Applied Mathematics E-Notes, 24(2024), 237-273 © ISSN 1607-2510
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Periodic Orbits In The Zero-Hopf Bifurcations Of 3-Dimensional Kolmogorov Systems Of Degree 3*

Chamseddine Bouaziz[†], Jaume Llibre[‡], Djamila Djedid[§], Amar Makhlouf[¶]

Received 3 April 2023

Abstract

We study the zero Hopf bifurcation of 3-dimensional Kolmogorov systems using first-order averaging theory. We find that one or two limit cycles can bifurcate from the singular point.

1 Introduction and Statements of the Main Results

We study the *zero-Hopf equilibria* of a 3-dimensional autonomous differential system, i.e. the equilibrium points of a 3-dimensional autonomous system of differential equations whose linear part has a zero eigenvalue and a pair of purely imaginary eigenvalues. For such an equilibrium, there is no general theory for knowing when this equilibrium bifurcates a small-amplitude periodic orbit, moving the parameters of the system. We provide an algorithm for solving this problem; see, for instance, [9, 15].

The Lotka-Volterra systems were initially considered independently by Lotka in 1925 [17] and by Volterra in 1926 [20], also known as the Predator-Prey equation, are a pair of first-order, nonlinear, differential equations frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. Later on, Kolmogorov [13] in 1936 extended these systems to arbitrary dimension and arbitrary degree, these kinds of systems are now called Kolmogorov systems.

It is known that the polynomial Lotka-Volterra differential systems in $\mathbb{R}^3$ of degree 2 cannot have isolated zero-Hopf equilibrium points in the set of all equilibrium points; see [16]. In the article [7], we have studied the periodic orbits bifurcating from a Hopf equilibrium of 2-dimensional polynomial Kolmogorov systems of arbitrary degrees.

In this paper, we shall study the zero-Hopf equilibrium points of the polynomial Kolmogorov differential systems in $\mathbb{R}^3$ of degree 3 via the averaging theory of first-order and we shall prove that there are 8 families of such equilibrium points. We shall study when these families of zero-Hopf equilibrium points have a zero-Hopf bifurcation, i.e. when a limit cycle bifurcates from such equilibrium points, moving the parameters of the system. We also give an example for each case, we plot their bifurcated limit cycles, and we study their stability.

The zero-Hopf bifurcation, for other differential systems, has been studied by many authors, for instance see [10, 11, 12, 14, 18]. In some cases of *zero-Hopf equilibria*, a zero-Hopf bifurcation could imply a local birth of "chaos" see, for instance, the articles (cf. [1, 2, 5, 6, 18]).

The Kolmogorov systems in $\mathbb{R}^3$ of degree 3 that we consider are

$$\begin{aligned}\dot{x} &= x(a_1(x-1)+b_1(y-1)+c_1(z-1)+d_1(x-1)^2+e_1(y-1)^2+f_1(z-1)^2),\\ \dot{y} &= y(a_2(x-1)+b_2(y-1)+c_2(z-1)+d_2(x-1)^2+e_2(y-1)^2+f_2(z-1)^2),\\ \dot{z} &= z(a_3(x-1)+b_3(y-1)+c_3(z-1)+d_3(x-1)^2+e_3(y-1)^2+f_3(z-1)^2),\end{aligned} \tag{1}$$

*Mathematics Subject Classifications: 37N25, 34C12, 34C8, 37G20.

†Department of Mathematics, University of Annaba, Laboratory LMA PO.Box.12;23 Annaba, Algeria
‡Departament de Matematiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
§Department of Mathematics, University of Annaba, Laboratory LMA PO.Box.12;23 Annaba, Algeria
¶Department of Mathematics, University of Annaba, Laboratory LMA PO.Box.12;23 Annaba, Algeria

where x, y and z are supposed positive and the dot denotes the derivative with respect to the time t.

Note that if there is an equilibrium point (a, b, c) with a, b and c positive, doing the rescaling $(x, y, z) \to (x/a, y/b, z/c)$ this equilibrium pass to the point $(1, 1, 1)$ and the dynamical behavior of the Kolmogorov persists. So it is not restrictive to consider that an equilibrium in the positive octant of $\mathbb{R}^3$ is the point $(1, 1, 1)$.

In the next proposition we characterize when the equilibrium point $(1, 1, 1)$ of the Kolmogorov system (1) is a zero-Hopf equilibrium point.

Proposition 1 *There are eight m-parameter families of Kolmogorov systems (1) for which the equilibrium point $(1, 1, 1)$ is a zero-Hopf equilibrium point. Namely:*

(i) $a_1 = b_1 = b_2 = c_1 = c_3 = 0,\ b_3c_2 = -\omega^2 < 0$;

(ii) $a_1 = -b_2, c_1 = c_2 = c_3 = 0,\ a_2b_1 + b_2^2 = -\omega^2 < 0$;

(iii) $a_1 = b_1 = c_1 = 0,\ c_3 = -b_2,\ b_3c_2 + b_2^2 = -\omega^2 < 0$;

(iv) $a_1 = -c_3, b_1 = b_2 = b_3 = 0,\ a_3c_1 + c_3^2 = -\omega^2 < 0$;

(v) $a_1 = -c_3,\ a_2 = -\dfrac{c_2c_3}{c_1},\ b_1 = b_2 = 0,\ a_3c_1 + b_3c_2 + c_3^2 = -\omega^2 < 0$;

(vi) $a_1 = -b_2 - c_3,\ b_3 = \dfrac{b_1c_3}{c_1},\ c_2 = \dfrac{b_2c_1}{b_1},\ a_2b_1 + a_3c_1 + b_2^2 + 2b_2c_3 + c_3^2 = -\omega^2 < 0$;

(vii) $a_1 = -b_2 - c_3,\ a_2 = -\dfrac{b_2(b_2 + c_3)}{b_1},\ c_2 = \dfrac{b_2c_1}{b_1},\ \dfrac{1}{b_1}(a_3b_1c_1 + b_2b_3c_1 + b_1c_3^2) = -\omega^2 < 0$;

(viii) $a_1 = -b_2 - c_3,\ a_3 = \dfrac{a_2(b_1c_3 - b_3c_1) + (b_2 + c_3)(b_2c_3 - b_3c_2)}{b_1c_2 - b_2c_1},\ \omega^2 = \dfrac{A}{b_1c_2 - b_2c_1} > 0,$ *where*

$$\begin{aligned} A &= -(-b_2^3c_1 + b_1b_2^2c_2 + b_2(-2b_3c_1c_2 + b_1c_2c_3) + c_2(b_1b_3c_2 - b_3c_1c_3 + b_1c_3^2) \\ &\quad + a_2(b_1^2c_2 - b_3c_1^2 + b_1c_1(c_3 - b_2))). \end{aligned}$$

Proposition 1 is proved in section 3. We define the following eight sets of conditions:

$$\begin{array}{ll}
(i) & a_1 = a_{11}\varepsilon,\ b_1 = b_{11}\varepsilon,\ b_2 = b_{21}\varepsilon,\ c_1 = c_{11}\varepsilon,\ c_3 = c_{31}\varepsilon,\ b_3c_2 < 0; \\
(ii) & a_1 = -(b_{20} + b_{21}\varepsilon),\ b_2 = b_{20} + b_{21}\varepsilon,\ c_1 = c_{11}\varepsilon,\ c_2 = c_{21}\varepsilon,\ c_3 = c_{31}\varepsilon,\ a_2b_1 + b_2^2 < 0; \\
(iii) & a_1 = a_{11}\varepsilon,\ b_2 = b_{20} + b_{21}\varepsilon,\ c_1 = c_{11}\varepsilon,\ b_1 = b_{11}\varepsilon,\ c_3 = -(b_{20} + b_{21}\varepsilon),\ b_3c_2 + b_2^2 < 0; \\
(iv) & a_1 = -(c_{30} + c_{31}\varepsilon),\ b_1 = b_{11}\varepsilon,\ b_2 = b_{21}\varepsilon,\ b_3 = b_{31}\varepsilon,\ c_3 = c_{30} + c_{31}\varepsilon,\ a_3c_1 + c_3^2 < 0; \\
(v) & a_1 = -(c_{30} + c_{31}\varepsilon),\ b_1 = b_{11}\varepsilon,\ b_2 = b_{21}\varepsilon,\ c_3 = c_{30} + c_{31}\varepsilon,\ a_2 = -\dfrac{c_2c_{30}}{c_1} - \dfrac{c_2c_{31}}{c_1}\varepsilon, \\
 & a_3c_1 + b_3c_2 + c_3^2 < 0; \\
(vi) & a_1 = -b_{20} - c_{30} - (b_{21} + c_{31})\varepsilon, b_2 = b_{20} + b_{21}\varepsilon, b_3 = \dfrac{b_1c_{30}}{c_1} + \dfrac{b_1c_{31}}{c_1}\varepsilon, c_2 = \dfrac{b_{20}c_1}{b_1} + \dfrac{b_{21}c_1}{b_1}\varepsilon, \\
 & c_3 = c_{30} + c_{31}\varepsilon,\ a_2b_1 + a_3c_1 + b_2^2 + 2b_2c_3 + c_3^2 < 0; \\
(vii) & a_1 = -b_{20} - c_{30} - (b_{21} + c_{31})\varepsilon,\ b_2 = b_{20} + b_{21}\varepsilon,\ a_2 = \dfrac{1}{b_1}(-b_{20}(b_{20} + c_{30}) - (b_{20}(b_{21} + c_{31}) \\
 & +b_{21}(b_{20} + c_{30})))\varepsilon,\ c_2 = \dfrac{b_{20}c_1}{b_1} + \dfrac{b_{21}c_1}{b_1}\varepsilon,\ c_3 = c_{30} + c_{31}\varepsilon,\ \dfrac{1}{b_1}(a_3b_1c_1 + b_2b_3c_1 + b_1c_3^2) < 0;
\end{array} \tag{2}$$

$(viii)$ $a_1=-b_2-c_{30}-c_{31}\varepsilon,\ c_3=c_{30}+c_{31}\varepsilon,\ \dfrac{A}{b_1c_2-b_2c_1}>0, a_3=\dfrac{1}{b_1c_2-b_2c_1}\Big(a_2(b_1c_{30}-b_3c_1)+(b_2+$ $c_{30})(b_2c_{30}-b_3c_2)\Big)+\dfrac{a_2b_1c_{31}+(b_2+c_{30})b_2c_{31}+c_{31}(b_2c_{30}-b_3c_2)}{b_1c_2-b_2c_1}\varepsilon.$

Theorem 1 *Assume that the family (r) of Kolmogorov systems (1) of the Proposition 1 is perturbed by the condition (r), for $r\in\{i,ii,iii,iv,v,vi,vii,viii\}$. Then, for $\varepsilon\neq 0$ sufficiently small, the perturbed system (1) has at least one periodic solution $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))$ bifurcating from the zero-Hopf equilibrium $(1,1,1)$ when $\varepsilon=0$.*

The structure of the rest of this paper is the following. In section 2, we present the basic results of the averaging theory that we need to prove Theorem 1. And in section 3, we prove Proposition 1 and Theorem 1.

2 The Averaging Theory for Periodic Orbits

The averaging theory is a classical and matured tool for studying the behaviour of the dynamics of nonlinear smooth dynamical systems, and in particular, their periodic orbits. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace, who provided an intuitive justification of the process. The first formalization of this procedure is due to Fatou [8] in 1928. Important practical and theoretical contributions to this theory were made by Krylov and Bogoliubov [4] in the 1930s, and Bogoliubov [3] in 1945. The averaging theory of first order for studying periodic orbits can be found in [19]; see also [11]. It can be summarized as follows.

Now, we shall present the basic results from the averaging theory that we need to prove the results of this paper. The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof, see Theorems 11.5 and 11.6 of Verhulst [19].

Consider the differential equation

$$\dot{\mathbf{x}}=\varepsilon F(t,\mathbf{x})+\varepsilon^2G(t,\mathbf{x},\varepsilon),\quad \mathbf{x}(0)=\mathbf{x}_0, \tag{3}$$

with $\mathbf{x}\in D$, where D is an open subset of $\mathbb{R}^n$, $t\geq 0$. Moreover we assume that both $F(t,\mathbf{x})$ and $G(t,\mathbf{x},\varepsilon)$ are T-periodic in t. We also consider in D the averaged differential equation

$$\dot{\mathbf{y}}=\varepsilon f(\mathbf{y}),\quad \mathbf{y}(0)=\mathbf{y}_0, \tag{4}$$

where

$$f(\mathbf{y})=\frac{1}{T}\int_0^T F(t,\mathbf{y})dt. \tag{5}$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with T-periodic solutions of equation (3).

Theorem 2 *Consider the two initial value problems (3) and (4). Suppose:*

(i) F, its Jacobian $\partial F/\partial x$, its Hessian $\partial^2F/\partial x^2$, G and its Jacobian $\partial G/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0,\infty)\times D$ and $\varepsilon\in(0,\varepsilon_0]$.

(ii) F and G are T-periodic in t (T independent of ε). Then the following statements hold.

(a) If p is an equilibrium point of the averaged equation (4) and

$$\det\left(\frac{\partial f}{\partial \mathbf{y}}\right)\Big|_{\mathbf{y}=p}\neq 0, \tag{6}$$

then there exists a T-periodic solution $\varphi(t,\varepsilon)$ of equation (3) such that $\varphi(0,\varepsilon)\to p$ as $\varepsilon\to 0$.

(b) The stability or instability of the limit cycle $\mathbf{x}(t,\varepsilon)$ is given by the stability or instability of the equilibrium point p of the averaged system (4). If p is a simple zero of the averaged system (4), the eigenvalues of the Jacobian matrix evaluated at p provides the linear stability, i.e. if some eigenvalue has a positive real part, then the limit cycle associated to the zero p is unstable; if all the eigenvalues have negative real part, then the limit cycle is stable.

3 Proofs

Proof of Proposition 1. The characteristic polynomial of the linear part of the Kolmogorov system (1) at the equilibrium point $(1,1,1)$ is

$$\begin{aligned} p(\lambda) &= -\lambda^3 + (a_1+b_2+c_3)\lambda^2 + (a_2b_1 - a_1b_2 + a_3c_1 + b_3c_2 - a_1c_3 - b_2c_3)\lambda \\ &\quad + a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1. \end{aligned}$$

Imposing that $p(\lambda) = -\lambda(\lambda^2+\omega^2)$, we obtain the system

$$\begin{aligned} a_1+b_2+c_3 &= 0, \\ a_2b_1 - a_1b_2 + a_3c_1 + b_3c_2 - a_1c_3 - b_2c_3 &= -\omega^2, \\ a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 &= 0. \end{aligned}$$

Solving this system we get the 8 families of zero-Hopf equilibria described in Proposition 1. This completes the proof. ■

Proof of Theorem 1. We shall prove that a periodic orbit bifurcates from the zero-Hopf equilibrium point $(1,1,1)$ of the Kolmogorov system (1) when the parameters of the system (1) are given by statement (i) of Proposition (1). The proof for the others zero-Hopf equilibrium are analogous, and we only indicate the main steps of their proofs.

We perturb the Kolmogorov system (1) with the parameters given in statement (i) of Proposition 1, as it is indicated in (2). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates doing the change of variables $x = X+1$, $y = Y+1$, $z = Z+1$. Then system (1) becomes

$$\begin{aligned} \dot{X} &= (1+X)\left(d_1X^2 + e_1Y^2 + f_1Z^2 + \varepsilon(a_{11}X + b_{11}Y + c_{11}Z)\right), \\ \dot{Y} &= (1+Y)\left(a_2X + d_2X^2 + e_2Y^2 + c_2Z + f_2Z^2 + \varepsilon b_{21}Y\right), \\ \dot{Z} &= (1+Z)\left(a_3X + d_3X^2 + b_3Y + e_3Y^2 + f_3Z^2 + \varepsilon c_{31}Z\right). \end{aligned} \tag{7}$$

In order to facilitate the application of the averaging theory for computing the zero-Hopf bifurcation we write the linear part of system (7) with $\varepsilon = 0$ at the equilibrium point $(0,0,0)$ in its real Jordan normal form, i.e. into the form

$$\begin{pmatrix} 0 & -\sqrt{-b_3c_2} & 0 \\ \sqrt{-b_3c_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then doing the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \dfrac{a_3}{\sqrt{-b_3c_2}} & -\sqrt{-\dfrac{b_3}{c_2}} & 0 \\ \dfrac{a_2}{c_2} & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{8}$$

whose inverse is

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ b_3\sqrt{-\dfrac{c_2}{b_3^3}} & 0 & -\dfrac{a_3}{b_3} \\ 0 & 1 & -\dfrac{a_2}{c_2} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

The differential system (7) writes

$$
\begin{aligned}
\dot{u} \;=\;& \frac{1}{\sqrt{-b_3c_2}}\left(a_3(1+w)\left(d_1w^2+\frac{e_1(\sqrt{-b_3c_2}u-a_3w)^2}{b_3^2}+f_1\left(v-\frac{a_2w}{c_2}\right)^2\right)\right.\\
&+\left(b_3+\sqrt{-b_3c_2}u-a_3w\right)\left(c_2+v+d_2w^2\frac{e_2\left(\sqrt{-b_3c_2}u-a_3w\right)^2}{b_3^2}+f_2\left(v-\frac{a_2w}{c_2}\right)^2\right)\\
&+\varepsilon\frac{1}{\sqrt{-b_3c_2}}\left[\frac{b_{21}}{b_3}\left(\sqrt{-b_3c_2}u-a_3w\right)\left(b_3+\sqrt{-b_3c_2}u-a_3w\right)\right.\\
&\left.+a_3\left(1+w\right)\left(a_{11}w+\frac{b_{11}}{b_3}\left(\sqrt{-b_3c_2}u-a_3w\right)+c_{11}\left(v-\frac{a_2w}{c_2}\right)^2\right)\right], \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
\dot{v} \;=\;& \frac{a_2(1+\omega)}{c_2}\left(d_1w^2+\frac{e_1}{b_3}\left(\sqrt{-b_3c_2}u-a_3w\right)^2+f_1\left(v-\frac{a_2w}{c_2}\right)^2\right)\\
&+\left(1+v-\frac{a_2w}{c_2}\right)\left(\sqrt{-b_3c_2}u+d_3w^2+\frac{e_3}{b_3^2}\left(\sqrt{-b_3c_2}u-a_3w\right)^2+f_3\left(v-\frac{a_2w}{c_2}\right)^2\right)\\
&+\varepsilon\frac{1}{c_2^2}\Big(c_{31}\left(c_2v-a_2w\right)\left(c_2+c_2v-a_2w\right)\\
&+a_2c_2(1+w)\left(a_{11}w+\frac{b_{11}}{b_3}\left(\sqrt{-b_3c_2}u-a_3w\right)+c_{11}\left(v-\frac{a_2w}{c_2}\right)\right)\Big), \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
\dot{w} \;=\;& (1+w)\left(d_1w^2+\frac{e_1}{b_3^2}(\sqrt{-b_3c_2}u-a_3w)^2+f_1\left(v-\frac{a_2w}{c_2}\right)^2\right)\\
&+(1+w)\left(a_{11}w+\frac{b_{11}}{b_3}\left(\sqrt{-b_3c_2}u-a_3w\right)+c_{11}\left(v-\frac{a_2w}{c_2}\right)\right). \qquad (11)
\end{aligned}
$$

