in \mathbb{S}^2 : y_3 = 0\}$. The *northern hemisphere* and the *southern hemisphere* are defined as

$$H_+ = \{(y_1, y_2, y_3) \in \mathbb{S}^2 : y_3 > 0\} \text{ and } H_- = \{(y_1, y_2, y_3) \in \mathbb{S}^2 : y_3 < 0\},$$

respectively. We identify $\mathbb{R}^2$ with the plane $(x_1, x_2, 1)$ in $\mathbb{R}^3$. Consider the central projections $f_\pm : \mathbb{R}^2 \to H_\pm$, where $f_\pm(x_1, x_2) = \pm\Delta(x_1, x_2)(x_1, x_2, 1)$, with $\Delta(x_1, x_2) = (x_1^2 + x_2^2 + 1)^{-\frac{1}{2}}$. These two maps define two copies of X, one copy X^+ in H_+ and the other copy X^- in H_-. Thus we have the vector field $X'^+ \cup X^-$ defined on $\mathbb{S}^2 \backslash \mathbb{S}^1$, where the equator $\mathbb{S}^1$ of the sphere $\mathbb{S}^2$ corresponds with the *infinity* of $\mathbb{R}^2$. In order to extend X' from $\mathbb{S}^2 \backslash \mathbb{S}^1$ to $\mathbb{S}^2$, we apply the rescaling $y_3^{n-1}X'$. The resulting analytic extension is the Poincaré compactified vector field $p(X)$. The projection of the closed northern hemisphere onto $y_3 = 0$ using the transformation $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ is called the *Poincaré disk* $\mathbb{D}$. The dynamics of $p(X)$ near $\mathbb{S}^1$ is the same as the dynamics of X near infinity in $\mathbb{R}^2$. To compute the expression of $p(X)$, we define the local charts of $\mathbb{S}^2$ by

$$U_i = \{(y_1, y_2, y_3) \in \mathbb{S}^2 : y_i > 0\} \text{ and } V_i = \{(y_1, y_2, y_3) \in \mathbb{S}^2 : y_i < 0\} \text{ for } i \in \{1, 2, 3\}.$$

Their corresponding local maps are $\phi_i : U_i \to \mathbb{R}^2$ and $\psi_i : V_i \to \mathbb{R}^2$ with

$$\phi_i(y_1, y_2, y_3) = \psi_i(y_1, y_2, y_3) = \left(\frac{y_m}{y_i}, \frac{y_n}{y_i}\right),$$

where $m \neq i$, $n \neq i$ and $m < n$. We denote by (u, v) the image of ϕ_i and ψ_i, for $i = 1, 2$ in each chart. Therefore, the expression of $p(X)$ in the local chart U_1 is given by

$$\dot{u} = v^n \left[Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right], \quad \dot{v} = -v^{n+1} P\left(\frac{1}{v}, \frac{u}{v}\right),$$

and the expression of $p(X)$ in the local chart U_2 is

$$\dot{u} = v^n \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right], \quad \dot{v} = -v^{n+1} Q\left(\frac{u}{v}, \frac{1}{v}\right).$$

The expression of $p(X)$ in V_1 and V_2 is identical to the expression of $p(X)$ in U_1 and U_2 except that it is multiplied by the factor $(-1)^{n-1}$. In the local charts U_1, U_2, V_1 and V_2, the coordinate $v = 0$ represents the points of $\mathbb{S}^1$. Hence, the singularities at infinity of $\mathbb{R}^2$. Note that $\mathbb{S}^1$ is invariant under the flow of $p(X)$.

2.5 The Markus-Neumann-Peixoto Theorem

Consider the polynomial vector field X. Let $p(X)$ be its compactification on $\mathbb{D}$ and ϕ be the flow associated to $p(X)$. The *separatrices* of $p(X)$ are:

1. All the orbits at infinity;
2. All the singular points;
3. All the limit cycles of X;
4. All the trajectories that are on the boundaries of the hyperbolic sectors of the finite and infinite singular points.

The set of all separatrices, denoted by $\mathcal{S}$ is closed. Each connected component of $\mathbb{D}\backslash\mathcal{S}$ is called a *canonical region* of the flow $(\mathbb{D}, \phi)$. The *separatrix configuration* $\mathcal{S}_c$ of the flow $(\mathbb{D}, \phi)$ is defined as the union of the separatrices $\mathcal{S}$, with one orbit from each canonical region. Two separatrix configurations $\mathcal{S}_c$ and $\mathcal{S}_c^*$ of the flows $(\mathbb{D}, \phi)$ and $(\mathbb{D}, \phi^*)$ are said to be *topologically equivalent* if there is a homeomorphism of $\mathbb{D}$ that transforms the orbits of $\mathcal{S}_c$ into those of $\mathcal{S}_c^*$ and preserves or reverses the orientation of all these orbits.

