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Common Domain Of Asymptotic Stability Of A Family Of Difference Equations*†

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Abstract

A necessary and sufficient condition is obtained for a family of difference equations to be asymptotically stable.

1. Introduction and Results

The following difference equation (see e.g. [1, 2] for its importance)

$$u_n = au_{n-\tau} + bu_{n-\sigma},\ n = 0, 1, 2, ... \tag{1.1}$$

where a, b are real numbers and τ, σ are positive integers, is said to be (globally) asymptotically stable if each of its solutions tends to zero.

When the delays τ and σ are given (fixed), whether the corresponding equation (1.1) is asymptotically stable clearly depends on the coefficients a and b. For this reason, we denote the set of all pairs (x, y) such that the equation

$$u_n = xu_{n-\tau} + yu_{n-\sigma},\ n = 0, 1, 2, ... \tag{1.2}$$

is asymptotically stable by $\Omega(x, y|\tau, \sigma)$.

It is well known that equation (1.1) is asymptotically stable if, and only if, all the (complex) roots of its characteristic equation

$$1 = a\lambda^{-\tau} + b\lambda^{-\sigma}, \tag{1.3}$$

are inside the open unit disk. In other words, the set $\Omega(x, y|\tau, \sigma)$ is also the set of pairs (x, y) such that all the (complex) roots of

$$1 = x\lambda^{-\tau} + y\lambda^{-\sigma} \tag{1.4}$$

has magnitude less than one.

By means of commercial software such as the MATLAB, it is not difficult to generate domains $\Omega(x, y|\tau, \sigma)$ in the x, y-plane for different values of the delays τ and σ. It is interesting to observe that the set

$$\{(x, y)|\, |x| + |y| < 1\}$$

is included in all of these computer generated domains. This motivates the following theorem.

Theorem 1. *Let $\Omega(x, y|\tau, \sigma)$ be the set of all pairs of the form (x, y) such that equation (1.2) is asymptotically stable. Then we have*

$$\cap_{\tau,\sigma\in N}\Omega(x, y|\tau, \sigma) = \{(x, y)|\, |x| + |y| < 1\},$$

where N is the set of all positive integers.

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One part of the proof is easy. Let μ be a nonzero root of equation (1.3). If $|a|+|b|<1$, then since

$$|a|+|b|<1\leq |a|\,|\mu|^{-\tau}+|b|\,|\mu|^{-\sigma},$$

we see that

$$|a|<|a|\,|\mu|^{-\tau}$$

or

$$|b|<|b|\,|\mu|^{-\sigma}.$$

But then $|\mu|^{\tau}<1$ or $|\mu|^{\sigma}<1$. In other words, $|\mu|<1$.

In order to complete our proof, we need the following preparatory lemma.

Lemma 2 (cf. [4, Lemma 2.1]). *Suppose a,b are real numbers such that $|a|+|b|\neq 0$, and τ and σ are two positive integers. Then the equation*

$$|a|\,x^{-\tau}+|b|\,x^{-\sigma}=1,\ x>0$$

has a unique solution in $(0,\infty)$.

Proof. Consider the function

$$f(x)=|a|\,x^{-\tau}+|b|\,x^{-\sigma},\ x>0.$$

Since f is continuous on $(0,\infty)$, $\lim_{x\to 0^+}f(x)=\infty$, $\lim_{x\to\infty}f(x)=0$ and

$$f'(x)=-\left(|a|\,\tau x^{-\tau-1}+|b|\,\sigma x^{-\sigma-1}\right)<0,\ x>0,$$

thus our proof follows from the intermediate value theorem. ∎

2. Proof of Main Result

Now if (a,b) belongs to $\cap_{\tau,\sigma\in N}\Omega(x,y|\tau,\sigma)$, then for each pair (τ,σ) of integers, each root μ of equation (1.3) satisfies $|\mu|<1$. Let us write $\mu=re^{\theta}$ and write (1.3) in the form

$$ar^{-\tau}\cos\tau\theta+br^{-\sigma}\cos\sigma\theta=1, \tag{2.1}$$

$$ar^{-\tau}\sin\tau\theta+br^{-\tau}\sin\sigma\theta=0. \tag{2.2}$$

There are several cases to consider: (i) $a=0$ or $b=0$; (ii)$a>0,b>0$; (iii)$a<0,b<0$; (iv)$a<0,b>0$; and (v) $a>0,b<0$. The first case is easily dealt with. In the second case, since the equation

$$ax^{-\tau}+bx^{-\sigma}=1$$

has a unique positive root ρ_1 by Lemma 1, $(r,\theta)=(\rho_1,0)$ is a solution of equations (2.1)-(2.2). This implies that $\rho_1=r=|\mu|<1$. But then

$$1=a\rho_1^{-\tau}+b\rho_1^{-\sigma}>a+b=|a|+|b|.$$

In the third case, since the equation

$$-ax^{-\tau}-bx^{-\sigma}=1$$

has a unique positive root ρ_2 by Lemma 1, it we pick $\tau=1$ and $\sigma=3$, then $(r,\theta)=(\rho_2,\pi)$ is a solution of equations (2.1)–(2.2). This implies $\rho_2=|\mu|<1$. But then

$$1=-a\rho_2^{-\tau}-b\rho_2^{-\sigma}>-a-b=|a|+|b|.$$

In the fourth case, since the equation

$$-ax^{-\tau} + bx^{-\sigma} = 1$$

has a unique positive root ρ_3 by Lemma 1, if we pick $\tau = 1$ and $\sigma = 2$, then $(r, \theta) = (\rho_3, \pi)$ is a solution of equations (2.1)–(2.2). This implies $\rho_3 = |\mu| < 1$. But then

$$1 = -a\rho_3^{-\tau} + b\rho_3^{-\sigma} > -a + b = |a| + |b|.$$

In the final case, since the equation

$$ax^{-\tau} - bx^{-\sigma} = 1$$

has a positive root ρ_4, if we pick $\tau = 2$ and $\sigma = 3$, then $(r, \theta) = (\rho_4, \pi)$ is a solution of equations (2.1)–(2.2). This implies $\rho_4 < 1$ and consequently

$$1 \geq a\rho_4^{-\tau} - b\rho_4^{-\sigma} > a - b = |a| + |b|.$$

The proof is complete.

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