Doing the rescaling of the variables $(u,v,w)=(\varepsilon U,\varepsilon V,\varepsilon W)$ system (9)–(11) in the new variables (U,V,W) writes

$$
\begin{aligned}
\dot{U} \;=\;& b_3\sqrt{-\frac{c_2}{b_3}}V+\varepsilon\frac{1}{\sqrt{-b_3c_2^5}b_3^2}\Big((a_3b_{11}+b_{21}b_3)\sqrt{-b_3c_2^5}b_3U-(a_3e_1+b_3e_2)b_3c_2^3U^2+a_3b_3^2c_{11}c_2^2V\\
&+\sqrt{-b_3c_2^7}b_3^2UV+(a_3f_1+b_3f_2)b_3^2c_2^2V^2-(a_2b_3c_{11}+a_3b_{11}c_2-a_{11}b_3c_2+b_{21}b_3c_2)a_3b_3c_2W\\
&-(a_3e_1+b_3e_2)2a_3\sqrt{-b_3c_2^5}UW-(a_3c_2^2+2a_2a_3f_1+2a_2b_3f_2)b_3^2c_2VW\\
&+(a_3b_3^3c_2^2d_1+b_3^2c_2^2d_2+a_3^3c_2^2e_1+a_3^2b_3c_2^2e_2+a_2^2b_3^2f_1+a_2^2b_3^3f_2)W^2\Big), \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
\dot{V} \;=\;& \sqrt{-b_3c_2}U+\varepsilon\frac{1}{b_3^2c_2^3}\Big(a_2b_{11}\sqrt{-b_3c_2^5}b_3U-(a_2e_1+c_2e_3)b_3c_2^3U^2+(a_2c_{11}+c_{31})\,b_3^2c_2^2V\\
&+\sqrt{-b_3c_2^7}b_3^2UV+(a_2f_1+c_2f_3)\,b_3^2c_2^2V^2-(a_2b_3c_{11}+a_3b_{11}c_2-a_{11}b_3c_2+b_3c_2c_{31})\,a_2b_3c_2W\\
&-\left(a_2b_3^2+2a_2a_3e_1+2a_3c_2e_3\right)\sqrt{-b_3c_2^5}UW-2(a_2f_1+c_2f_3)a_2b_3^2c_2VW\\
&+\left(a_2b_3^2c_2^2d_1+b_3^2c_2^3d_3+a_2a_3^2c_2^2e_1+a_3^2c_2^3e_3+a_2^3b_3^2f_1+a_2^2b_3^2c_2f_3\right)W^2\Big), \qquad (13)
\end{aligned}
$$

$$
\dot{W} \;=\; \varepsilon\frac{1}{b_3^2c_2^3}\Big(b_{11}\sqrt{-b_3c_2^5}b_3U-b_3c_2^3e_1U^2+b_3^2c_{11}c_2^2V+b_3^2c_2^2f_1V^2
$$

$$
\begin{aligned}
&- \big(a_2 b_3^2 c_{11} c_2 + a_3 b_{11} b_3 c_2^2 + a_{11} b_3^2 c_2^2\big) W - 2a_3 \sqrt{-b_3 c_2^5} e_1 UW - 2a_2 b_3^2 c_2 f_1 VW \\
&+ \big(b_3^2 c_2^2 d_1 + a_3^2 c_2^2 e_1 + a_2^2 b_3^2 f_1\big) W^2\Big) \qquad (14)
\end{aligned}
$$

Now we pass from the differential system (12)–(14) to cylindrical coordinates (r, W) defined by $U = r\cos\theta$ and $V = r\sin\theta$, and we obtain

$$
\begin{aligned}
\dot{r} \;=\;& \frac{\varepsilon}{b_3^2 c_2^2}\Big(\big(\sqrt{-b_3 c_2^3}(a_3 e_1 + b_3 e_2)\cos^3\theta + b_3 c_2 (b_3 c_2 - e_3 c_2 - a_2 e_1)\sin\theta\cos^2\theta \\
&+ \sqrt{-b_3 c_2}\, b_3\,(b_3 c_2 - a_3 f_1 - b_3 f_2)\sin^2\theta\cos\theta + b_3^2(a_2 f_1 + c_2 f_3)\sin^3\theta\big) r^2 \\
&+ b_3 c_2\big((a_3 b_{11} + b_{21} b_3) c_2 \cos^2\theta + \sqrt{-b_3 c_2}(a_2 b_{11} - a_3 c_{11})\sin\theta\cos\theta \\
&+ b_3(a_2 c_{11} + c_2 c_{31})\sin^2\theta\big) r - W\big(2a_3 c_2^2(a_3 e_1 + b_3 e_2)\cos^2\theta + \sqrt{-b_3 c_2}\big(a_2 c_2 b_3^2 \\
&- 2a_2 f_2 b_3^2 - a_3 c_2^2 b_3 - 2a_2 a_3 f_1 b_3 + 2a_2 a_3 c_2 e_1 + 2a_3 c_2^2 e_3\big)\sin\theta\cos\theta \\
&+ 2a_2 b_3^2(a_2 f_1 + c_2 f_3)\sin^2\theta\big) r + W\big(a_3\sqrt{-b_3 c_2}(a_2 b_3 c_{11} + a_3 b_{11} c_2 - a_{11} b_3 c_2 + b_{21} b_3 c_2)\cos\theta \\
&- a_2 b_3(a_2 b_3 c_{11} + a_3 b_{11} c_2 - a_{11} b_3 c_2 + b_3 c_2 c_{31})\sin\theta\big) \\
&- W^2\big(\sqrt{-b_3 c_2}\big(c_2^2 e_1 a_3^3 + b_3 c_2^2 e_2 a_3^2 + b_3^2 c_2^2 d_1 a_3 + a_2^2 b_3^2 f_1 a_3 + b_3^3 c_2^2 d_2 + a_2^2 b_3^3 f_2\big)\cos\theta \\
&- b_3\big(b_3^2 f_1 a_2^3 + b_3^2 c_2 f_3 + a_2^2 + b_3^2 c_2^2 d_1 a_2 + a_3^2 c_2^2 e_1 a_2 + b_3^2 c_2^3 d_3 \\
&+ a_3^2 c_2^3 e_3\big)\sin\theta\big)\Big), \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\theta} \;=\;& \sqrt{-b_3 c_2} + \frac{\varepsilon}{b_3^2 c_2^2}\Big(\frac{1}{b_3 c_2 r}\big(b_3\big(b_3^2 f_1 a_2^3 + b_3^2 c_2 f_3 a_2^2 + b_3^2 c_2^2 d_1 a_2 + a_3^2 c_2^2 e_1 a_2 + b_3^2 c_2^3 d_3 \\
&+ a_3^2 c_2^3 e_3\big)\cos\theta + \sqrt{-b_3 c_2}\big(c_2^2 e_1 a_3^3 + b_3 c_2^2 e_2 a_3^2 + b_3^2 c_2^2 d_1 a_3 + a_2^2 b_3^2 f_1 a_3 + b_3^3 c_2^2 d_2 \\
&+ a_2^2 b_3^3 f_2\big)\sin\theta\big) W^2 - \cos\theta\big(\sqrt{-b_3 c_2^3}\big(a_2 b_3^2 + 2a_2 a_3 e_1 + 2a_3 c_2 e_3\big)\cos\theta \\
&- 2\big(-a_2^2 f_1 b_3^2 - a_2 c_2 f_3 b_3^2 + a_3 c_2^2 e_2 b_3 + a_3^2 c_2^2 e_1\big)\sin\theta\big) W + \big(-\sqrt{b_3 c_2}\, b_3\big(a_3 c_2^2 + 2a_2 a_3 f_1 \\
&+ 2a_2 b_3 f_2\big)\sin^2\theta - \frac{\sqrt{-b_3 c_2}}{r}\big(b_{11} c_2 a_3^2 + a_2 b_3 c_{11} a_3 - a_{11} b_3 c_2 a_3 + b_{21} b_3 c_2 a_3\big)\sin\theta \\
&\frac{1}{r}\big(a_2^2 c_{11} b_3^2 - a_{11} a_2 c_2 b_3^2 + a_2 c_2 c_{31} b_3^2 + a_2 a_3 b_{11} c_2 b_3\big)\cos\theta\big) W + b_3 c_2\big(a_2 b_{11}\sqrt{-b_3 c_2}\cos^2\theta \\
&+ (a_2 b_3 c_{11} - a_3 b_{11} c_2 - b_{21} b_3 c_2 + b_3 c_2 c_{31})\sin\theta\cos\theta + a_3\sqrt{-b_3}\, c_{11}\sqrt{c_2}\sin^2\theta\big) \\
&- \frac{1}{2} c_2 r\big(-b_3\big(b_3 c_2^2 - 2e_3 c_2^2 - 2a_2 e_1 c_2 - b_3 f_3 c_2 - a_2 b_3 f_1\big)\cos^3\theta - \sqrt{-b_3 c_2}\big(2c_2 b_3^2 - f_2 b_3^2 \\
&- 2c_2 e_2 b_3 - a_3 f_1 b_3 - 2a_3 c_2 e_1\big)\sin\theta\cos^2\theta + b_3^2\big(c_2^2 - f_3 c_2 - a_2 f_1\big)\sin^2\theta\cos\theta + b_3^2\big(c_2^2 - f_3 c_2 \\
&- a_2 f_1\big)\cos\theta - \sqrt{-b_3 c_2}\, b_3(a_3 f_1 + b_3 f_2)\sin^3\theta - \sqrt{-b_3}\, b_3\sqrt{c_2}(a_3 f_1 + b_3 f_2)\sin\theta\big)\Big), \qquad (16)
\end{aligned}
$$

$$
\begin{aligned}
\dot{W} \;=\;& \frac{\varepsilon}{b_3^2 c_2^2}\Big(b_3 r\big(b_{11}\sqrt{-b_3 c_2}\cos\theta + b_3 c_{11}\sin\theta\big) c_2^2 + b_3 r^2\big(b_3 f_1 \sin^2\theta - c_2 e_1\cos^2\theta\big) c_2^2 \\
&- b_3 c_2(a_2 b_3 c_{11} + a_3 b_{11} c_2 - a_{11} b_3 c_2) W - 2rW\big(a_2 f_1 \sin\theta\, b_3^2 \\
&+ a_3\sqrt{-b_3}\, c_2^{3/2} e_1 \cos\theta\big) c_2 + \big(c_2^2 d_1 b_3^2 + a_2^2 f_1 b_3^2 + a_3^2 c_2^2 e_1\big) W^2\Big). \qquad (17)
\end{aligned}
$$

We change the independent variable from t to θ, and denoting the derivative with respect to θ by a dot, then the differential system (15)–(17) becomes

$$
\dot{r} \;=\; \frac{\varepsilon}{(-b_3 c_2)^{5/2}}\Big(\big(\sqrt{-b_3 c_2^3}(a_3 e_1 + b_3 e_2)\cos^3\theta + b_3 c_2(b_3 c_2 - e_3 c_2 - a_2 e_1)\sin\theta\cos^2\theta + \sqrt{-b_3 c_2}\, b_3(b_3 c_2 -
$$

$$a_3f_1 - b_3f_2)\sin^2\theta\cos\theta + b_3^2(a_2f_1 + c_2f_3)\sin^3\theta\big)r^2 + b_3c_2\big((a_3b_{11} + b_{21}b_3)c_2\cos^2\theta + \sqrt{-b_3c_2}(a_2b_{11} - a_3c_{11})\sin\theta\cos\theta + b_3(a_2c_{11} + c_2c_{31})\sin^2\theta\big)r - W\big(2a_3c_2^2(a_3e_1 + b_3e_2)\cos^2\theta + \sqrt{-b_3c_2}\big(a_2c_2b_3^2 - 2a_2 f_2b_3^2 - a_3c_2^2b_3 - 2a_2a_3f_1b_3 + 2a_2a_3c_2e_1 + 2a_3c_2^2e_3\big)\sin\theta\cos\theta + 2a_2b_3^2(a_2f_1 + c_2f_3)\sin^2\theta\big)r + W\big(a_3 \sqrt{-b_3c_2}(a_2b_3c_{11} + a_3b_{11}c_2 - a_{11}b_3c_2 + b_{21}b_3c_2)\cos\theta - a_2b_3(a_2b_3c_{11} + a_3b_{11}c_2 - a_{11}b_3c_2 + b_3c_2c_{31}) \sin\theta\big) - W^2\big(\sqrt{-b_3c_2}\big(c_2^2e_1a_3^3 + b_3c_2^2e_2a_3^2 + b_3^2c_2^2d_1a_3 + a_2^2b_3^2f_1a_3 + b_3^3c_2^2d_2 + a_2^2b_3^3f_2\big)\cos\theta - b_3\big(b_3^2f_1a_2^3 + b_3^2c_2f_3a_2^2 + b_3^2c_2^2d_1a_2 + a_3^2c_2^2e_1a_2 + b_3^2c_2^3d_3 + a_3^2c_2^3e_3\big)\sin\theta\big)\Big) = F_1(\theta, r, W)\,, \tag{18}$$

$$\dot{W} = \frac{\varepsilon}{(-b_3c_2)^{5/2}}\Big(b_3r\big(b_{11}\sqrt{-b_3c_2}\cos\theta + b_3c_{11}\sin\theta\big)c_2^2 + b_3r^2\big(b_3f_1\sin^2\theta - c_2e_1\cos^2\theta\big)c_2^2 - b_3c_2(a_2b_3c_{11} + a_3b_{11}c_2 - a_{11}b_3c_2)W - 2rW\big(a_2f_1\sin\theta b_3^2 + a_3\sqrt{-b_3}c_2^{3/2}e_1\cos\theta\big)c_2 + \big(c_2^2d_1b_3^2 + a_2^2f_1b_3^2 + a_3^2c_2^2e_1\big)W^2\Big) = F_2(\theta, r, W). \tag{19}$$

We shall apply the averaging theory described in Theorem 2 to the differential system (18)–(19). Using the notation of section 2 we have $t = \theta$, $T = 2\pi$, $\mathbf{x} = (r, W)^T$ and

$$F(\theta, r, W) = \begin{pmatrix} F_1(\theta, r, W) \\ F_2(\theta, r, W) \end{pmatrix} \quad and \quad f(r, W) = \begin{pmatrix} f_1(r, W) \\ f_2(r, W) \end{pmatrix}.$$

It is immediate to check that system (18)–(19) satisfies all the assumptions of Theorem 2. Now we compute the integrals (5), i.e.

$$f_1(r, W) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta, r, W)d\theta = \frac{r}{2(-b_3c_2)^{5/2}}\big(-2a_2^2b_3^2f_1W + a_2b_3^2c_2(c_{11} - 2f_3W) + c_2^2(b_3^2(b_{21} + c_{31}) - 2a_3^2e_1W + a_3b_3(b_{11} - 2e_2W))\big)\,,$$

$$f_2(r, W) = \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta, r, W)d\theta = \frac{1}{2(-b_3c_2)^{5/2}}\big(b_3c_2^2(-c_2e_1 + b_3f_1)r^2 + 2b_3c_2(-a_2b_3c_{11} - a_3b_{11}c_2 + a_{11}b_3c_2)W + 2(a_3^2c_2^2e_1 + b_3^2(c_2^2d_1 + a_2^2f_1))W^2\big).$$

The system $f_1(r, W) = f_2(r, W) = 0$ has a unique solution (r^*, W^*) with $r^* > 0$, namely

$$(r^*, W^*) = \left(\frac{1}{T_1}\sqrt{\frac{C_1N_1T_1 - D_1N_1^2}{R_1}}, -\frac{N_1}{T_1}\right)$$

if $T_1 > 0$, and $R_1(C_1N_1T_1 - D_1N_1^2) > 0,$ and the Jacobian (6) at (r^*, W^*) takes the value

$$N_1(C_1T_1 - D_1N_1)/(2T_1b_3^5c_2^5) \neq 0$$

where

$$C_1 = 2b_3c_2(-a_2b_3c_{11} - a_3b_{11}c_2 + a_{11}b_3c_2), \qquad D_1 = 2\big(a_3^2c_2^2e_1 + b_3^2(a_2^2f_1 + c_2^2d_1)\big),$$
$$N_1 = a_2b_3^2c_2c_{11} + c_2^2\big(b_3^2(b_{21} + c_{31}) + a_3b_3b_{11}\big), \qquad T_1 = -2a_2^2b_3^2f_1 - 2a_2b_3^2c_2f_3 + c_2^2\big(-2a_3^2e_1 - 2a_3b_3e_2\big),$$
$$R_1 = b_3c_2^2(b_3f_1 - c_2e_1).$$

Then Theorem 2 guarantees for $\varepsilon > 0$ sufficiently small the existence of a periodic solution $(r(\theta, \varepsilon), W(\theta, \varepsilon))$ of system (18)–(19) such that $(r(0, \varepsilon), W(0, \varepsilon)) \to (r^*, W^*)$ when $\varepsilon \to 0$. So for $\varepsilon > 0$ sufficiently small system (15)–(17) has the periodic solution $(r(t, \varepsilon), (t, \varepsilon), W(t, \varepsilon))$ with $(t, \varepsilon) = \sqrt{-b_3c_2}\,t + O(\varepsilon)$, such that $(r(0, \varepsilon), (0, \varepsilon), W(0, \varepsilon)) \to (r^*, 0, W^*)$ when $\varepsilon \to 0$. Consequently system (12)–(14) has the periodic solution

$$(U(t, \varepsilon), V(t, \varepsilon), W(t, \varepsilon)) = \Big(r^*\cos(\sqrt{-b_3c_2}\,t) + O(\varepsilon), r^*\sin(\sqrt{-b_3c_2}\,t) + O(\varepsilon), W^* + O(\varepsilon)\Big)\,,$$

for $\varepsilon > 0$ sufficiently small. Therefore system (9)–(11) for $\varepsilon > 0$ sufficiently small has the periodic solution

$$(u(t,\varepsilon), v(t,\varepsilon), w(t,\varepsilon)) = \left(\varepsilon r^* \cos(\sqrt{-b_3c_2}\,t) + O(\varepsilon^2), \varepsilon r^* \sin(\sqrt{-b_3c_2}\,t) + O(\varepsilon^2), \varepsilon W^* + O(\varepsilon^2)\right). \quad (20)$$

Finally for $\varepsilon > 0$ sufficiently small system (7) has the periodic solution $(X(t,\varepsilon), Y(t,\varepsilon), Z(t,\varepsilon))$ obtained from (20) through the change of variables (8). This periodic solution tends to the origin of coordinates when $\varepsilon \to 0$. Therefore there is a periodic solution starting at the zero-Hopf equilibrium point $(1,1,1)$ when $\varepsilon = 0$. This completes the proof of Theorem 1 under the condition (i). ■

Example 1 *Consider the Kolmogorov system*

$$\begin{cases} \dot{x} = & x(2(x-1)^2 + (z-1)^2), \\ \dot{y} = & y(-1 + z + 2(y-1)^2 + (z-1)^2), \\ \dot{z} = & z(x - y + (x-1)^2 + (y-1)^2). \end{cases} \quad (21)$$

This system in the new variables (X, Y, Z) writes

$$\begin{cases} \dot{X} = & (X+1)(2X^2 + 2X\varepsilon + Z^2), \\ \dot{Y} = & (Y+1)(2Y^2 + Y\varepsilon + Z^2 + Z), \\ \dot{Z} = & (Z+1)(X^2 + Y^2 + Z\varepsilon + X - Y). \end{cases}$$

The corresponding system associated to system (18)–(19) satisfies

$$\begin{aligned} F_1(\theta, r, W) = & \ (-3\cos(\theta)^3 + 2\cos(\theta)^2\sin(\theta) + \cos(\theta))r^2 + (4\cos(\theta)^2 W - 3\cos(\theta)\sin(\theta)W + 1)r \\ & +2\sin(\theta)W^2 + \cos(\theta)W, \\ F_2(\theta, r, W) = & \ \sin(\theta)^2 r^2 + 2W^2 + 2W. \end{aligned}$$

To look for the limit cycles we must solve the system given by the averaged function

$$f(r, W) = (f_1(r, W), f_2(r, W)) = (0, 0)$$

where

$$f_1(r, W) = r(2W + 1) \quad \text{and} \quad f_2(r, W) = 2W^2 + \frac{1}{2}r^2 + 2W. \quad (22)$$

This system has four solutions for (r, W) given by $(0,0)$, $(0,-1)$, $(1,-1/2)$, $(-1,-1/2)$. The solution $(0,0)$ does not provide any periodic orbit because it corresponds to the equilibrium point localized at the origin. The averaging theory of first order provides two good solutions: $(0,-1)$ and the $(1,-1/2)$. Since the determinants (6) at these two solutions are 2 and -2 and thus non-zero, respectively, the Kolmogorov system (21) has two limit cycles bifurcating from the origin provided by the averaging theory of first order. We plot these two limit cycles for $\varepsilon = 10^{-5}$ in Figure 1, where we provide the initial conditions of the two limit cycles that we have drawn. The software used for doing all the figures is Maple 20.