Theorem 2 (Markus-Neumann-Peixoto) *Consider two Poincaré compactified $p(X)$ and $p(Y)$ in the Poincaré disk $\mathbb{D}$ of two polynomial vector fields X and Y with finitely many singularities. Then the phase portraits of $p(X)$ and $p(Y)$ are topologically equivalent if and only if their separatrix configurations are topologically equivalent.*

Proof. See [29, 30, 8]. ■

2.6 Poincaré Return Map

Let $X = (P, Q)$ be a planar vector field of class C^r, where $r \in \mathbb{N}$, and $\phi_t(x)$ be the flow associated to X. Consider $\gamma = \{\phi_t(p), 0 \leq t \leq T\}$ a periodic orbit of X through a point $p \in \mathbb{R}^2$, with period $T > 0$. Let $\Sigma \subset \mathbb{R}^2$ be a transverse section of γ at p. For each point $q \in \Sigma$ sufficiently close to p, the orbit $\phi_t(q)$ of X through q intersects Σ. More precisely, if Σ is small enough, then there exists a function $\tau\colon \Sigma \to \mathbb{R}^+$ of class C^r, such that $\phi_{\tau(q)}(q) \in \Sigma$ is the first intersection of $\phi_t(q)$ with Σ. The C^r-map $\Pi\colon \Sigma \to \Sigma$ given by $\Pi(q) = \phi_{\tau(q)}(q)$, is known as the *Poincaré return map*. Since a limit cycle is an isolated periodic orbit, it follows that γ is a limit cycle if and only if q is an isolated zero of Π. Moreover, γ is hyperbolic if and only if $\Pi'(q) \neq 1$. Let

$$\operatorname{div}(x,y) = \frac{\partial P}{\partial x}(x,y) + \frac{\partial Q}{\partial y}(x,y).$$

It follows from the *Liouville's formula* that

$$\Pi'(q) = \exp\left(\int_0^T \operatorname{div}(\gamma(t))\, dt\right). \tag{10}$$

Hence, if

$$r(\Gamma) = \int_0^T \operatorname{div}(\gamma(t))\, dt, \tag{11}$$

then it follows that γ is hyperbolic and stable (resp. unstable) if $r(\Gamma) < 0$ (resp. $r(\Gamma) > 0$). For more details, see Section 3.4 of [33]. In the special case of an algebraic limit cycle, it follows that

$$r(\Gamma) = \int_0^T K(\gamma(t))\, dt, \tag{12}$$

where $K(x,y)$ is the cofactor defined in (8). For more details in this special case, see [18].

3 Proof of Theorem 1

Let us look at statements (a) and (b). To see that

$$H(x,y) = \frac{\left(bx^2+ay\right)^2 + x^2}{\left(bx^2+ay\right)^2 + x^2 - h} \exp\left(w \arctan\left(a\frac{y}{x} + bx\right)\right) \tag{13}$$

is an autonomous first integral of X, it is sufficient to observe that the equation

$$P(x,y)\frac{\partial H}{\partial x}(x,y) + Q(x,y)\frac{\partial H}{\partial y}(x,y) = 0,$$

is satisfied. Similarly, to see that

$$I(x,y,t) = \frac{\left(bx^2+ay\right)^2 + x^2}{\left(bx^2+ay\right)^2 + x^2 - h} \exp(2ahwt) \tag{14}$$

is a non-autonomous first integral of X, it is sufficient to observe that the equation

$$P(x,y)\frac{\partial I}{\partial x}(x,y,t) + Q(x,y)\frac{\partial I}{\partial y}(x,y,t) + \frac{\partial I}{\partial t}(x,y,t) = 0,$$

is also satisfied. Let us now look at statement (c). To see that $(x_i(t), y_i(t))$ are indeed exact solutions of X, it is sufficient to observe that the equations

$$\dot{x}_i(t) = P(x_i(t), y_i(t)), \quad \dot{y}_i(t) = Q(x_i(t), y_i(t)),$$

are satisfied, for $i \in \{1,2\}$. From equations (13) and (14), we get

$$\exp\left(w \arctan\left(a\frac{y}{x} + bx\right)\right) = \exp(2ahwt). \tag{15}$$

If we isolate y in equation (15), we obtain

$$y(x,t) = \frac{x}{a}\big(-bx + \tan(2aht)\big). \tag{16}$$

Replacing (16) in the first equation of (5), we obtain the Bernoulli differential equation

$$\dot{x} = \frac{aw}{\cos^2(2aht)}x^3 - \big[ah\big(w + 2\tan(2aht)\big)\big]\,x. \tag{17}$$