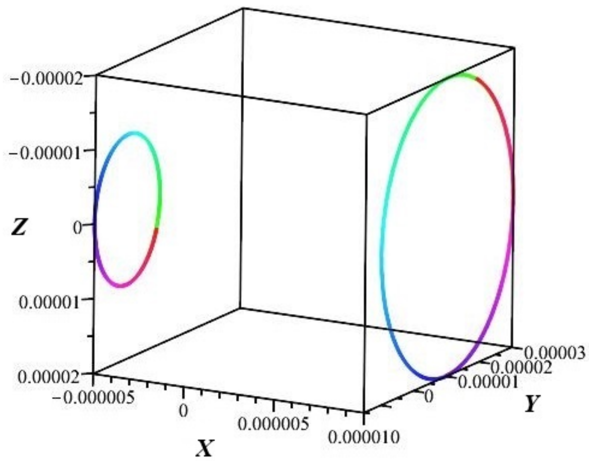

Figure 1: 1st LC: $X(0) = \varepsilon,\ Y(0) = \varepsilon,\ Z(0) = 2\varepsilon$. 2nd LC: $X(0) = -\varepsilon/2,\ Y(0) = -3\varepsilon/2,\ Z(0) = 0$.

Since the eigenvalues of the Jacobian matrix of (f_1, f_2) *at the singular points* $(0, -1)$ *and* $(1, -1/2)$ *are* $(-2, -1)$ *and* $\pm\sqrt{2}$*, respectively, by Theorem 2 the limit cycles are stable and unstable. Going back through the changes of variables as we did in the proof of Theorem 1 we obtain that the limit cycles bifurcating from the equilibrium point* $(1, 1, 1)$ *of system (21) are*

$$
\begin{aligned}
(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) &= (1-\varepsilon+O(\varepsilon^2), 1-\varepsilon+O(\varepsilon^2), 1+O(\varepsilon^2)),\\
(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) &= (1-\varepsilon/2+O(\varepsilon^2), 1-\varepsilon(\cos t+1/2)+O(\varepsilon^2), 1+\varepsilon\sin t+O(\varepsilon^2)),
\end{aligned}
$$

respectively. This completes Example 1.

Now we perturb the Kolmogorov system (1) with the parameters given in statement (ii) of Proposition 1, as it is indicated in (2). We translate the equilibrium point $(1, 1, 1)$ *to the origin of coordinates doing the change of variables* $x = X + 1$, $y = Y + 1$, $z = Z + 1$*. Then system (1) becomes*

$$
\begin{aligned}
\dot{X} &= (X+1)\big((-2Xb_{21}+Zc_{11})\varepsilon + d_1X^2 + e_1Y^2 + f_1Z^2 - b_{20}X + b_1Y\big),\\
\dot{Y} &= (Y+1)\big((Yb_{21}+Zc_{21})\varepsilon + d_2X^2 + e_2Y^2 + f_2Z^2 + a_2X + b_{20}Y\big),\\
\dot{Z} &= (Z+1)\big(X^2d_3 + Y^2e_3 + Z^2f_3 + Z\varepsilon c_{31} + Xa_3 + Yb_3\big).
\end{aligned}
\tag{23}
$$

In order to facilitate the application of the averaging theory for computing the zero-Hopf bifurcation we write the linear part of system (23) with $\varepsilon = 0$ *at the equilibrium point* $(0, 0, 0)$ *in its real Jordan normal form, i.e. into the form*

$$
\begin{pmatrix}
0 & -\sqrt{-a_2b_1-b_{20}^2} & 0\\
\sqrt{-a_2b_1-b_{20}^2} & 0 & 0\\
0 & 0 & 0
\end{pmatrix}.
$$

Then doing the change of variables

$$
\begin{pmatrix} u\\ v\\ w \end{pmatrix} =
\begin{pmatrix}
-\dfrac{b_{20}}{b_1\sqrt{-a_2b_1-b_{20}^2}} & \dfrac{1}{\sqrt{-a_2b_1-b_{20}^2}} & 0\\
\dfrac{1}{b_1} & 0 & 0\\
-\dfrac{a_2b_3-a_3b_{20}}{a_2b_1+b_{20}^2} & -\dfrac{a_3b_1+b_{20}b_3}{a_2b_1+b_{20}^2} & 1
\end{pmatrix}
\begin{pmatrix} X\\ Y\\ Z \end{pmatrix},
$$

whose inverse is

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & b_1 & 0 \\ \sqrt{-a_2b_1 - b_{20}^2} & b_{20} & 0 \\ -\dfrac{a_3b_1 + b_{20}b_3}{\sqrt{-a_2b_1 - b_{20}^2}} & b_3 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

And by following the same steps as the first case (i) indicated in (2), we get the system

$$\begin{aligned}
\dot{r} \;=\; & -\frac{\varepsilon}{b_1\big(-a_2b_1 - b_{20}^2\big)^2}\Big(\sqrt{-a_2b_1 - b_{20}^2}\Big(\Big((2(a_3b_1 + b_{20}b_3))b_3(b_1f_2 - b_{20}f_1) + (a_2b_1 + b_{20}^2)((a_2 - b_{20})b_1^2 \\
& +2b_{20}^2(b_1 - e_1) + 2b_1b_{20}e_2) - (a_3b_1 + b_{20}b_3)^2 f_1 - (a_2b_1 + b_{20}^2)^2 e_1\Big)\cos^2\theta\sin\theta r^2 + r\big(\cos^2\theta\big((2(a_3b_1 \\
& +b_{20}b_3))W(b_1f_2 - b_{20}f_1) + (a_3b_1 + b_{20}b_3)(b_1c_{21} - b_{20}c_{11}) + (a_2b_1 + b_{20}^2)b_1b_{21}) + (a_2b_1 + b_{20}^2)(2Wb_3f_1 \\
& -2b_1b_{21} + b_3c_{11})\sin^2\theta\big)\big) + (-a_2b_1 - b_{20}^2)\big(\big(b_3f_1(2a_3b_1 + 3b_{20}b_3) - b_{20}^2(b_1e_2 - b_{20}e_1) - b_1b_3^2f_2 + (a_2b_1 \\
& +b_{20}^2)(b_1^2 - b_1b_{20} + 2b_{20}e_1) - b_1^2(b_1d_2 - b_{20}d_1)\big)\cos\theta\sin^2\theta r^2 + (2W((a_3b_1 + b_{20}b_3)f_1 - b_3(b_1f_2 - b_{20}f_1)) \\
& -b_1(3b_{20}b_{21} + b_3c_{21}) + c_{11}(a_3b_1 + 2b_{20}b_3))\cos\theta\sin\theta r - \cos\theta W(W(b_1f_2 - b_{20}f_1) + b_1c_{21} - b_{20}c_{11})) \\
& -(-a_2b_1 - b_{20}^2)^{3/2}(\sin\theta W(Wf_1 + c_{11}) + \sin^3\theta r^2(b_1^2d_1 + b_{20}^2e_1 + b_3^2f_1)) \\
& -\cos^3\theta r^2((a_3b_1 + b_{20}b_3)^2(b_1f_2 - b_{20}f_1) + (a_2b_1 + b_{20}^2)^2(b_1b_{20} + b_1e_2 - b_{20}e_1)\Big)\Big) \\
=\; & F_1(\theta, r, W), \qquad (24)
\end{aligned}$$

$$\begin{aligned}
\dot{W} \;=\; & -\frac{\varepsilon}{(-a_2b_1 - b_{20}^2)^{5/2}}\Big(\sqrt{-a_2b_1 - b_{20}^2}\Big(\Big(W\Big(2(a_3b_1 + b_{20}b_3)((a_3b_1 + b_{20}b_3)f_2 + (a_2b_3 - a_3b_{20})f_1 \\
& -(a_2b_1 + b_{20}^2)f_3) - (a_2b_1 + b_{20}^2)^2b_3\Big) + (a_3b_1 + b_{20}b_3)((a_3b_1 + b_{20}b_3)c_{21} + (a_2b_3 - a_3b_{20})c_{11}) \\
& +(a_3b_1 + b_{20}b_3)(a_2b_1 + b_{20}^2)(b_{21} - c_{31})\Big)\cos\theta r + \Big(2(a_3b_1 + b_{20}b_3)^2b_3f_2 \\
& +(a_3b_1 + b_{20}b_3)(2(a_2b_3 - a_3b_{20})b_3f_1 + (a_2b_1 + b_{20}^2)(b_1(a_2 - a_3) + b_{20}(2b_{20} - b_3 + 2e_2) - 2b_3f_3)\Big) \\
& +(a_2b_3 - a_3b_{20})(a_2b_1 + b_{20}^2)(b_1^2 + 2b_{20}e_1) - (a_2b_1 + b_{20}^2)^2(2b_{20}e_3 + b_3^2)\Big)\sin\theta\cos\theta r^2\Big) \\
& -(-a_2b_1 - b_{20}^2)(\sin\theta r(W(2b_3((a_3b_1 + b_{20}b_3)f_2 + (a_2b_3 - a_3b_{20})f_1) - (a_2b_1 + b_{20}^2)(a_3b_1 + b_{20}b_3 + 2b_3f_3)\Big) \\
& +(a_3b_1 + b_{20}b_3)(b_{20}b_{21} + b_3c_{21}) - (a_2b_3 - a_3b_{20})(2b_1b_{21} - b_3c_{11}) - (a_2b_1 + b_{20}^2)b_3c_{31}) \\
& +W\Big((a_3b_1 + b_{20}b_3)(Wf_2 + c_{21}) + (a_2b_3 - a_3b_{20})(Wf_1 + c_{11}) - (a_2b_1 + b_{20}^2)(Wf_3 + c_{31})\Big)\Big) \\
& -\Big(r^2\Big((a_3b_1 + b_{20}b_3)^3f_2 - (a_2b_1 + b_{20}^2)^3e_3\Big) + (a_2b_1 + b_{20}^2)^2r^2((b_{20} - b_3 + e_2)(a_3b_1 + b_{20}b_3) \\
& +(a_2b_3 - a_3b_{20})e_1) + (a_3b_1 + b_{20}b_3)^2r^2\Big((a_2b_3 - a_3b_{20})f_1 - (a_2b_1 + b_{20}^2)f_3)\Big)\cos\theta - \Big(\\
& -(a_2b_1 + b_{20}^2)r^2((a_2b_1b_{20} + b_1^2d_2 + b_{20}^3 + b_{20}^2e_2 + b_3^2f_2)(a_3b_1 + b_{20}b_3) \\
& +(a_2b_3 - a_3b_{20})(b_1^2d_1 + b_{20}^2e_1 + b_3^2f_1)) \\
& +(a_2b_1 + b_{20}^2)^2r^2(a_3b_1b_3 + b_1^2d_3 + b_{20}^2e_3 + (b_{20} + f_3)b_3^2)\Big)\sin^2\theta\Big) \\
=\; & F_2(, r, W). \qquad (25)
\end{aligned}$$

We shall apply the averaging theory described in Theorem 2 to the differential system (24)–(25). Using the notation of section 2 we have $t = \theta$, $T = 2\pi$, $\mathbf{x} = (r, W)^T$ *and*

$$F(\theta, r, W) = \begin{pmatrix} F_1(\theta, r, W) \\ F_2(\theta, r, W) \end{pmatrix} \quad \text{and} \quad f(r, W) = \begin{pmatrix} f_1(r, W) \\ f_2(r, W) \end{pmatrix}.$$

It is immediate to check that system (24)–(25) satisfies all the assumptions of Theorem 2. Now we compute the integrals (5), i.e.

$$\begin{aligned} f_1(r,W) = &\quad \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta,r,W)d \ = -\frac{r(T_2W+N_2)}{2\big(-a_2b_1-b_{20}^2\big)^{3/2}},\\ f_2(r,W) = &\quad \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta,r,W)d \ = \frac{D_2W^2+R_2r^2+C_2W}{2\big(-a_2b_1-b_{20}^2\big)^{5/2}}. \end{aligned}$$

The system $f_1(r,W) = f_2(r,W) = 0$ has a unique solution (r^*,W^*) with $r^* > 0$, namely

$$(r^*,W^*) = \left(\frac{1}{T_2}\sqrt{\frac{C_2N_2T_2-D_2N_2^2}{R_2}}, -\frac{N_2}{T_2}\right)$$

if $T_2 > 0$, and $R_2(C_2N_2T_2 - D_2N_2^2) > 0$, and the Jacobian (6) at (r^*,W^*) takes the value

$$N_2(C_2T_2 - D_2N_2)/2T_2(a_2b_1 + b_{20}^2)^4 \neq 0$$

where

$$\begin{aligned} C_2 = &\quad -2\big(a_2b_1+b_{20}^2)((a_3b_1+b_{20}b_3)c_{21}+(a_2b_3-a_3b_{20})c_{11}-(a_2b_1+b_{20}^2)c_{31}\big),\\ D_2 = &\quad -2(a_2b_1+b_{20}^2)\big((a_3b_1+b_{20}b_3)f_2+(a_2b_3-a_3b_{20})f_1-(a_2b_1+b_{20}^2)f_3\big),\\ N_2 = &\quad -b_{21}(a_2b_1+b_{20}^2)+(a_3b_1+b_{20}b_3)c_{21}+(a_2b_3-a_3b_{20})c_{11},\\ R_2 = &\quad (a_2b_1+b_{20}^2)^2\big(a_3b_1b_3+b_1^2d_3+b_{20}^2e_3+(a_2b_3-a_3b_{20})e_1+(a_3b_1+b_{20}b_3)(b_{20}-b_3+e_2)\\ &\quad +(b_{20}+f_3)b_3^2)+(a_3b_1+b_{20}b_3)^2((a_2b_3-a_3b_{20})f_1-(a_2b_1+b_{20}^2)f_3)+(a_3b_1+b_{20}b_3)^3f_2\\ &\quad -(a_2b_1+b_{20}^2)^3e_3-(a_2b_1+b_{20}^2)((a_2b_1b_{20}+b_1^2d_2+b_{20}^3+b_{20}^2e_2+b_3^2f_2)(a_3b_1+b_{20}b_3)\\ &\quad +(a_2b_3-a_3b_{20})(b_1^2d_1+b_{20}^2e_1+b_3^2f_1)\big),\\ T_2 = &\quad 2\big((b_3(a_2f_1+b_{20}f_2)+a_3(b_1f_2-b_{20}f_1))\big). \end{aligned}$$

Finally, we apply Theorem 2 like in the first case (i), then for $\varepsilon > 0$ sufficiently small system (23) has a periodic solution $(x(\theta),y(\theta),z(\theta)) = (\varepsilon X(\theta),\varepsilon Y(\theta),\varepsilon Z(\theta))$ which tends to the origin of coordinates when $\varepsilon \to 0$. Therefore, it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when $\varepsilon = 0$. This completes the proof of Theorem 1 under the condition (ii).

Example 2 *Consider the Kolmogorov system*

$$\begin{cases} \dot{x} = & x\Big(-x+y-\dfrac{161}{162}(x-1)^2+(z-1)^2\Big),\\ \dot{y} = & y(-2x+1+y+(z-1)^2),\\ \dot{z} = & z(-y+1). \end{cases} \tag{26}$$

This system in the new variables (X,Y,Z) writes

$$\begin{cases} \dot{X} = & (X+1)\Big(\dfrac{1}{9}Z\varepsilon-\dfrac{161}{162}X^2+Z^2-X+Y\Big),\\ \dot{Y} = & (Y+1)(Z^2-2X+Y),\\ \dot{Z} = & -(Z+1)Y. \end{cases}$$

The corresponding system associated to system (24)–(25) satisfies

$$F_1(\theta,r,W) = -r\Big(\frac{1}{9}\cos^2(\theta)-\Big(-2W-\frac{1}{9}\Big)\sin^2(\theta)\Big)-\frac{163}{162}\cos(\theta)\sin^2(\theta)r^2-\Big(-2W-\frac{2}{9}\Big)\cos(\theta)\sin(\theta)r$$
$$-\frac{1}{9}\cos(\theta)W+\sin(\theta)W\Big(W+\frac{1}{9}\Big)+\frac{1}{162}\sin^3(\theta)r^2+\cos^3(\theta)r^2$$
$$F_2(\theta,r,W) = -\Big(-W-\frac{2}{9}\Big)\cos(\theta)r+\sin(\theta)r\Big(-3W-\frac{2}{9}\Big)+W\Big(W+\frac{2}{9}\Big)-r^2\cos^2(\theta)+\frac{82}{81}r^2\sin^2(\theta).$$

To look for the limit cycles we must solve the system given by the averaged function

$$f(r,W) = (f_1(r,W), f_2(r,W)) = (0,0)$$

where

$$f_1(r,W) = -\frac{1}{2}r\left(2W+\frac{2}{9}\right) \quad \textit{and} \quad f_2(r,W) = W^2+\frac{1}{162}r^2+\frac{2}{9}W. \tag{27}$$

This system has four solutions for (r,W) *given by* $(0,0)$, $(0,-2/9)$, $(\sqrt{2},-1/9)$, $(-\sqrt{2},-1/9)$. *As in Example 1 we have two good solutions, the* $(0,-2/9)$ *and the* $(\sqrt{2},-1/9)$. *Since the determinants (6) at these two solutions are* $-2/81$ *and* $2/81$ *and thus non-zero, respectively, the Kolmogorov system (26) has two limit cycles bifurcating from the origin provided by the averaging theory of first order. We plot these two limit cycles for* $\varepsilon = 10^{-5}$ *in Figure 2.*

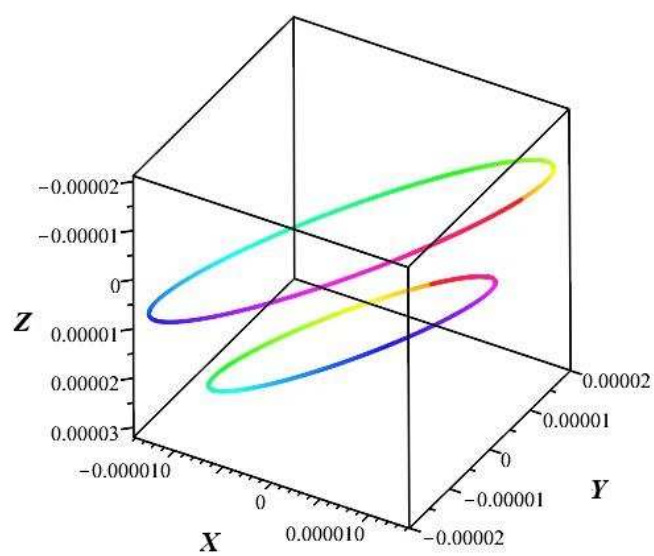

Figure 2: 1st LC: $X(0)=\varepsilon,\ Y(0)=\varepsilon,\ Z(0)=7\varepsilon/9$. 2nd LC: $X(0)=0,\ Y(0)=\varepsilon\sqrt{2},\ Z(0)=(\sqrt{2}-1/9)\varepsilon$.

Since the eigenvalues of the Jacobian matrix of (f_1,f_2) *at the singular points* $(0,-2/9)$ *and* $(\sqrt{2},-1/9)$ *are* $(-2/9,1/9)$ *and* $\pm i\sqrt{2}/9$, *respectively, by Theorem 2 the limit cycles are unstable and stable. Going back through the changes of variables as we did in the proof of Theorem 1 we obtain that the limit cycles bifurcating from the equilibrium point* $(1,1,1)$ *of system (26) are*

$$(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon)) = (1+O(\varepsilon^2),1+O(\varepsilon^2),1-2\varepsilon/9+O(\varepsilon^2)),$$

$$(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon)) = (1+\varepsilon\sqrt{2}\sin t+O(\varepsilon^2),1+\varepsilon\sqrt{2}(\cos t+\sin t)+O(\varepsilon^2),1+\varepsilon\sqrt{2}(\cos t-\sin t-1/9)+O(\varepsilon^2)),$$

respectively. This completes Example 2.

Now we perturb the Kolmogorov system (1) with the parameters given in statement (iii) of Proposition 1, as it is indicated in (2). We translate the equilibrium point $(1,1,1)$ *to the origin of coordinates doing the change of variables* $x = X+1, y = Y+1, z = Z+1$*. Then system (1) becomes*

$$\begin{aligned} \dot{X} &= (X+1)\big((Xa_{11}+Yb_{11}+Zc_{11})\varepsilon + X^2 d_1 + Y^2 e_1 + Z^2 f_1\big),\\ \dot{Y} &= (Y+1)\big(X^2 d_2 + Y^2 e_2 + Y\varepsilon b_{21} + Z^2 f_2 + Xa_2 + Yb_{20} + Zc_2\big),\\ \dot{Z} &= (Z+1)(X^2 d_3 + Y^2 e_3 + Z^2 f_3 - Z\varepsilon b_{21} + Xa_3 + Yb_3 - Zb_{20}). \end{aligned} \tag{28}$$

In order to facilitate the application of the averaging theory for computing the zero-Hopf bifurcation we write the linear part of system (28) with $\varepsilon = 0$ *at the equilibrium point* $(0,0,0)$ *in its real Jordan normal form, i.e. into the form*

$$\begin{pmatrix} 0 & -\sqrt{-b_{20}^2 - b_3c_2} & 0\\ \sqrt{-b_{20}^2 - b_3c_2} & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}.$$

Then doing the change of variables

$$\begin{pmatrix} u\\ v\\ w \end{pmatrix} = \begin{pmatrix} -\dfrac{a_2b_{20}+a_3c_2}{b_{20}^2+b_3c_2} & -1 & 0\\ \dfrac{a_2}{\sqrt{-b_{20}^2-b_3c_2}} & \dfrac{b_{20}}{\sqrt{-b_{20}^2-b_3c_2}} & \dfrac{c_2}{\sqrt{-b_{20}^2-b_3c_2}}\\ \dfrac{1}{b_{20}^2+b_3c_2} & 0 & 0 \end{pmatrix} \begin{pmatrix} X\\ Y\\ Z \end{pmatrix},$$

whose inverse is

$$\begin{pmatrix} X\\ Y\\ Z \end{pmatrix} = \begin{pmatrix} 0 & 0 & b_{20}^2+b_3c_2\\ -1 & 0 & -a_2b_{20}-a_3c_2\\ \dfrac{b_{20}}{c_2} & \dfrac{\sqrt{-b_{20}^2-b_3c_2}}{c_2} & -a_2b_3+a_3b_{20} \end{pmatrix} \begin{pmatrix} u\\ v\\ w \end{pmatrix}.$$

And by following the same steps as the first case (i) indicated in (2), we get the system

$$\begin{aligned} \dot{r} \;=\; & \frac{\varepsilon}{c_2^2(-b_{20}^2-b_3c_2)^{3/2}}\Big((-b_{20}^2-b_3c_2)\Big(\cos(\theta)\sin^2(\theta)r^2(f_2(3b_{20}^2+b_3c_2)+f_1(3a_2b_{20}+a_3c_2)\\ & -2b_{20}c_2(b_{20}-f_3)-c_2^2(b_{20}+b_3)) + (a_2f_1-b_{20}c_2+b_{20}f_2+c_2f_3)\sqrt{-b_{20}^2-b_3c_2}\sin^3(\theta)r^2\Big)\\ & +sqrt(-b_{20}^2-b_3c_2)\Big(\cos^2(\theta)\sin(\theta)r^2\Big((b_{20}^2+b_3c_2)(2b_{20}f_2-c_2^2)-b_{20}^3(c_2-f_2)-b_{20}c_2^2(b_3-e_2)\\ & +b_{20}f_1(3a_2b_{20}+2a_3c_2)c_2^2(a_2e_1+c_2e_3)+b_{20}^2c_2f_3\Big) - c_2\cos(\theta)\sin(\theta)r\Big(\Big((b_{20}^2+b_3c_2)b_{20}c_2(a_2-a_3)\\ & -(-a_2b_3+a_3b_{20})(2b_{20}(a_2f_1+c_2f_3)-2b_{20}^2(c_2-f_2)-c_2^2(b_{20}+b_3)) + (a_2b_{20}+a_3c_2)((b_{20}^2+b_3c_2)c_2\\ & -2(-a_2b_3+a_3b_{20})f_1) - 2(b_{20}^2+b_3c_2)(-a_2b_3+a_3b_{20})f_2 - (a_2b_{20}+a_3c_2)c_2(b_{20}(b_{20}-b_3+2e_2)\\ & +2a_2e_1+2c_2e_3)\Big)W - 2b_{20}(a_2c_{11}-b_{21}c_2)+c_2(a_2b_{11}-a_3c_{11})\Big)\\ & +c_2^2\sin(\theta)W\Big(\Big((a_2b_{20}+a_3c_2)^2(a_2e_1+b_{20}^2+b_{20}e_2+c_2e_3)-(a_2b_{20}+a_3c_2)(b_{20}^2(a_2b_{20}+a_3c_2)\\ & -b_3c_2(a_2b_3-a_3b_{20})) + (b_{20}^2+b_3c_2)^2(a_2d_1+b_{20}d_2+c_2d_3)\\ & +(-a_2b_3+a_3b_{20})^2(a_2f_1-b_{20}c_2+b_{20}f_2+c_2f_3)+(b_{20}^2+b_3c_2)(-a_2b_3+a_3b_{20})a_3c_2)W\\ & +(b_{20}^2+b_3c_2)a_2a_{11}-(a_2b_{20}+a_3c_2)(a_2b_{11}+b_{20}b_{21})\\ & +(-a_2b_3+a_3b_{20})(a_2c_{11}-b_{21}c_2)\Big)\Big) + c_2r\Big(\Big((b_{20}^2+b_3c_2)((a_2b_{20}+a_3c_2)c_2(b_{20}+b_3)\\ & -(b_{20}^2+b_3c_2)a_3c_2 - 2(-a_2b_3+a_3b_{20})(a_2f_1-b_{20}c_2+b_{20}f_2+c_2f_3)\Big)\sin^2(\theta) \end{aligned}$$

$$
\begin{aligned}
&+\Big((a_2b_{20}+a_3c_2)((b_{20}^2+b_3c_2)(b_{20}+2e_2)c_2+2(-a_2b_3+a_3b_{20})b_{20}f_1)\\
&+(b_{20}^2+b_3c_2)(-a_2b_3+a_3b_{20})(2b_{20}f_2-c_2^2)+2(a_2b_{20}+a_3c_2)^2c_2e_1\\
&-(b_{20}^2+b_3c_2)^2a_2c_2\Big)\cos^2(\theta)\Big)W+\cos^2(\theta)\Big((a_2b_{20}+a_3c_2)(b_{20}c_{11}-b_{11}c_2)\\
&-(b_{20}^2+b_3c_2)b_{21}c_2\Big)-(b_{20}^2+b_3c_2)\sin^2(\theta)(a_2c_{11}-b_{21}c_2)\Big)\\
&-c_2^2\Big(\Big((a_2b_{20}+a_3c_2)\Big((a_2-d_1)(b_{20}^2+b_3c_2)^2+(-a_2b_3+a_3b_{20})((b_{20}^2+b_3c_2)c_2\\
&-(-a_2b_3+a_3b_{20})f_1)\Big)-(a_2b_{20}+a_3c_2)^2(b_{20}^2+b_3c_2)(e_2+b_{20})\\
&-(b_{20}^2+b_3c_2)((b_{20}^2+b_3c_2)^2d_2+(-a_2b_3+a_3b_{20})^2f_2\Big)-(a_2b_{20}+a_3c_2)^3e_1\Big)W+(a_2b_{20}+a_3c_2)^2b_{11}\\
&-(a_2b_{20}+a_3c_2)\big((a_{11}-b_{21})(b_{20}^2+b_3c_2)+(-a_2b_3+a_3b_{20})c_{11}\big)\Big)\cos(\theta)W+((a_2b_{20}+a_3c_2)(b_{20}^2f_1+c_2^2e_1)\\
&+(b_{20}^2+b_3c_2)(b_{20}^2f_2+c_2^2e_2))\cos^3(\theta)r^2\Big)\\
=\;&F_1(\theta,r,W), \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
\dot W\;=\;&-\frac{\varepsilon}{c_2^2(-b_{20}^2-b_3c_2)^{3/2}}\Big(\sqrt{-b_{20}^2-b_3c_2}\big(c_2\sin(\theta)r(2(-a_2b_3+a_3b_{20})Wf_1+c_{11})+2\cos(\theta)\sin(\theta)r^2b_{20}f_1\big)\\
&+c_2\cos(\theta)r\big(2W((a_2b_{20}+a_3c_2)c_2e_1+(-a_2b_3+a_3b_{20})b_{20}f_1)+b_{20}c_{11}-b_{11}c_2\big)+c_2^2W\Big(W\big((a_2b_{20}\\
&+a_3c_2)^2e_1+(b_{20}^2+b_3c_2)^2d_1+(-a_2b_3+a_3b_{20})^2f_1\big)-(a_2b_{20}+a_3c_2)b_{11}+(b_{20}^2+b_3c_2)a_{11}+(-a_2b_3\\
&+a_3b_{20})c_{11}\Big)+r^2\Big(\cos^2(\theta)(b_{20}^2f_1+c_2^2e_1+(b_{20}^2+b_3c_2)f_1)-(b_{20}^2+b_3c_2)f_1\Big)\Big)\\
=\;&F_2(\theta,r,W). \qquad (30)
\end{aligned}
$$

We shall apply the averaging theory described in Theorem 2 to the differential system (29)–(30). Using the notation of section 2 we have $t=\theta$, $T=2\pi$, $\mathbf{x}=(r,W)^T$ *and*

$$F(\theta,r,W)=\begin{pmatrix}F_1(\theta,r,W)\\F_2(\theta,r,W)\end{pmatrix}\quad\text{and}\quad f(r,W)=\begin{pmatrix}f_1(r,W)\\f_2(r,W)\end{pmatrix}.$$

It is immediate to check that system (29)–(30) satisfies all the assumptions of Theorem 2. Now we compute the integrals (5), i.e.

$$f_1(r,W)=\frac{r(T_3W+N_3)}{2c_2\big(-b_{20}^2-b_3c_2\big)^{3/2}},\qquad f_2(r,W)=-\frac{D_3W^2-R_3r^2+C_3W}{2c_2^2\big(-b_{20}^2-b_3c_2\big)^{3/2}}$$

The system $f_1(r,W)=f_2(r,W)=0$ *has a unique solution* (r^*,W^*) *with* $r^*>0$, *namely*

$$(r^*,W^*)=\left(\frac{1}{T_3}\sqrt{\frac{D_3N_3^2-C_3N_3T_3}{R_3}},\ -\frac{N_3}{T_3}\right),$$

if $T_3>0$, *and* $R_3(D_3N_3^2-C_3N_3T_3)>0$, *and the Jacobian (6) at* (r^*,W^*) *takes the value*

$$-N_3(C_3T_3-D_3N_3)/2c_2^3(b_{20}^2+b_3c_2)^3T_3\neq 0$$

where

$$\begin{aligned}
C_3 = &\ -2c_2^2\big((a_2b_{20}+a_3c_2)b_{11}-(b_{20}^2+b_3c_2)a_{11}-(-a_2b_3+a_3b_{20})c_{11}\big),\\
R_3 = &\ -b_{20}^2f_1-c_2^2e_1+(b_{20}^2+b_3c_2)f_1,\\
N_3 = &\ (a_2b_{20}+a_3c_2)(b_{20}c_{11}-b_{11}c_2)-(b_{20}^2+b_3c_2)a_2c_{11},\\
D_3 = &\ 2c_2^2\Big((a_2b_{20}+a_3c_2)^2e_1+\big(b_{20}^2+b_3c_2\big)^2d_1+(-a_2b_3+a_3b_{20})^2f_1\Big),\\
T_3 = &\ \Big(2\big(a_2b_{20}+a_3c_2\big)^2c_2e_1+(a_2b_{20}+a_3c_2)\big((b_{20}^2+b_3c_2)c_2(2b_{20}+b_3+2e_2)+2(-a_2b_3+a_3b_{20})b_{20}f_1\big)\\
&-\big(b_{20}^2+b_3c_2\big)^2c_2(a_2+a_3)-\big(b_{20}^2+b_3c_2\big)(-a_2b_3+a_3b_{20})(2a_2f_1-c_2(2b_{20}-c_2-2f_3))\Big).
\end{aligned}$$

Finally, we apply Theorem 2 like in the first case (i), then for $\varepsilon > 0$ sufficiently small system (28) has a periodic solution $(x(\theta), y(\theta), z(\theta)) = (\varepsilon X(\theta), \varepsilon Y(\theta), \varepsilon Z(\theta))$ which tends to the origin of coordinates when $\varepsilon \to 0$. Therefore, it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when $\varepsilon = 0$. This completes the proof of Theorem 1 under the condition (iii).

Example 3 *Consider the Kolmogorov system*

$$\begin{cases}
\dot{x} = & x\Big(-\dfrac{1}{16}(x-1)^2+(z-1)^2\Big),\\
\dot{y} = & y\big(x-2-y+2z+(x-1)^2+(y-1)^2+(z-1)^2\big),\\
\dot{z} = & z\big(-y+z+(x-1)^2+(z-1)^2\big).
\end{cases} \tag{31}$$

This system in the new variables (X, Y, Z) writes

$$\begin{cases}
\dot{X} = & (X+1)\Big(\dfrac{1}{9}Z\varepsilon-\dfrac{1}{16}X^2+Z^2\Big),\\
\dot{Y} = & (Y+1)\big(X^2+Y^2+Z^2+X-Y+2Z\big),\\
\dot{Z} = & (Z+1)\big(X^2+Z^2-Y+Z\big).
\end{cases}$$

The corresponding system associated to system (29)–(30) satisfies

$$\begin{aligned}
F_1(\theta,r,W) = &\ -\frac{1}{9}\cos(\theta)\sin(\theta)r(18W+1)+\Big(\frac{1}{144}(279W^2+144r^2+16W)\Big)\sin(\theta)-\frac{1}{144}\cos(\theta)\big(567W^2\\
&+72r^2+16W\big)+2rW(\cos^2(\theta)+1)-\cos^3(\theta)r^2+\frac{1}{18}r,\\
F_2(\theta,r,W) = &\ \frac{1}{2}\cos(\theta)\sin(\theta)r^2+\frac{1}{18}r(18W+1)(-\sin(\theta)+\cos(\theta))-\frac{15}{16}W^2-\frac{1}{4}r^2-\frac{1}{9}W.
\end{aligned}$$

To look for the limit cycles we must solve the system given by the averaged function

$$f(r,W) = (f_1(r,W), f_2(r,W)) = (0,0)$$

where

$$f_1(r,W) = \frac{1}{4}r\left(12W+\frac{2}{9}\right) \quad \textit{and} \quad f_2(r,W) = -\frac{15}{16}W^2-\frac{1}{4}r^2-\frac{1}{9}W. \tag{32}$$

This system has four solutions for (r, W) given by $(0, 0)$, $(0, -16/135)$, $(1/12, -1/54)$, $(-1/12, -1/54)$. As in Example 1 we have two good solutions, the $(0, -16/135)$ and the $(1/12, -1/54)$. Since the determinants (6) at these two solutions are $-1/30$ and $1/96$ and thus non-zero, respectively, the Kolmogorov system (31) has two limit cycles bifurcating from the origin provided by the averaging theory of first order. We plot these two limit cycles for $\varepsilon = 10^{-5}$ in Figure 3.