If $h > 0$, the solutions of (17) are given by

$$x_i(t) = (-1)^i\sqrt{\frac{h\cos^2(2aht)}{1 + hC\exp(2ahwt)}}, \tag{18}$$

for $i \in \{1, 2\}$ and $C \geq 0$. Replacing (18) in equation (16), we get

$$y_i(t) = -\frac{\sqrt{h}\cos^2(2aht)\left(b\sqrt{h} + (-1)^{i+1}\tan(2aht)\sqrt{\dfrac{1 + hC\exp(2ahwt)}{\cos^2(2aht)}}\right)}{a(1 + hC\exp(2ahwt))}.$$

Hence, we have obtained the exact solutions. We now look at statement (d). Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$F(x,y) = (bx^2 + ay)^2 + x^2 - h,$$

and observe that if $h > 0$, then

$$P(x,y)\frac{\partial F}{\partial x}(x,y) + Q(x,y)\frac{\partial F}{\partial y}(x,y) = K(x,y)F(x,y), \tag{19}$$

where

$$K(x,y) = 2aw(b^2x^4 + a^2y^2 + x^2 + 2abx^2y). \tag{20}$$

Therefore, if $h > 0$, then the curve $F^{-1}(0) = \Gamma$ is an invariant algebraic curve for X. Observe that Γ is given by

$$y^{\pm}(x) = \frac{-bx^2 \pm \operatorname{sign}(a)\sqrt{h - x^2}}{a},$$

for $x \in [-\sqrt{h}, \sqrt{h}]$. Hence, Γ is an oval. Since the origin is the unique singularity of X and it does not lie on Γ, thus Γ must be an algebraic limit cycle for X. Let $T > 0$ be the period of Γ and let $\gamma(t)$ be the parameterization of Γ given by the flow of X. From equation (12), where K is given by (19) and (20), it follows that $K(x,y) > 0$ (resp. $K(x,y) < 0$) if $aw > 0$ (resp. $aw < 0$). Therefore, we conclude that $r(\Gamma) \neq 0$ (i.e., Γ is hyperbolic) and the sign of aw determines the sign of $r(\Gamma)$. This proves statement (i).

Let us now look at statement (ii). Let $I_1\colon \mathbb{R}^2\backslash\Gamma \to \mathbb{R}$ be given by

$$I_1(x,y) = \frac{\big(bx^2 + ay\big)^2 + x^2}{\big(bx^2 + ay\big)^2 + x^2 - h}, \tag{21}$$

and observe that $I(x,y,t) = I_1(x,y)e^{2ahwt}$, where I is the non-autonomous first integral of X, given by (14). Suppose $aw < 0$ and let $(\overline{x}(t), \overline{y}(t))$ be any orbit of X. Since $I(\overline{x}(t), \overline{y}(t), t)$ is constant and

$$\lim_{t\to+\infty} e^{2ahwt} = 0,$$

it follows that

$$\lim_{t\to+\infty} |I_1(\overline{x}(t), \overline{y}(t))| \to \infty.$$

Therefore, it follows from (21) that the denominator of I_1 approaches zero, i.e.,

$$\lim_{t\to+\infty}\left(b\overline{x}^2(t)+a\overline{y}(t)\right)^2+\overline{x}^2(t)-h=0.$$

Hence, it follows that $(\overline{x}(t),\overline{y}(t))\to\Gamma$, as $t\to+\infty$. The case $aw>0$ follows similarly.

To prove statement (iii), observe that if X had another limit cycle, then Γ could not be the global sink or source of X. Finally, we now look at the phase portraits of X. First, we observe that X is invariant under the following change of variables and parameters:

(1) $(a,b,w)\to(-a,-b,-w)$,

(2) $(x,y,b)\to(-x,-y,-b)$,

(3) $(y,t,a)\to(-y,-t,-a)$.

Therefore, it follows from (1) that we can assume $w>0$. Hence, it follows from (2) that we can assume $b>0$ and thus it follows from (3) that we can also assume $a>0$. It is not hard to see that the origin is the unique singularity of X and that its Jacobian matrix is given by

$$DX(0,0)=\begin{pmatrix}-ahw & -2a^2h\\ 2h & -ahw\end{pmatrix}.$$

Therefore, the eigenvalues of $DX(0,0)$ are $\lambda^{\pm}=-awh\pm i2a$, where $i^2=-1$. Hence, it follows from Subsection 2.2 that the origin is a hyperbolic focus if $h\neq0$, and its stability is determined by the sign of $-aw$. We now look at infinity. The origin of the second chart of Poincaré compactification is the unique singularity at infinity. Moreover, after performing a quasihomogeneous blow-up with weights $(\alpha,\beta)=(1,2)$ and a homogeneous blow-up, its local phase portrait is given by Figure 2. Therefore, it follows that the