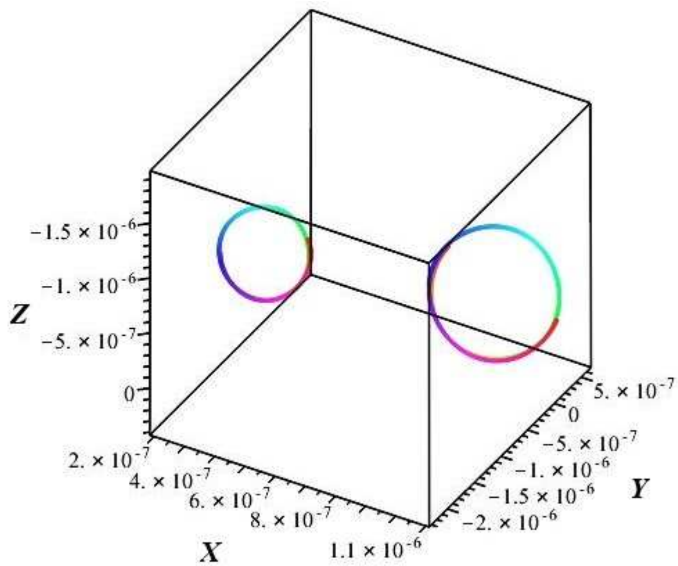

Figure 3: 1st LC: $X(0) = \varepsilon/9,\ Y(0) = -343\varepsilon/1755,\ Z(0) = -343\varepsilon/1755$. 2nd LC: $X(0) = \varepsilon/54,\ Y(0) = -11\varepsilon/108,\ Z(0) = -13\varepsilon/216$.

Since the eigenvalues of the Jacobian matrix of (f_1, f_2) *at the singular points* $(0, -16/135)$ *and* $(1/12, -1/54)$ *are* $(-3/10, 1/9)$ *and* $(-11 \pm i\sqrt{743})/288$*, respectively, by Theorem 2 the limit cycles are unstable and stable. Going back through the changes of variables as we did in the proof of Theorem 1 we obtain that the limit cycles bifurcating from the equilibrium point* $(1, 1, 1)$ *of system (31) are*

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = (1 + 16\varepsilon/135 + O(\varepsilon^2), 1 - 16\varepsilon/135 + O(\varepsilon^2), 1 - 16\varepsilon/135 + O(\varepsilon^2)),$$

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = \left(1 + \frac{\varepsilon}{54} + O(\varepsilon^2), 1 - \frac{\varepsilon(9\cos t + 2)}{108} + O(\varepsilon^2), 1 - \frac{\varepsilon(9\cos t - 9\sin t + 4)}{216} + O(\varepsilon^2)\right),$$

respectively. This completes Example 3.

Now we perturb the Kolmogorov system (1) with the parameters given in statement (iv) of Proposition 1, as it is indicated in (2). We translate the equilibrium point $(1, 1, 1)$ *to the origin of coordinates doing the change of variables* $x = X + 1, y = Y + 1, z = Z + 1$*. Then system (1) becomes*

$$\begin{aligned}
\dot{X} &= (X+1)\Big((-2Xc_{31} + Yb_{11})\varepsilon + X^2 d_1 + Y^2 e_1 + Z^2 f_1 - Xc_{30} + Zc_1\Big),\\
\dot{Y} &= (Y+1)\Big(X^2 d_2 + Y^2 e_2 + Y\varepsilon b_{21} + Z^2 f_2 + Xa_2 + Zc_2\Big),\\
\dot{Z} &= (Z+1)\Big((Yb_{31} + Zc_{31})\varepsilon + X^2 d_3 + Y^2 e_3 + Z^2 f_3 + Xa_3 + Zc_{30}\Big).
\end{aligned} \tag{33}$$

In order to facilitate the application of the averaging theory for computing the zero-Hopf bifurcation we write the linear part of system (33) with $\varepsilon = 0$ *at the equilibrium point* $(0, 0, 0)$ *in its real Jordan normal form, i.e. into the form*

$$\begin{pmatrix} 0 & -\sqrt{-a_3c_1 - c_{30}^2} & 0 \\ \sqrt{-a_3c_1 - c_{30}^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then doing the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\dfrac{c_{30}}{c_1\sqrt{-a_3c_1-c_{30}^2}} & 0 & \dfrac{1}{\sqrt{-a_3c_1-c_{30}^2}} \\ \dfrac{1}{c_1} & 0 & 0 \\ a_2c_{30}-a_3c_2 & a_3c_1+c_{30}^2 & -a_2c_1-c_2c_{30} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

whose inverse is

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 & c_1 & 0 \\ -\dfrac{a_2c_1+c_2c_{30}}{\sqrt{-a_3c_1-c_{30}^2}} & c_2 & \dfrac{1}{a_3c_1+c_{30}^2} \\ \sqrt{-a_3c_1-c_{30}^2} & c_{30} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

And by following the same steps as the first case (i) indicated in (2), we get the system

$$\begin{aligned}
\dot{r} \;=\; & \frac{\varepsilon}{c_1\big(-a_3c_1-c_{30}^2\big)^{7/2}}\Big(\big(-a_3c_1-c_{30}^2\big)^{3/2}\big(\big((a_3c_1+c_{30}^2)(a_2b_{11}c_1+2b_{11}c_2c_{30}-b_{31}c_1c_2-3c_1c_{30}c_{31}) \\
& +2W(a_2c_1e_1-c_1c_2e_3+2c_2c_{30}e_1)\big)\cos(\theta)\sin(\theta)r+(c_1(a_2(c_1e_3-c_{30}e_1)(a_2c_1+2c_2c_{30}) \\
& +a_3(c_1c_{30}+c_1f_3-c_{30}f_1)(a_3c_1+2c_{30}^2)+c_{30}^2(c_2^2e_3+c_{30}^3+c_{30}^2f_3))-c_2^2c_{30}^3e_1-c_{30}^5f_1)\cos^3(\theta)r^2\big) \\
& -\big(-a_3c_1-c_{30}^2\big)^{5/2}\big(3c_2^2c_{30}e_1+3c_{30}^3f_1+c_1(2a_2c_2e_1+(c_1^2-c_1c_{30}+2c_{30}f_1)a_3 \\
& -(c_1d_3-c_{30}^2-c_{30}d_1)c_1-c_2^2e_3-c_{30}^3-c_{30}^2f_3)\big)r^2\cos(\theta)\sin^2(\theta) \\
& -(-c_1(a_3(a_2c_1(a_3c_1+2c_{30}^2)(a_2c_1e_1-2c_1c_2e_3+4c_2c_{30}e_1) \\
& -a_3c_1(a_3c_1(a_3c_1^2-a_3c_1f_1-c_1^2c_{30}+5c_1c_{30}^2+2c_1c_{30}f_3-6c_{30}^2f_1) \\
& -c_1c_{30}(5c_1c_{30}^2-2c_2^2e_3-9c_{30}^3-6c_{30}^2f_3) \\
& -3c_2^2c_{30}^2e_1-12c_{30}^4f_1)+c_1c_{30}^3(5c_1c_{30}^2-4c_2^2e_3-7c_{30}^3-6c_{30}^2f_3)+6c_2^2c_{30}^4e_1+10c_{30}^6f_1) \\
& +c_{30}^4(c_1(a_2^2e_1-2a_2c_2e_3+c_{30}^3)+2c_{30}(2a_2c_2e_1-c_2^2e_3-c_{30}^3-c_{30}^2f_3))) \\
& -6e_1c_2^2c_{30}^6-3c_{30}^8f_1)\cos^2(\theta)\sin(\theta)r^2+(-a_3c_1-c_{30}^2)\Big(r\big((2W(a_2c_1^2e_3-a_2c_1c_{30}e_1 \\
& -a_3c_1c_2e_1+c_1c_2c_{30}e_3-2c_2c_{30}^2e_1)-(a_3c_1+c_{30}^2)(a_2b_{11}c_1c_{30}-a_2b_{31}c_1^2+a_3b_{11}c_1c_2-3a_3c_1^2c_{31} \\
& +2b_{11}c_2c_{30}^2-b_{31}c_1c_2c_{30}-3c_1c_{30}^2c_{31}))\cos^2(\theta) \\
& +2Wc_2e_1(a_3c_1+c_{30}^2)+(a_3c_1+c_{30}^2)^2(b_{11}c_2-2c_1c_{31})\big)+\sin(\theta)W(b_{11}(a_3c_1+c_{30}^2)+We_1)\Big) \\
& +\big(-a_3c_1-c_{30}^2\big)^3(c_1^2d_1+c_2^2e_1+c_{30}^2f_1)r^2\sin^3(\theta)+\sqrt{-a_3c_1-c_{30}^2}((-b_{11}c_{30}+b_{31}c_1)(a_3c_1+c_{30}^2) \\
& +W(c_1e_3-c_{30}e_1))\cos(\theta)W\Big)=F_1(\theta,r,W),
\end{aligned} \tag{34}$$

$$\begin{aligned}
\dot{W} \;=\; & \frac{\varepsilon}{(-a_3c_1-c_{30}^2)^{5/2}}\Big(W\left(a_3c_1+c_{30}^2\right)\left(a_2(b_{11}c_{30}-b_{31}c_1)-a_3(b_{11}c_2-b_{21}c_1)+b_{21}c_{30}^2-b_{31}c_2c_{30}\right) \\
& -W^2\left(a_2(c_1e_3-c_{30}e_1)-a_3(c_1e_2-c_2e_1)+c_2c_{30}e_3-c_{30}^2e_2\right) \\
& -\left(-a_3c_1-c_{30}^2\right)^{3/2}\sin(\theta)\cos(\theta)r^2\big(2c_{30}(c_{30}(c_2^2(a_2e_1-c_2e_3) \\
& +c_{30}(c_{30}(a_2f_1+c_2(c_2-c_{30}-f_3)+c_{30}f_2)-c_2(a_3f_1-c_2e_2))) \\
& -a_3c_2^3e_1)+c_1(c_{30}(2c_2(a_2^2e_1-2a_2c_2e_3-a_3^2f_1+a_3c_2e_2) \\
& +c_{30}(2a_2(a_3f_1+c_2c_{30}+c_2e_2-c_{30}^2-c_{30}f_3)-a_3(3c_2c_{30}+2c_2f_3-4c_{30}f_2))) \\
& -a_3c_2^2(2a_2e_1-3c_{30}^2)+c_1(a_3c_1(a_2(a_2-a_3+c_{30})-a_3c_2)-a_2(a_2(2c_2e_3-c_{30}^2) \\
& -a_3(2c_2c_{30}+2c_2e_2-3c_{30}^2-2c_{30}f_3)-c_{30}^3)+a_3((c_2^2-c_2c_{30}+2c_{30}f_2)a_3-c_2c_{30}^2))))\big)
\end{aligned}$$

$$
\begin{aligned}
&-\left(-a_3c_1-c_{30}^2\right)\Big(r^2\big(\big(a_3c_1^4(a_2d_3-a_3d_2)+c_1^3(a_2^2(a_2e_3-a_3e_2)-a_3^2(a_3f_2-c_2d_1)\\
&+a_3c_{30}(c_2d_3-2c_{30}d_2)-a_2(a_3^2(2c_2-2c_{30}-f_3)+c_{30}(a_3d_1-c_{30}d_3)))c_1^2(-a_3^3c_2f_1\\
&+a_3^2(c_2^2(2c_{30}+e_2)-c_2c_{30}(2c_{30}+f_3)+c_{30}(a_2f_1+4c_{30}f_2))-a_3(a_2(c_2^2e_3-2c_2c_{30}(2c_{30}+e_2)\\
&+c_{30}^2(4c_{30}+3f_3))+c_2(a_2^2e_1+c_{30}^2d_1))+c_{30}(a_2^3e_1-a_2^2(3c_2e_3-c_{30}e_2)\\
&+c_{30}^2(a_2d_1-c_2d_3+c_{30}d_2)))-c_1(-a_3^2c_2(c_2^2e_1+3c_{30}^2f_1)-a_3c_{30}(e_1c_2^2a_2-3c_{30}^2f_1a_2\\
&+e_3c_2^3-5c_{30}^3f_2-c_2c_{30}(c_2(4c_{30}+3e_2)-4c_{30}^2-3c_{30}f_3))-2a_2c_{30}^2(2c_2^2e_3+c_{30}^3+f_3c_{30}^2\\
&-c_2c_{30}(c_{30}+e_2))+2a_2^2c_2c_{30}^2e_1)-2c_{30}^2(c_2c_{30}(c_2(a_2e_1+c_{30}^2+c_{30}e_2)-a_3c_{30}f_1\\
&-c_2^2e_3-c_{30}^3-f_3c_{30}^2)-a_3c_2^3e_1+c_{30}^3f_1a_2+c_{30}^4f_2)\big)\cos^2(\theta)\\
&+(a_3c_1+c_{30}^2)(a_3a_2c_1^2c_2+c_{30}^4f_2+c_2^2c_{30}^3-c_2c_{30}^4\\
&-c_2c_{30}^3f_3-e_3c_2^3c_{30}+e_2c_2^2c_{30}^2+c_{30}^3f_1a_2-d_3a_2c_1^3+c_1^2c_{30}^2d_2-a_2c_1c_{30}^3+a_3c_1^3d_2-a_3c_2^3e_1\\
&-a_2a_3c_1^2c_{30}-f_1a_3c_2c_{30}^2+f_2a_3c_1c_{30}^2-a_3c_1^2c_2d_1+a_3c_1c_2^2c_{30}-a_3c_1c_2c_{30}^2+e_1c_2^2a_2c_{30}+a_2c_1^2c_{30}d_1\\
&-c_1^2c_2c_{30}d_3-a_2e_3c_1c_2^2-a_2f_3c_1c_{30}^2+a_2c_1c_2c_{30}^2+a_3c_1e_2c_2^2)\big)\\
&+\sin(\theta)r\big(W(c_1((a_3c_1-2c_2e_3+c_{30}^2)a_2+a_3c_2(c_{30}+2e_2))\\
&+_c2(c_{30}(2a_2e_1-2c_2e_3+c_{30}^2+2c_{30}e_2)-2a_3c_2e_1))+(a_3c_1+c_{30}^2)(a_2(b_{11}c_2c_{30}-b_{31}c_1c_2-3c_1c_{30}c_{31})\\
&-a_3c_2(b_{11}c_2-(b_{21}+2c_{31})c_1)+c_2c_{30}(c_{30}(b_{21}-c_{31})-b_{31}c_2)))\Big)\\
&-\sqrt{-a_3c_1-c_{30}^2}((c_1^2(2a_2^2e_3-2a_2a_3e_2-a_3^2c_2)-2c_1(a_2(c_{30}(a_2e_1-2c_2e_3+c_{30}e_2)-a_3c_2e_1)\\
&+a_3c_2c_{30}(c_{30}+e_2))-c_2c_{30}(c_{30}(2a_2e_1-2c_2e_3+c_{30}^2+2c_{30}e_2)-2a_3c_2e_1)\big)W\\
&-\big(a_3c_1+c_{30}^2\big)(a_2c_1+c_2c_{30})\big(a_2(b_{11}c_{30}-b_{31}c_1)-a_3(b_{11}c_2-b_{21}c_1+c_1c_{31})+c_{30}^2(b_{21}-c_{31})\\
&-b_{31}c_2c_{30})\big)\cos(\theta)r\Big)=F_2(\theta,r,W).
\end{aligned}
\tag{35}
$$

We shall apply the averaging theory described in Theorem 2 to the differential system (34)–(35). Using the notation of section 2 we have $t=\theta$, $T=2\pi$, $\mathbf{x}=(r,W)^T$ and

$$
F(\theta,r,W)=\begin{pmatrix} F_1(\theta,r,W)\\ F_2(\theta,r,W)\end{pmatrix} \quad \text{and} \quad f(r,W)=\begin{pmatrix} f_1(r,W)\\ f_2(r,W)\end{pmatrix}.
$$

It is immediate to check that system (34)–(35) satisfies all the assumptions of Theorem 2. Now we compute the integrals (5), i.e.

$$
f_1(\theta,r,W)=\frac{(T_4W-N_4)r}{2\big(-a_3c_1-c_{30}^2\big)^{5/2}},\quad f_2(r,W)=\frac{2D_4W^2-R_4r^2-2C_4W}{2\big(-a_3c_1-c_{30}^2\big)^{3/2}}
$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$
(r^*,\,W^*)=\left(\frac{1}{T_4}\sqrt{\frac{2D_4N_4^2-2C_4N_4T_4}{R_4}},\,\frac{N_4}{T_4}\right),
$$

if $T_4>0$, and $R_4(D_4N_4^2-C_4N_4T_4)>0$, and the Jacobian (6) at (r^*,W^*) takes the value $-N_4(C_4T_4-$

$D_4N_4)/T_4(a_3c_1+c_{30}^2)^4 \neq 0$ *where*

$$
\begin{aligned}
N_4 = {} & \big(a_3c_1+c_{30}^2\big)\big(a_2b_{11}c_{30}-a_2b_{31}c_1-a_3b_{11}c_2+a_3c_1c_{31}-b_{31}c_2c_{30}+c_{30}^2c_{31}\big),\\
T_4 = {} & 2(a_2c_1e_3-a_2c_{30}e_1+a_3c_2e_1+c_2c_{30}e_3),\\
D_4 = {} & \frac{a_2c_1e_3-a_2c_{30}e_1-a_3c_1e_2+a_3c_2e_1+c_2c_{30}e_3-c_{30}^2e_2}{a_3c_1+c_{30}^2},\\
C_4 = {} & a_2b_{11}c_{30}-a_2b_{31}c_1-a_3b_{11}c_2+a_3b_{21}c_1+b_{21}c_{30}^2-b_{31}c_2c_{30},\\
R_4 = {} & c_1\Big(a_3^3c_1(c_2f_1-c_1f_2)-a_3^2\big((a_2c_{30}f_1-c_2^2e_2-c_2c_{30}f_3+2c_{30}^2f_2-2a_2f_3c_1)c_1+c_1^2(a_2f_3-c_1d_2+c_2d_1)\\
& +c_2(c_2^2e_1-c_{30}^2f_1)\big)-a_3\big(c_1^2(a_2^2e_2-a_2c_{30}d_1+c_2c_{30}d_3-2c_{30}^2d_2)-c_1(a_2^2c_2e_1-a_2(c_2^2e_3+2c_2c_{30}e_2\\
& -c_{30}^2f_3)-c_2c_{30}^2d_1)+a_2(c_1^3d_3-3c_2^2c_{30}e_1+c_{30}^3f_1)+c_{30}(c_2^3e_3-c_2^2c_{30}e_2-c_2c_{30}^2f_3+c_{30}^3f_2)\big)+a_2^2\big(a_2c_1^2\\
& e_3-2c_2c_{30}^2e_1\big)-c_1c_{30}\big(a_2^3e_1-3a_2^2c_2e_3+a_2^2c_{30}e_2-a_2c_{30}^2d_1+c_2c_{30}^2d_3-c_{30}^3d_2\big)-a_2c_{30}^2\big(c_1^2d_3-2c_2^2e_3\\
& +2c_2c_{30}e_2\big)\Big).
\end{aligned}
$$

Finally, we apply Theorem 2 like in the first case (i), then for $\varepsilon>0$ sufficiently small system (33) has a periodic solution $(x(\theta),y(\theta),z(\theta))=(\varepsilon X(\theta),\varepsilon Y(\theta),\varepsilon Z(\theta))$ which tends to the origin of coordinates when $\varepsilon\to0$. Therefore, it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when $\varepsilon=0$. This completes the proof of Theorem 1 under the condition (iv).

Example 4 *Consider the Kolmogorov system*

$$
\begin{cases}
\dot{x} = & x\big(-x+z+(y-1)^2+(z-1)^2\big),\\
\dot{y} = & y\big(1-z+(x-1)^2+(y-1)^2\big),\\
\dot{z} = & z\big(-2x+1+z+(x-1)^2+(z-1)^2\big).
\end{cases}
\tag{36}
$$

This system in the new variables (X,Y,Z) writes

$$
\begin{cases}
\dot{X} = & \big(X+1\big)\Big(\frac{1}{9}Y\varepsilon+Y^2+Z^2-X+Z\Big),\\
\dot{Y} = & \big(Y+1\big)\big(X^2+Y^2-Z\big),\\
\dot{Z} = & \big(Z+1\big)\big(X^2+Z^2-2X+Z\big).
\end{cases}
$$

The corresponding system associated to system (34)–(35) satisfies

$$
F_1(\theta,r,W)=2r^2\big(4\cos(\theta)^2+1\big)\sin(\theta)+\frac{r(18W-1)\big(\sin(\theta)-\cos(\theta)\big)^2}{9}+\frac{W(9W-1)\big(\sin(\theta)-\cos(\theta)\big)}{9},
$$

$$
F_2(\theta,r,W)=\frac{2w}{9}-3W^2+2\sin(\theta)\cos(\theta)r^2-r^2\big(6-4\cos(\theta)^2\big)-\sin(\theta)r\Big(7W-\frac{2}{9}\Big)-\Big(-5W+\frac{2}{9}\Big)\cos(\theta)r.
$$

To look for the limit cycles we must solve the system given by the averaged function

$$
f(r,W)=(f_1(r,W),f_2(r,W))=(0,0)
$$

where

$$
f_1(r,W)=\frac{1}{2}\Big(4W-\frac{2}{9}\Big)r \quad \textit{and} \quad f_2(r,W)=-3W^2-4r^2+\frac{2}{9}W.
\tag{37}
$$

This system has four solutions for (r,W) given by $(0,0)$, $(0,2/27)$, $(1/36,1/18)$, $(-1/36,1/18)$. As in Example 11 we have two good solutions, the $(0,2/27)$ and the $(1/36,1/18)$. Since the determinants (6) at these *two solutions are $-2/243$ and $1/81$ and thus non-zero, respectively, the Kolmogorov system (36) has two limit cycles bifurcating from the origin provided by the averaging theory of first order. We plot these two limit cycles for $\varepsilon = 10^{-5}$ in Figure 4.*

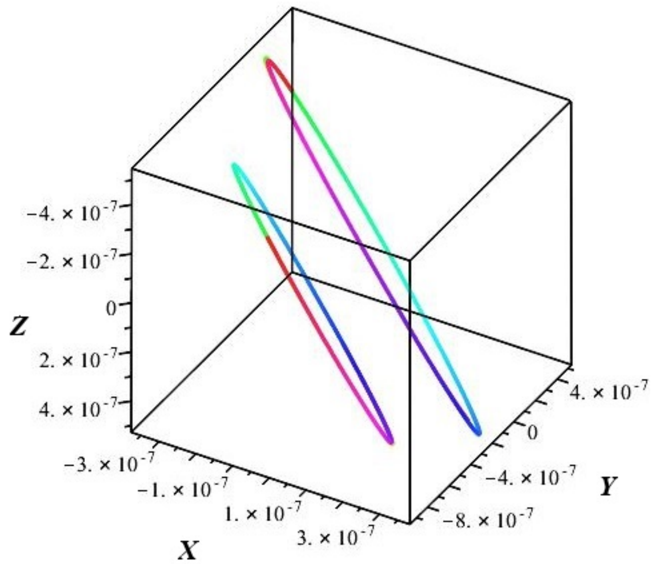

Figure 4: 1st LC: $X(0) = \varepsilon/27,\ Y(0) = -\varepsilon/27,\ Z(0) = \varepsilon/27$. 2nd LC: $X(0) = 0,\ Y(0) = -\varepsilon/36,\ Z(0) = \varepsilon/36$.

Since the eigenvalues of the Jacobian matrix of (f_1, f_2) at the singular points $(0, 2/27)$ and $(1/36, 1/18)$ are $(-2/9, 1/27)$ and $(-1 \pm i\sqrt{3})/18$, respectively, by Theorem 2 the limit cycles are unstable and stable. Going back through the changes of variables as we did in the proof of Theorem 1 we obtain that the limit cycles bifurcating from the equilibrium point $(1, 1, 1)$ of system (36) are

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = (1 + O(\varepsilon^2), 1 - 2\varepsilon/27 + O(\varepsilon^2), 1 + O(\varepsilon^2)),$$

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = \left(1 + \frac{\varepsilon \sin t}{36} + O(\varepsilon^2), 1 + \frac{\cos t - \sin t - 2}{36}\varepsilon + O(\varepsilon^2), 1 + \frac{\cos t + \sin t}{36}\varepsilon + O(\varepsilon^2)\right),$$

respectively. This completes Example 4.

Now we perturb the Kolmogorov system (1) with the parameters given in statement (v) of Proposition 1, as it is indicated in (2). We translate the equilibrium point $(1, 1, 1)$ to the origin of coordinates doing the change of variables $x = X + 1$, $y = Y + 1$, $z = Z + 1$. Then system (1) becomes

$$\begin{aligned} \dot{X} &= (X+1)\Big(\big(-2Xc_{31} + Yb_{11}\big)\varepsilon + X^2 d_1 + Y^2 e_1 + Z^2 f_1 - Xc_{30} + Zc_1\Big), \\ \dot{Y} &= (Y+1)\Big(\Big(\frac{-c_2 c_{31} X}{c_1} + b_{21} Y\Big)\varepsilon - \frac{c_2 c_{30} X}{c_1} + c_2 Z + d_2 X^2 + e_2 Y^2 + f_2 Z^2\Big), \\ \dot{Z} &= (Z+1)\big(X^2 d_3 + Y^2 e_3 + Z^2 f_3 + c_{31} Z\varepsilon + Xa_3 + Yb_3 + Zc_{30}\big). \end{aligned} \tag{38}$$

In order to facilitate the application of the averaging theory for computing the zero-Hopf bifurcation we write the linear part of system (38) with $\varepsilon = 0$ at the equilibrium point $(0, 0, 0)$ in its real Jordan normal form, i.e. into the form

$$\begin{pmatrix} 0 & -\sqrt{-a_3 c_1 - b_3 c_2 - c_{30}^2} & 0 \\ \sqrt{-a_3 c_1 - b_3 c_2 - c_{30}^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then doing the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \dfrac{a_3c_1+c_{30}^2}{c_1\big(a_3c_1+b_3c_2+c_{30}^2\big)b_3} & \dfrac{1}{a_3c_1+b_3c_2+c_{30}^2} & 0 \\ \dfrac{c_{30}}{c_1\sqrt{-a_3c_1-b_3c_2-c_{30}^2}\,b_3} & 0 & -\dfrac{1}{b_3\sqrt{-a_3c_1-b_3c_2-c_{30}^2}} \\ -\dfrac{c_2}{c_1\big(a_3c_1+b_3c_2+c_{30}^2\big)} & \dfrac{1}{a_3c_1+b_3c_2+c_{30}^2} & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

whose inverse is

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} c_1b_3 & 0 & -c_1b_3 \\ b_3c_2 & 0 & a_3c_1+c_{30}^2 \\ c_{30}b_3 & -b_3\sqrt{-a_3c_1-b_3c_2-c_{30}^2} & -c_{30}b_3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

And by following the same steps as the first case (i) indicated in (2), we get the system

$$\begin{aligned}
\dot r \;=\;& \frac{\varepsilon}{c_1\big(-a_3c_1-b_3c_2-c_{30}^2\big)^{5/2}}\Big(\big(-a_3c_1-b_3c_2-c_{30}^2\big)^{3/2}\big(-\big(W\big(a_3c_1\big(b_3(c_1^2-c_1c_2-c_1c_{30}+2c_{30}f_1)\\
&+2c_2(c_1e_3-c_{30}e_1)\big)-b_3c_1\big(c_{30}(b_3c_2-2b_3f_2-2c_1d_1)-c_{30}^2(c_1-c_2-c_{30}-2f_3)+2c_1^2d_3\big)\\
&+2c_{30}^2(2b_3c_{30}f_1+c_1c_2e_3-c_2c_{30}e_1)\big)-c_{30}(b_{11}c_2-3c_1c_{31})\big)r\cos(\theta)\sin(\theta)\\
&+(a_3c_1(c_1^2-c_1c_{30}+2c_{30}f_1)+b_3c_1(c_2^2-c_2c_{30}+2c_{30}f_2)-c_1(c_2^2e_3+c_{30}^3+c_{30}^2f_3)\\
&+c_{30}(c_{30}+d_1)c_1^2-d_3c_1^3+c_{30}c_2^2e_1+3c_{30}^3f_1)r^2\cos^2(\theta)\sin(\theta)b_3\\
&+(c_1c_{30}+c_1f_3-c_{30}f_1)(a_3c_1+b_3c_2+c_{30}^2)r^2\sin^3(\theta)b_3\big)\\
&-\big(-a_3c_1-b_3c_2-c_{30}^2\big)\Big(r(((a_3c_1+c_{30}^2)(2c_2e_1(a_3c_1+c_{30}^2)\\
&-b_3(c_1^2c_{30}+2c_1^2d_1-2c_1c_2e_2-c_1c_{30}^2-2c_1c_{30}f_3+4c_{30}^2f_1))-b_3^2(2c_1^3d_2+c_1^2c_2c_{30}+2c_2c_{30}^2f_1\\
&-c_1c_{30}(c_2c_{30}+2c_2f_3-2c_{30}f_2)))W+b_3c_1c_2(b_{21}-2c_{31})+(b_{11}c_2-3c_1c_{31})(a_3c_1+c_{30}^2))\cos^2(\theta)\\
&+r((c_1^2-c_1c_{30}-2c_1f_3+2c_{30}f_1)(b_3c_{30}(a_3c_1+c_{30}^2)+b_3^2c_2c_{30})W\\
&+c_1c_{31}(a_3c_1+b_3c_2+c_{30}^2))+(\cos(\theta)W\big(\big((a_3c_1+c_{30}^2)\big(b_3^2(c_1^2d_1+c_{30}^2f_1)+b_3c_1e_2(a_3c_1+c_{30}^2)\\
&+e_1(a_3c_1+c_{30}^2)^2\big)+c_1b_3^3(c_1^2d_2+c_{30}^2f_2)\big)W+(a_3c_1+c_{30}^2)(b_3c_1(b_{21}+2c_{31})+b_{11}(a_3c_1+c_{30}^2))\\
&+b_3^2c_1c_2c_{31}\big)\big)/(b_3)+b_3\cos^3(\theta)r^2((c_1^2d_1+c_2^2e_1+c_{30}^2f_1)(a_3c_1+c_{30}^2)+b_3c_1(c_1^2d_2+c_2^2e_2+c_{30}^2f_2))\Big)\\
&+\big(-a_3c_1-b_3c_2-c_{30}^2\big)^2(b_3c_1(c_2-f_2)+a_3c_1(c_1-f_1)+2c_1c_{30}(c_{30}+f_3)\\
&-c_1^2c_{30}-3c_{30}^2f_1)r^2\sin^2(\theta)\cos(\theta)b_3\\
&+(\sqrt{-a_3c_1-b_3c_2-c_{30}^2}\sin(\theta)W\big(\big((c_1^3d_3-c_1^2c_{30}d_1+c_1c_{30}^2f_3-c_{30}^3f_1)(b_3^3c_2+b_3^2(a_3c_1+c_{30}^2))\\
&+b_3c_2a_3^2c_1^2(c_1e_3-2c_{30}e_1)+(c_1e_3-c_{30}e_1)\big((a_3c_1+c_{30}^2)^3+b_3c_2(4a_3c_1c_{30}^2+c_{30}^4)\big)\big)W\\
&-b_{11}c_{30}(a_3c_1+c_{30}^2)^2-b_3c_{30}(3c_1c_{31}(a_3c_1+b_3c_2+c_{30}^2)+b_{11}c_2(2a_3c_1+c_{30}^2))\big)\big)\Big)\\
=\;& F_1(\theta,r,W),
\end{aligned} \tag{39}$$

$$\begin{aligned}
\dot W \;=\;& \frac{\varepsilon}{c_1\big(-a_3c_1-b_3c_2-c_{30}^2\big)^{3/2}}\Big(\sqrt{-a_3c_1-b_3c_2-c_{30}^2}\big(\sin(\theta)Wrb_3\big(2b_3c_2c_{30}f_1-c_1c_{30}(2b_3f_2-c_2c_{30})\\
&+c_1^2c_2(a_3+b_3)\big)-\cos(\theta)\sin(\theta)r^2b_3^2\big(c_2(c_1^2+2c_{30}f_1)-c_1(c_2^2+2c_{30}f_2)\big)\big)-\big(c_1^3(a_3^2e_2+b_3^2d_2)\\
&-c_1c_{30}^2(2a_3c_2e_1-b_3^2f_2-c_{30}^2e_2)-c_1^2(a_3^2c_2e_1-2a_3c_{30}^2e_2+b_3^2c_2d_1)-c_2c_{30}^2(b_3^2f_1+c_{30}^2e_1)\big)W^2\\
&-\big((a_3c_1+c_{30}^2)(-b_{11}c_2+b_{21}c_1)-b_3c_1c_2c_{31}\big)W-r^2b_3^2\big(\big((c_1f_2-c_2f_1)(a_3c_1+b_3c_2)+c_1(c_2^2e_2+2c_{30}^2f_2)
\end{aligned}$$

$$
\begin{aligned}
& -c_2(c_2^2e_1+2c_{30}^2f_1)+c_1^2(c_1d_2-c_2d_1)\big)\cos(\theta)^2-(c_1f_2-_c 2f_1)(a_3c_1+b_3c_2+c_{30}^2)\big) \\
& -\cos(\theta)rb_3\big(\big(2c_2(c_1e_2-c_2e_1)(a_3c_1+c_{30}^2)-2b_3\big(c_1^2(c_1d_2-c_2d_1)+c_{30}^2(c_1f_2-c_2f_1)\big)\big)W \\
& -c_2(b_{11}c_2-b_{21}c_1-c_1c_{31})\big)\Big) \\
= \; & F_2(\theta,r,W). \qquad (40)
\end{aligned}
$$

We shall apply the averaging theory described in Theorem 2 to the differential system (39)–(40). Using the notation of section 2 we have $t=\theta$, $T=2\pi$, $\mathbf{x}=(r,W)^T$ and

$$
F(\theta,r,W)=\begin{pmatrix} F_1(\theta,r,W) \\ F_2(\theta,r,W)\end{pmatrix} \quad \textit{and} \quad f(r,W)=\begin{pmatrix} f_1(r,W) \\ f_2(r,W)\end{pmatrix}.
$$

It is immediate to check that system (39)–(40) satisfies all the assumptions of Theorem 2. Now we compute the integrals (5), i.e.

$$
f_1(r,W)=-\frac{r\big(T_5W+N_5\big)}{2c_1\big(-a_3c_1-b_3c_2-c_{30}^2\big)^{3/2}}, \quad f_2(r,W)=-\frac{D_5W^2-R_5r^2+C_5W}{2c_1\big(-a_3c_1-b_3c_2-c_{30}^2\big)^{3/2}}
$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$
(r^*,W^*)=\left(\frac{1}{T_5}\sqrt{\frac{D_5N_5^2-C_5N_5T_5}{R_5}},-\frac{N_5}{T_5},\right)
$$

if $T_5>0$, and $R_5(D_5N_5^2-C_5N_5T_5)>0$, and the Jacobian (6) at (r^*,W^*) takes the value $N_5(C_5T_5-D_5N_5)/2T_5(a_3c_1+b_3c_2+c_{30}^2)^3c_1^2\neq 0$ where

$$
\begin{aligned}
D_5 = \; & \big(2b_3^2\big(c_1^2(c_1d_2-c_2d_1)+c_{30}^2(c_1f_2-c_2f_1)\big)+(c_1e_2-c_2e_1)\big(2a_3c_1\big(a_3c_1+2c_{30}^2\big)+2c_{30}^4\big)\big), \\
N_5 = \; & \big(a_3c_1+c_{30}^2\big)(b_{11}c_2-c_1c_{31})+b_3b_{21}c_1c_2, \\
R_5 = \; & b_3^2\big(c_2c_1(b_3f_2-c_2e_2)+a_3c_1(c_1f_2-c_2f_1)-c_1^2(c_1d_2-c_2d_1)-c_2^2(b_3f_1-c_2e_1)\big), \\
C_5 = \; & \big(2\big(a_3c_1+c_{30}^2\big)(b_{21}c_1-b_{11}c_2)-2b_3c_1c_2c_{31}\big), \\
T_5 = \; & \Big(b_3c_1\big(c_1c_{30}-c_{30}^2-2c_{30}f_3-2c_1d_1+2c_2e_2\big)\big(a_3c_1+c_{30}^2\big)+2c_2e_1\big(a_3c_1+c_{30}^2\big)^2-b_3^2\big(2c_1^3d_2-c_{30}c_2\big(c_1^2 \\
& +2c_{30}f_1\big)+c_{30}c_1(c_2c_{30}+2c_2f_3+2c_{30}f_2)\big)\Big).
\end{aligned}
$$

Finally, we apply Theorem 2 like in the first case (i), then for $\varepsilon>0$ sufficiently small system (38) has a periodic solution $(x(\theta),y(\theta),z(\theta))=(\varepsilon X(\theta),\varepsilon Y(\theta),\varepsilon Z(\theta))$ which tends to the origin of coordinates when $\varepsilon\to 0$. Therefore, it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when $\varepsilon=0$. This completes the proof of Theorem 1 under the condition (v).