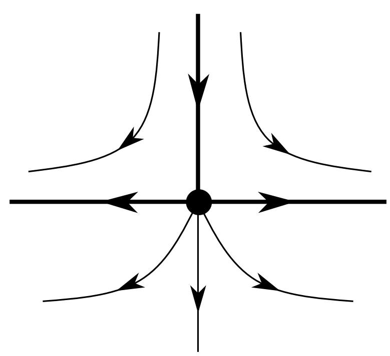
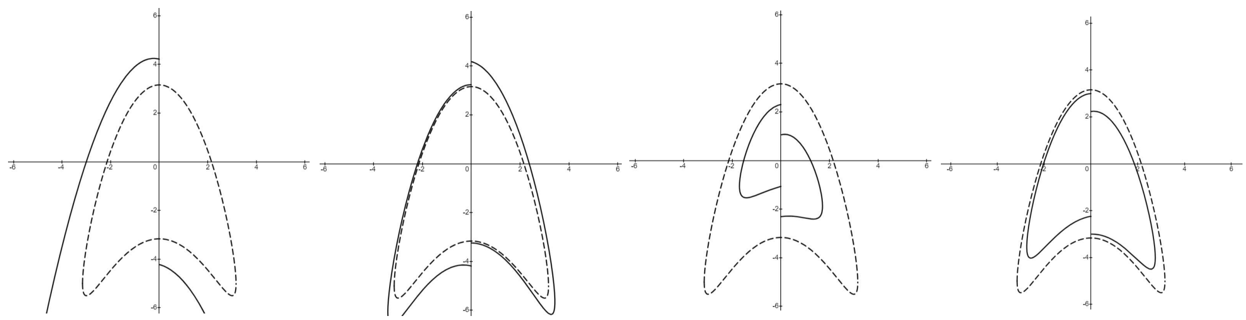


Figure 2: Local phase portrait at the origin of the second chart of the Poincaré compactification.

phase portrait of X is shown in Figure 1.

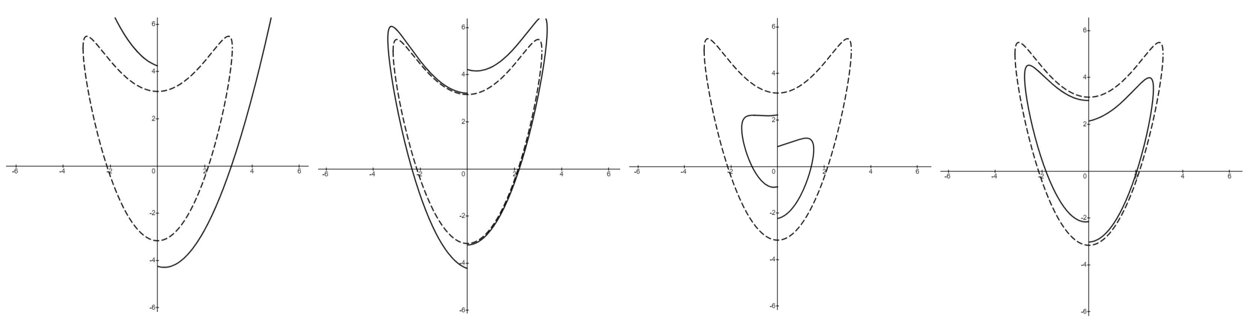
The level curves of the first integral H for the vector field X, represented by a continuous line, are defined by $H(x,y)=c$, where $c\in\mathbb{R}^*$ and H is defined in equation (13). These curves take the form shown in Figures 3 and 4 under the conditions $a,w,h\in\mathbb{R}_+^*$ and $b\in\mathbb{R}^*$. It is noteworthy that in Figure 3, when a, w, $h>0$ and $b>0$, the level curves for $|c|\gg0$ and $c\approx0$ complement each other. Furthermore, depending on whether $c>0$ (resp. $c<0$), the level curves are situated within the unbounded (or bounded) region delimited by the algebraic limit cycle Γ, depicted with a dashed line. The level curve corresponding to $c=0$ precisely coincides with the origin. It's important to observe also that Γ lies precisely in the region where H is not well-defined, specifically in the region where the denominator of H vanishes.

In a similar manner, we obtain the level curves of the first integral H and the algebraic limit cycle Γ for X when a, w, $h>0, b<0$ and $c\in\mathbb{R}^*$, as illustrated in Figure 4.



$c > 0,\ c \approx 0.$ $c \gg 0.$ $c < 0,\ c \approx 0.$ $c \ll 0.$

Figure 3: Level curves $H(x, y) = c$ and algebraic limit cycle Γ for the vector field X when $a,\ b,\ w,\ h > 0$ and $c \in \mathbb{R}^*$.



$c > 0,\ c \approx 0.$ $c \gg 0.$ $c < 0,\ c \approx 0.$ $c \ll 0.$

Figure 4: Level curves of H and algebraic limit cycle Γ for X when $a,\ w,\ h > 0, b < 0$ and $c \in \mathbb{R}^*$.

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