Example 5 *Consider the Kolmogorov system*

$$
\left\{\begin{aligned}
\dot{x} = \; & x\big(-x-1+2z+(x-1)^2+(y-1)^2+2(z-1)^2\big), \\
\dot{y} = \; & y\big(-2x-2+4z\big), \\
\dot{z} = \; & z\Big(-2x+\frac{1}{2}+\frac{1}{2}y+z\Big).
\end{aligned}\right. \qquad (41)
$$

This system in the new variables (X,Y,Z) writes

$$
\left\{\begin{aligned}
\dot{X} = \; & (X+1)\big((-2X+Y)\varepsilon+X^2+Y^2+2Z^2-X+2Z\big), \\
\dot{Y} = \; & (Y+1)\big((-2X+Y)\varepsilon-2X+4Z\big), \\
\dot{Z} = \; & (Z+1)\Big(Z\varepsilon-2X+\frac{1}{2}Y+Z\Big).
\end{aligned}\right.
$$

The corresponding system associated to system (39)–(40) satisfies

$$
\begin{aligned}
F_1(\theta,r,W) = {} & -\frac{1}{2}(31W+2)r\cos(\theta)\sin(\theta)+4r^2\cos^2(\theta)\sin(\theta)-\frac{1}{2}r(93W+2)\cos^2(\theta)-\frac{1}{2}r(-3W-2)\\
& -\cos(\theta)W\Big(-\frac{63}{2}W+2\Big)+\frac{33}{2}\cos(\theta)^3r^2-\frac{1}{2}r^2\sin^2(\theta)\cos(\theta)+\sin(\theta)W\Big(-\frac{11}{2}W+8\Big),\\
F_2(\theta,r,W) = {} & -2\sin(\theta)Wr+21W^2-W-\frac{1}{8}r^2\big(-80\cos^2(\theta)-8\big)-30\cos(\theta)rW.
\end{aligned}
$$

To look for the limit cycles we must solve the system given by the averaged function

$$f(r,W)=(f_1(r,W),f_2(r,W))=(0,0)$$

where

$$f_1(r,W)=-\frac{1}{4}r\,(87W-2) \quad \textit{and} \quad f_2(r,W)=21W^2+6r^2-W. \tag{42}$$

This system has four solutions for (r,W) *given by* $(0,0)$, $(0,1/21)$, $(\sqrt{15}/87,2/87)$, $(-\sqrt{15}/87,2/87)$*. As in Example 1 we have two good solutions, the* $(0,1/21)$ *and the* $(\sqrt{15}/87,2/87)$*. Since the determinants (6) at these two solutions are* $-15/28$ *and* $15/29$ *and thus non-zero, respectively, the Kolmogorov system (41) has two limit cycles bifurcating from the origin provided by the averaging theory of first order. We plot these two limit cycles for* $\varepsilon=10^{-5}$ *in Figure 5.*

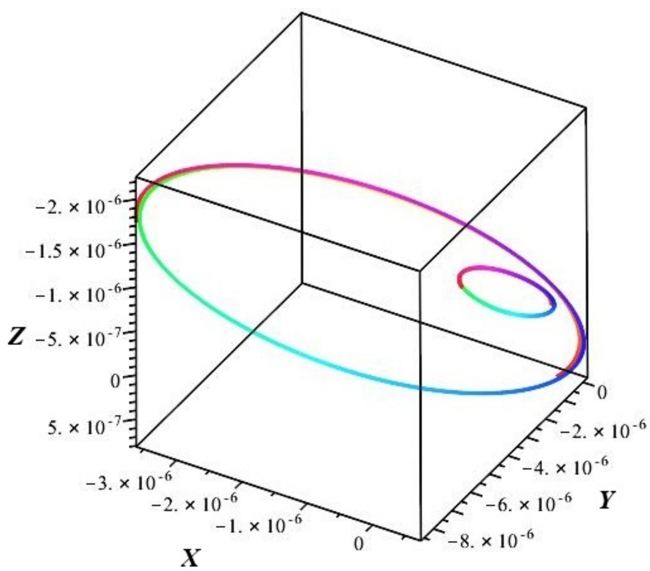

Figure 5: 1st LC: $X(0)=\varepsilon/21$, $Y(0)=-\varepsilon/21$, $Z(0)=\varepsilon/14$. 2nd LC: $X(0)=\dfrac{(\sqrt{15}-2)\varepsilon}{87}$, $Y(0)=\dfrac{2(\sqrt{15}-3)\varepsilon}{87}$, $Z(0)=\dfrac{(\sqrt{15}-2)\varepsilon}{174}$.

Since the eigenvalues of the Jacobian matrix of (f_1,f_2) *at the singular points* $(0,1/21)$ *and* $(\sqrt{15}/87,2/87)$ *are* $(-15/28,1)$ *and* $(-1\pm i\sqrt{1739})/58$*, respectively, by Theorem 2 the limit cycles are unstable and stable. Going back through the changes of variables as we did in the proof of Theorem 1 we obtain that the limit cycles bifurcating from the equilibrium point* $(1,1,1)$ *of system (41) are*

$$
\begin{aligned}
&(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))=(1-\varepsilon/21+O(\varepsilon^2),1-\varepsilon/7+O(\varepsilon^2),1-\varepsilon/42+O(\varepsilon^2)),\\
&(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon))=\big(1+\varepsilon(\sqrt{15}\cos t-2)/87+O(\varepsilon^2),\,1+2\varepsilon(\sqrt{15}\cos t-3)/87+O(\varepsilon^2)\,,\\
&\qquad 1+\varepsilon(\sqrt{15}(\cos t-\sin t)-2)/174+O(\varepsilon^2)\big)\,,
\end{aligned}
$$

respectively. This completes Example 5.

Now we perturb the Kolmogorov system (1) with the parameters given in statement (vi) of Proposition 1, as it is indicated in (2). We translate the equilibrium point $(1,1,1)$ *to the origin of coordinates doing the change of variables* $x = X+1$, $y = Y+1$, $z = Z+1$. *Then system (1) becomes*

$$\begin{aligned}
\dot{X} &= (X+1)\big((-2b_{21}-2c_{31})X\varepsilon + (-b_{20}-c_{30})X + b_1Y + c_1Z + d_1X^2 + e_1Y^2 + f_1Z^2\big),\\
\dot{Y} &= (Y+1)\left(\left(\frac{b_{21}Y + b_{21}c_1Z}{b_1}\right)\varepsilon + a_2X + b_{20}Y + \frac{b_{20}c_1Z}{b_1} + d_2X^2 + e_2Y^2 + f_2Z^2\right),\\
\dot{Z} &= (Z+1)\left(\left(\frac{2b_1c_{31}Y}{c_1} + c_{31}Z\right)\varepsilon + a_3X + \frac{b_1c_{30}Y}{c_1} + c_{30}Z + d_3X^2 + e_3Y^2 + f_3Z^2\right).
\end{aligned} \tag{43}$$

In order to facilitate the application of the averaging theory for computing the zero-Hopf bifurcation we write the linear part of system (43) with $\varepsilon = 0$ *at the equilibrium point* $(0,0,0)$ *in its real Jordan normal form, i.e. into the form*

$$\begin{pmatrix} 0 & -\sqrt{-(K_6+L_6)} & 0\\ \sqrt{-(K_6+L_6)} & 0 & 0\\ 0 & 0 & 0 \end{pmatrix},$$

where $K_6 = a_2b_1 + b_{20}^2 + b_{20}c_{30}$ *and* $L_6 = a_3c_1 + b_{20}c_{30} + c_{30}^2$. *Then doing the change of variables*

$$\begin{pmatrix} u\\ v\\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0\\ \dfrac{b_{20}+c_{30}}{\sqrt{-(K_6+L_6)}} & -\dfrac{b_1}{\sqrt{-(K_6+L_6)}} & -\dfrac{c_1}{\sqrt{-(K_6+L_6)}}\\ \dfrac{a_2b_1c_{30}-a_3b_{20}c_1}{-(K_6+L_6)c_1} & \dfrac{b_1L_6}{-(K_6+L_6)c_1} & \dfrac{K_6}{K_6+L_6} \end{pmatrix} \begin{pmatrix} X\\ Y\\ Z \end{pmatrix},$$

whose inverse is

$$\begin{pmatrix} X\\ Y\\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0\\ \dfrac{b_{20}}{b_1} & \dfrac{K_6}{b_1\sqrt{-(K_6+L_6)}} & -\dfrac{c_1}{b_1}\\ \dfrac{c_{30}}{c_1} & \dfrac{L_6}{c_1\sqrt{-(K_6+L_6)}} & 1 \end{pmatrix} \begin{pmatrix} u\\ v\\ w \end{pmatrix}.$$

And by following the same steps as the first case (i) indicated in (2), we get the system

$$\begin{aligned}
\dot{r} \;=\;& \frac{\varepsilon}{(-K_6-L_6)^2b_1^2c_1^2}(((2c_1^3e_3K_6b_{20} + (((a_2-b_{20}-c_{30}-d_1)K_6 - L_6(-a_3+b_{20}+c_{30}+d_1))b_1^2\\
&+((c_{30}+2e_2+2b_{20})K_6 + L_6b_{20})b_{20}b_1 - e_1(K_6^2 + b_{20}(2c_{30}+3b_{20})K_6 + L_6b_{20}^2))c_1^2 + b_1^2c_{30}(c_{30}K_6\\
&+(2c_{30}+2f_3+b_{20})L_6)c_1 - (-2c_{30}f_2L_6b_1 + (c_{30}^2K_6 + L_6^2 + (2b_{20}c_{30}+3c_{30}^2)L_6)f_1)b_1^2)r^2\cos^3(\theta)\\
&-c_1r(2Wc_1^3e_3K_6 - W(((-2e_2-b_{20})K_6 - L_6b_{20})b_1 + 2e_1((c_{30}+2b_{20})K_6 + L_6b_{20}))c_1^2\\
&-b_1^2((Wc_{30}+3b_{21}+4c_{31})K_6 + L_6(Wc_{30}+2Wf_3+3b_{21}+3c_{31}))c_1\\
&+W(-2f_2L_6b_1 + f_1(2c_{30}K_6 + 2(2c_{30}+b_{20})L_6))b_1^2)\cos^2(\theta) + (-e_1W^2(K_6+L_6)c_1^4 - 2c_1^3e_3r^2K_6b_{20}\\
&+((((c_{30}-a_2+b_{20})K_6 + L_6(-a_3+b_{20}+c_{30}))b_1^2 - ((c_{30}+2e_2+2b_{20})K_6 + L_6b_{20})b_{20}b_1\\
&+(K_6+2b_{20}(b_{20}+c_{30}))e_1K_6)r^2 - W^2f_1b_1^2(K_6+L_6))c_1^2 - b_1^2r^2c_{30}(c_{30}K_6 + (2c_{30}+2f_3+b_{20})L_6)c_1\\
&+b_1^2(-2c_{30}f_2b_1 + (L_6+2c_{30}(b_{20}+c_{30}))f_1)r^2L_6)\cos(\theta) + c_1(2Wc_1^3e_3K_6\\
&-W(((-2e_2-b_{20})K_6 - L_6b_{20})b_1 + 2e_1K_6(b_{20}+c_{30}))c_1^2 - b_1^2((Wc_{30}+b_{21}+2c_{31})K_6\\
&+L_6(Wc_{30}+2Wf_3+b_{21}+c_{31}))c_1 + 2Wb_1^2(-f_2b_1 + f_1(b_{20}+c_{30}))L_6)r)\sqrt{-K_6-L_6}\\
&+\sin(\theta)(((d_3(K_6+L_6)b_1^2 + e_3(K_6b_{20}^2 + L_6b_{20}^2 + K_6^2))c_1^3 + (d_2(K_6+L_6)b_1^3\\
&-(K_6+L_6)(K_6+L_6+(d_1-a_2)b_{20} + (d_1-a_3)c_{30})b_1^2 + ((e_2+b_{20})K_6^2\\
&+(L_6+b_{20}(c_{30}+e_2+b_{20}))b_{20}K_6 + L_6b_{20}^2(c_{30}+e_2+b_{20}))b_1 - e_1((c_{30}+3b_{20})K_6^2
\end{aligned}$$

$$
\begin{aligned}
&+(2L_6+b_{20}(b_{20}+c_{30}))b_{20}K_6+L_6b_{20}^2(b_{20}+c_{30})))c_1^2+((L_6+c_{30}(c_{30}+f_3+b_{20}))c_{30}K_6\\
&+((c_{30}+f_3)L_6+c_{30}^2(c_{30}+f_3+b_{20}))L_6)b_1^2c_1-b_1^2(-f_2(c_{30}^2K_6+L_6(c_{30}^2+L_6))b_1\\
&+((2L_6+c_{30}(b_{20}+c_{30}))c_{30}K_6+L_6((3c_{30}+b_{20})L_6+c_{30}^2(b_{20}+c_{30})))f_1))r^2\cos^2(\theta)\\
&-c_1(K_6+L_6)r(2Wc_1^3e_3b_{20}-W((-a_2+a_3)b_1^2-b_{20}(c_{30}+2e_2+b_{20})b_1\\
&+2(K_6+b_{20}(b_{20}+c_{30}))e_1)c_1^2-b_1^2((Wc_{30}+3b_{21}+4c_{31})b_{20}\\
&+c_{30}(Wc_{30}+2Wf_3+3b_{21}+3c_{31}))c_1+2Wb_1^2(-c_{30}f_2b_1+(L_6+c_{30}(b_{20}+c_{30}))f_1))\cos(\theta)\\
&+e_3W^2(K_6+L_6)c_1^5-W^2(-e_2b_1+e_1(b_{20}+c_{30}))(K_6+L_6)c_1^4\\
&+(-e_3K_6^2r^2+Wb_1^2(K_6+L_6)(Wf_3-c_{31}))c_1^3+((((-e_2-b_{20})K_6-L_6b_{20})b_1\\
&+e_1K_6(b_{20}+c_{30}))K_6r^2-W^2(K_6+L_6)b_1^2(-f_2b_1+f_1(b_{20}+c_{30})))c_1^2\\
&-(c_{30}K_6+(c_{30}+f_3)L_6)b_1^2r^2L_6c_1+b_1^2r^2(-f_2b_1+f_1(b_{20}+c_{30}))L_6^2))\\
=\;&F_1(\theta,r,W), \qquad (44)
\end{aligned}
$$

$$
\begin{aligned}
\dot W \;=\;& \frac{\varepsilon}{b_1^2c_1^3(-K_6-L_6)^{\frac{5}{2}}}(r\sin(\theta)(-(b_{20}(a_3(K_6+L_6)b_1^2+2K_6(a_3b_{20}e_1+K_6e_3))c_1^3-(a_2c_{30}(K_6+L_6)b_1^2\\
&+K_6L_6(a_2-a_3)b_1+b_{20}(((c_{30}+2e_2+2b_{20})L_6+2c_{30}e_1a_2)K_6+L_6^2b_{20}))b_1c_1^2\\
&+b_1^2c_{30}(c_{30}K_6^2+2(c_{30}+f_3+b_{20}/2)L_6K_6+2f_1L_6a_3b_{20})c_1-2c_{30}L_6b_1^3(a_2c_{30}f_1+L_6f_2))r\cos(\theta)\\
&+c_1(2WK_6(a_3b_{20}e_1+K_6e_3)c_1^3-Wb_1(((2e_2+b_{20})L_6+2c_{30}e_1a_2)K_6+L_6^2b_{20})c_1^2\\
&-b_1^2((Wc_{30}+2c_{31})K_6^2+L_6(Wc_{30}+2Wf_3-b_{21}+c_{31})K_6+2Wf_1L_6a_3b_{20}-b_{21}L_6^2)c_1\\
&+2WL_6b_1^3(a_2c_{30}f_1+L_6f_2)))\sqrt{-K_6-L_6}+r^2(((K_6+L_6)(a_3b_{20}d_1+K_6d_3)b_1^2\\
&+(K_6b_{20}^2+L_6b_{20}^2+K_6^2)(a_3b_{20}e_1+K_6e_3))c_1^3-((K_6+L_6)(a_2c_{30}d_1+L_6d_2)b_1^2\\
&-(K_6+L_6)(K_6a_3c_{30}-L_6a_2b_{20})b_1+((e_2+b_{20})L_6+c_{30}e_1a_2)K_6^2+(L_6^2+b_{20}(c_{30}+e_2+b_{20})L_6\\
&+c_{30}e_1a_2b_{20})b_{20}K_6+((c_{30}+e_2+b_{20})L_6+c_{30}e_1a_2)b_{20}^2L_6)b_1c_1^2+((L_6+c_{30}(c_{30}+f_3+b_{20}))c_{30}K_6^2\\
&+((c_{30}+f_3)L_6^2+c_{30}^2(c_{30}+f_3+b_{20})L_6+c_{30}^2f_1a_3b_{20})K_6+f_1L_6a_3b_{20}(c_{30}^2+L_6))b_1^2c_1\\
&-(a_2c_{30}f_1+L_6f_2)b_1^3(c_{30}^2K_6+L_6(c_{30}^2+L_6)))\cos^2(\theta)-c_1(K_6+L_6)r(2Wb_{20}(a_3b_{20}e_1+K_6e_3)c_1^3\\
&-((WK_6a_3+WL_6a_2-2a_3b_{20}(b_{21}+c_{31}))b_1+W((c_{30}+2e_2+b_{20})L_6+2c_{30}e_1a_2)b_{20})b_1c_1^2\\
&-(2a_2c_{30}(b_{21}+c_{31})b_1+((Wc_{30}+2c_{31})b_{20}+c_{30}(Wc_{30}+2Wf_3+c_{31}))K_6-b_{21}(b_{20}+c_{30})L_6\\
&+2Wc_{30}f_1a_3b_{20})b_1^2c_1+2Wc_{30}b_1^3(a_2c_{30}f_1+L_6f_2))\cos(\theta)+W^2(K_6+L_6)(a_3b_{20}e_1+K_6e_3)c_1^5\\
&-W^2b_1(K_6+L_6)(a_2c_{30}e_1+L_6e_2)c_1^4+((-K_6^2a_3b_{20}e_1-K_6^3e_3)r^2+W(K_6+L_6)b_1^2((Wf_3-c_{31})K_6\\
&+Wf_1a_3b_{20}))c_1^3-b_1(-(((e_2+b_{20})L_6+c_{30}e_1a_2)K_6+L_6^2b_{20})K_6r^2+W^2b_1^2(K_6+L_6)(a_2c_{30}f_1+L_6f_2))c_1^2\\
&-(c_{30}K_6^2+L_6(c_{30}+f_3)K_6+f_1L_6a_3b_{20})b_1^2r^2L_6c_1+r^2L_6^2b_1^3(a_2c_{30}f_1+L_6f_2))\\
=\;&F_2(\theta,r,W). \qquad (45)
\end{aligned}
$$

We shall apply the averaging theory described in Theorem 2 to the differential system (44)–(45). Using the notation of section 2 we have $t=\theta$, $T=2\pi$, $\mathbf{x}=(r,W)^T$ and

$$
F(\theta,r,W)=\begin{pmatrix} F_1(\theta,r,W)\\ F_2(\theta,r,W)\end{pmatrix} \quad \text{and} \quad f(r,W)=\begin{pmatrix} f_1(r,W)\\ f_2(r,W)\end{pmatrix}.
$$

It is immediate to check that system (44)–(45) satisfies all the assumptions of Theorem 2. Now we compute the integrals (5), i.e.

$$
f_1(r,W)=-\frac{r\big(T_6W+N_6\big)}{2c_1b_1^2\big(-K_6-L_6\big)^{3/2}}, \quad f_2(r,W)=\frac{D_6W^2-R_6r^2+C_6W}{2b_1^2c_1^3\big(-K_6-L_6\big)^{5/2}}.
$$

The system $f_1(r,W)=f_2(r,W)=0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$(r^*,W^*)=\left(\frac{1}{T_6}\sqrt{\frac{D_6N_6^2-C_6N_6T_6}{R_6}},\ -\frac{N_6}{T_6}\right),$$

if $T_6>0$, and $R_6(D_6N_6^2-C_6N_6T_6)>0$, and the Jacobian (6) at (r^*,W^*) takes the value $N_6\big(C_6T_6-D_6N_6\big)/2\,T_6b_1^4c_1^4\big(K_6+L_6\big)^4\neq 0$ where

$$\begin{aligned}
D_6 =\ & 2c_1^2\big(K_6+L_6\big)\big(b_1^2b_{20}c_1(a_3f_1+b_{20}f_3+c_{30}f_3)+b_1^3\big(a_2c_1f_3-a_2c_{30}f_1-f_2L_6\big)+b_1c_1^2\big(a_2c_1e_3\\
&-a_2c_{30}e_1-e_2L_6\big)+b_{20}c_1^3\big(a_3e_1+b_{20}e_3+c_{30}e_3\big)\big),\\
T_6 =\ & 2b_1^2L_6(b_1f_2+c_1f_3)-2c_1^2(b_1e_2+c_1e_3)K_6+2\big(b_1^2f_1+c_1^2e_1\big)(a_2b_1c_{30}-a_3b_{20}c_1)\\
&+b_1c_1(b_1c_{30}-b_{20}c_1)\,(K_6+L_6)\,,\\
R_6 =\ & \Big(-a_2b_1^4\big(a_2(c_1^3d_3-c_1^2c_{30}d_1+c_1c_{30}^2f_3-c_{30}^3f_1)-(c_1^2d_2+c_{30}^2f_2)L_6\big)+b_1^3c_1\big(a_2^3c_1(c_1e_3-c_{30}e_1)\\
&-a_2^2c_1e_2L_6-a_2a_3c_{30}f_1(a_3c_1+3b_{20}c_{30}+c_{30}^2)-a_2c_1d_1\big((b_{20}-c_{30})(a_3c_1-b_{20}c_{30})-3b_{20}c_{30}^2-c_{30}^3\big)\\
&+a_2f_3\big(a_3c_1(a_3c_1+2b_{20}c_{30}+c_{30}^2)-b_{20}^2c_{30}^2-b_{20}c_{30}^3\big)-a_2c_1^2d_3\big(a_3c_1+2b_{20}^2+3b_{20}c_{30}+c_{30}^2\big)-L_6\big((a_3c_1\\
&+2b_{20}c_{30}+c_{30}^2)(a_3f_2-c_1d_2)-b_{20}^2c_1d_2\big)\big)+b_1b_{20}^2c_1^3\big(a_2(a_3e_1(b_{20}+3c_{30})-e_3(a_3c_1-b_{20}^2-3b_{20}c_{30}\\
&-2c_{30}^2))+a_3e_2\big(a_3c_1+b_{20}c_{30}+c_{30}^2\big)\big)+c_1^2b_1^2b_{20}\big(+a_3^3c_1f_1+a_2^2(e_1(a_3c_1-b_{20}c_{30}-2c_{30}^2)+c_1e_3(2b_{20}\\
&+3c_{30}))-a_2e_2(b_{20}+2c_{30})L_6+a_3^2(f_3(b_{20}+c_{30})c_1-c_1^2d_1+c_{30}f_1(2b_{20}+c_{30}))-a_3(b_{20}+c_{30})(c_1^2d_3\\
&+c_1d_1(b_{20}+c_{30})-c_{30}f_3(2b_{20}+c_{30}))-c_1d_3(b_{20}+c_{30})^3\big)-a_3b_{20}^3c_1^4(a_3e_1+b_{20}e_3+c_{30}e_3)\Big),\\
N_6 =\ & -b_1^2b_{21}c_1(K_6+L_6)-b_1^2c_1c_{31}L_6,\\
C_6 =\ & \big(-2b_1^2c_1^3c_{31}K_6\big(K_6+L_6\big)\big).
\end{aligned}$$

Finally, we apply Theorem 2 like in the first case (i), then for $\varepsilon>0$ sufficiently small system (43) has a periodic solution $(x(\theta),y(\theta),z(\theta))=(\varepsilon X(\theta),\varepsilon Y(\theta),\varepsilon Z(\theta))$ which tends to the origin of coordinates when $\varepsilon\to 0$. Therefore, it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when $\varepsilon=0$. This completes the proof of Theorem 1 under the condition (vi).

Example 6 *Consider the Kolmogorov system*

$$\left\{\begin{aligned}
\dot x =\ & x\big(-2x+y+z+(x-1)^2+(y-1)^2+2(z-1)^2\big),\\
\dot y =\ & y\big(-2x+y+z-(x-1)^2+3(y-1)^2+(z-1)^2\big),\\
\dot z =\ & z\big(-3x+1+y+z+(z-1)^2\big).
\end{aligned}\right. \tag{46}$$

This system in the new variables (X,Y,Z) writes

$$\left\{\begin{aligned}
\dot X =\ & (X+1)\big(X^2-4X\varepsilon+Y^2+2Z^2-2X+Y+Z\big),\\
\dot Y =\ & (Y+1)\big((Y+Z)\varepsilon-2X+Y+Z-X^2+3Y^2+Z^2\big),\\
\dot Z =\ & (Z+1)\big((2Y+Z)\varepsilon-3X+Y+Z+Z^2\big).
\end{aligned}\right.$$

The corresponding system associated to system (44)–(45) satisfies

$$\begin{aligned}
F_1(\theta,r,W) =\ & -\sin(\theta)\cos(\theta)r(-3W+13)-3\cos(\theta)\sin^2(\theta)r^2-\sin^2(\theta)r(4W-2)+\sin^3(\theta)r^2\\
&-(-W-1)W\sin(\theta)+2\cos^2(\theta)r(W-2)+3\cos(\theta)W^2+4\cos^3(\theta)r^2,\\
F_2(\theta,r,W) =\ & -2r^2\cos(\theta)\sin(\theta)-(3W-1)r\sin(\theta)-W^2-(-6W+6)r\cos(\theta)+r^2.
\end{aligned}$$

To look for the limit cycles we must solve the system given by the averaged function

$$f(r,W) = (f_1(r,W), f_2(r,W)) = (0,0)$$

where

$$f_1(r,W) = -\frac{1}{2}(2W+2)r \quad \text{and} \quad f_2(r,W) = -W^2 + r^2. \tag{47}$$

This system has three solutions for (r,W) given by $(0,0)$, $(1,-1)$, $(-1,-1)$. Here we have only one good solution, the $(1,-1)$. Since the determinant (6) at this solution is 2 and thus non-zero, the Kolmogorov system (46) has one limit cycle bifurcating from the origin provided by the averaging theory of first order. We plot this limit cycle for $\varepsilon = 10^{-5}$ in Figure 6.

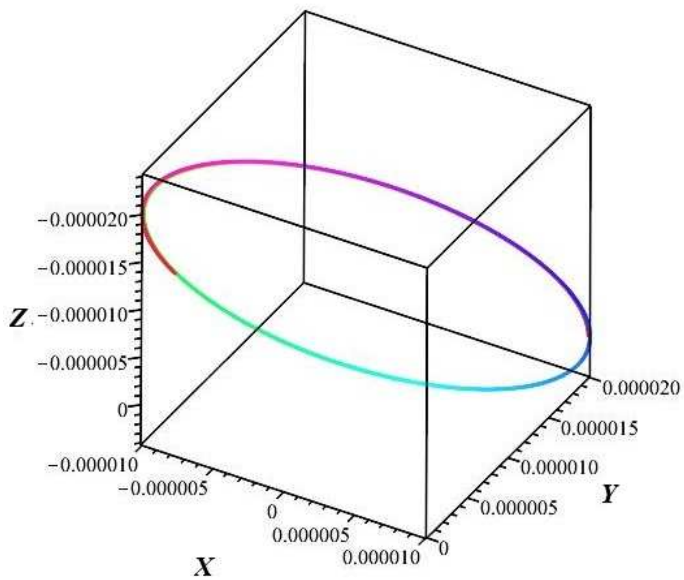

Figure 6: $X(0) = \varepsilon$, $Y(0) = 2\varepsilon$, $Z(0) = 0$.

Since the eigenvalues of the Jacobian matrix of (f_1, f_2) at the singular point $(1,-1)$ are $1 \pm i$, by Theorem 2 this limit cycle is unstable. Going back through the changes of variables as we did in the proof of Theorem 1 we obtain that the limit cycle bifurcating from the equilibrium point $(1,1,1)$ of system (46) is

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = (1 + \varepsilon\cos t + O(\varepsilon^2), 1 + (1+\cos t)\varepsilon + O(\varepsilon^2), 1 + (\cos t - \sin t - 1)\varepsilon + O(\varepsilon^2)).$$

This completes Example 6.

Now we perturb the Kolmogorov system (1) with the parameters given in statement (vii) of Proposition 1, as it is indicated in (2). We translate the equilibrium point $(1,1,1)$ to the origin of coordinates doing the change of variables $x = X+1$, $y = Y+1$, $z = Z+1$. Then system (1) becomes

$$\begin{aligned}
\dot{X} &= (X+1)\big((-2b_{21} - 2c_{31})X\varepsilon + (-b_{20} - c_{30})X + b_1 Y + c_1 Z + d_1 X^2 + e_1 Y^2 + f_1 Z^2\big),\\
\dot{Y} &= (Y+1)\left(\left(\left(-\frac{b_{20}c_{31}}{b_1} - \frac{b_{20}(b_{21}+c_{31}) + b_{21}(b_{20}+c_{30})}{b_1}\right)X + b_{21}Y + \frac{b_{21}c_1 Z}{b_1}\right)\varepsilon - \frac{b_{20}(b_{20}+c_{30})X}{b_1}\right.\\
&\quad \left. + b_{20}Y + \frac{b_{20}c_1 Z}{b_1} + d_2 X^2 + e_2 Y^2 + f_2 Z^2\right),\\
\dot{Z} &= (Z+1)\big(X^2 d_3 + Y^2 e_3 + Z^2 f_3 + Z\varepsilon c_{31} + X a_3 + Y b_3 + Z c_{30}\big).
\end{aligned} \tag{48}$$

In order to facilitate the application of the averaging theory for computing the zero-Hopf bifurcation we write the linear part of system (48) with $\varepsilon=0$ *at the equilibrium point* $(0,0,0)$ *in its real Jordan normal form, i.e. into the form*

$$\begin{pmatrix} 0 & -\dfrac{\sqrt{-b_1A_2}}{b_1} & 0 \\ \dfrac{\sqrt{-b_1A_2}}{b_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $A_2=a_3b_1c_1+b_1c_{30}^2+b_{20}b_3c_1$*. Then doing the change of variables*

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}=\begin{pmatrix} -\dfrac{a_3c_1+b_{20}c_{30}+c_{30}^2}{b_1A_2} & \dfrac{b_1c_{30}-b_3c_1}{b_1A_2} & 0 \\ -\dfrac{b_{20}+c_{30}}{b_1\sqrt{-b_1A_2}} & \dfrac{1}{\sqrt{-b_1A_2}} & \dfrac{c_1}{b_1\sqrt{-b_1A_2}} \\ -\dfrac{b_{20}}{b_1A_2} & \dfrac{1}{A_2} & 0 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{49}$$

whose inverse is

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}=\begin{pmatrix} -b_1^2 & 0 & b_1(b_1c_{30}-b_3c_1) \\ -b_1b_{20} & 0 & (a_3c_1+b_{20}c_{30}+c_{30}^2)b_1 \\ -\dfrac{c_{30}b_1^2}{c_1} & \dfrac{b_1\sqrt{-b_1A_2}}{c_1} & -b_1(a_3b_1+b_{20}b_3+b_3c_{30}) \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}.$$

And by following the same steps as the first case (i) indicated in (2), we get the system

$$\begin{aligned}
\dot r \;=\;& \frac{\varepsilon}{c_1^2A_2\sqrt{-b_1A_2}}\Big(\sqrt{-b_1A_2}\Big(-\sin(\theta)r\cos(\theta)c_1\big(\big(\big(a_3c_1+b_{20}c_{30} \\
&+c_{30}^2\big)b_1c_1((b_1-b_{20})(b_1b_{20}+b_1c_{30}-b_3c_1)+b_1(b_1c_{30}-2b_{20}e_2-b_3c_{30})+2b_{20}(b_{20}e_1-c_1e_3+c_{30}e_1)) \\
&-(b_1c_{30}-b_3c_1)b_1c_1\big((b_{20}+c_{30})\big(b_1b_{20}+b_1c_{30}-2b_1d_1-b_{20}^2\big)+b_1(a_3c_{30}+2b_1d_2+2c_1d_3)\big) \\
&-(a_3b_1+b_{20}b_3+b_3c_{30})b_1\big(2b_1\big(\big(a_3c_1+b_{20}c_{30}+c_{30}^2\big)f_1-(b_1c_{30}-b_3c_1)f_2 \\
&-c_{30}(b_1f_2-b_{20}f_1+c_1c_{30}+c_1f_3-c_{30}f_1)\big)-c_1^2\big((a_3-b_{20}-c_{30})b_1+b_{20}^2+b_{20}b_3\big)\big)\big)W-3b_1c_1c_{30}c_{31} \\
&-b_{21}c_1(2b_1b_{20}+4b_1c_{30}-b_3c_1)+b_{21}b_1(b_{20}+c_{30})c_1\big) \\
&+b_1^2A_2\sin^3(\theta)r^2(b_1f_2-b_{20}f_1+c_1c_{30}+c_1f_3-c_{30}f_1)+b_1\sin(\theta)r^2\cos^2(\theta)(b_1^2(c_1^3(a_3-d_3) \\
&+3c_{30}^2f_1(b_{20}+c_{30})+c_1c_{30}(2a_3f_1+2b_3f_2-c_{30}^2-c_{30}f_3)-c_1^2(c_{30}(a_3-b_{20}-c_{30}-d_1)-b_{20}d_1)\big) \\
&-b_1^3\big(c_1^2d_2+3c_{30}^2f_2\big)-b_{20}c_1^2(b_1b_{20}(c_{30}+e_2)-b_{20}(b_{20}e_1+b_3c_1-c_1e_3+c_{30}e_1)+b_1b_3c_{30})) \\
&-c_1^2\sin(\theta)\Big(\Big(b_1\Big(\big(a_3c_1+b_{20}c_{30}+c_{30}^2\big)^2(b_1b_{20}+b_1e_2-b_{20}e_1+c_1e_3-c_{30}e_1) \\
&+(b_1c_{30}-b_3c_1)^2(b_1d_2+b_{20}(b_{20}+2c_{30}-d_1)+c_1d_3+c_{30}^2-c_{30}d_1) \\
&+(a_3b_1+b_{20}b_3+b_3c_{30})^2(b_1f_2-b_{20}f_1+c_1c_{30}+c_1f_3-c_{30}f_1)\Big) \\
&-\big(a_3c_1+b_{20}c_{30}+c_{30}^2\big)b_1((b_1c_{30}-b_3c_1)(b_{20}+c_{30})(b_{20}+b_1)+(a_3b_1+b_{20}b_3+b_3c_{30})c_1(b_{20}+b_3)) \\
&-(b_1c_{30}-b_3c_1)(a_3b_1+b_{20}b_3+b_3c_{30})b_1c_1(a_3-b_{20}-c_{30})\Big)W^2+\big(b_{21}\big(\big(a_3c_1+b_{20}c_{30}+c_{30}^2\big)b_1 \\
&+(b_{20}+2c_{30})(b_1c_{30}-b_3c_1)-(a_3b_1+b_{20}b_3+b_3c_{30})c_1\big)+c_{31}(2(b_1c_{30}-b_3c_1)c_{30} \\
&-(a_3b_1+b_{20}b_3+b_3c_{30})c_1)-b_{21}(b_1c_{30}-b_3c_1)(b_{20}+c_{30})\big)W\big)\big) \\
&+b_1^2(a_3b_1c_1+b_1c_{30}^2+b_{20}b_3c_1)\cos(\theta)\sin^2(\theta)r^2(b_1f_1(a_3c_1+3b_{20}c_{30}+3c_{30}^2) \\
&-c_1^2\big((a_3-b_{20}-c_{30})b_1+b_{20}^2+b_{20}b_3\big)-b_1(3b_1c_{30}f_2-c_1(b_3f_2-2c_{30}^2-2c_{30}f_3))\big)\big) \\
&+c_1^2\cos(\theta)\Big(\Big(-(a_3c_1+b_{20}c_{30}+c_{30}^2)(b_1c_{30}-b_3c_1)b_1((a_3c_1+b_{20}c_{30}+c_{30}^2)b_1(b_1-b_{20}-e_2)
\end{aligned}$$

$$
\begin{aligned}
&-(b_1c_{30}-b_3c_1)((b_{20}+c_{30}-d_1)b_1-b_{20}^2-b_{20}c_{30})-(a_3b_1+b_{20}b_3+b_3c_{30})c_1(b_1-b_{20}))-b_1^2\Big((a_3c_1\\
&+b_{20}c_{30}+c_{30}^2)^3e_1-(b_1c_{30}-b_3c_1)^3d_2\Big)-(a_3b_1+b_{20}b_3+b_3c_{30})^2b_1^2\big((a_3c_1+b_{20}c_{30}+c_{30}^2)f_1\\
&-(b_1c_{30}-b_3c_1)f_2)\Big)W^2+\big((b_1c_{30}-b_3c_1)b_{21}((2a_3c_1+2b_{20}c_{30}+3c_{30}^2)b_1-b_3c_1c_{30})\\
&+2(b_1c_{30}-b_3c_1)c_{31}A_2-b_{21}(b_1c_{30}-b_3c_1)^2(b_{20}+c_{30}))W\Big)+b_1^2\big((b_1c_{30}-b_3c_1)\big(b_1^2c_1^2d_2+b_1^2c_{30}^2f_2\\
&+b_{20}^2c_1^2e_2\big)-\big(a_3c_1+b_{20}c_{30}+c_{30}^2\big)(b_1^2c_1^2d_1+b_1^2c_{30}^2f_1+b_{20}^2c_1^2e_1))\cos^3(\theta)r^2\\
&+b_1c_1r\Big(\Big(\Big(\big(a_3c_1+b_{20}c_{30}+c_{30}^2\big)^2b_1c_1(b_1^2+2b_{20}e_1)-(b_1c_{30}-b_3c_1)^2c_1(2b_1^2d_2-b_{20}^3-b_{20}^2c_{30})\\
&-(a_3c_1+b_{20}c_{30}+c_{30}^2)(b_1c_{30}-b_3c_1)b_1c_1\big((b_{20}+c_{30}-2d_1)b_1+b_{20}^2+2b_{20}e_2\big)\\
&-(a_3b_1+b_{20}b_3+b_3c_{30})((a_3c_1+b_{20}c_{30}+c_{30}^2)b_1^2(c_1^2+2c_{30}f_1)-(b_1c_{30}-b_3c_1)\big(2b_1^2c_{30}f_2+b_{20}^2c_1^2\big)\big)\Big)W\\
&-b_{21}c_1\big(2\big(a_3c_1+b_{20}c_{30}+c_{30}^2\big)b_1+(b_1c_{30}-b_3c_1)c_{30}+A_2\big)\\
&-c_1c_{31}(2(a_3c_1+b_{20}c_{30}+c_{30}^2)b_1-2(b_1c_{30}-b_3c_1)b_{20}+A_2)+b_{21}(b_1c_{30}-b_3c_1)(b_{20}+c_{30})c_1)\cos^2(\theta)\\
&+b_1(a_3b_1c_1+b_1c_{30}^2+b_{20}b_3c_1)((a_3c_1+b_{20}c_{30}+c_{30}^2)c_1(b_{20}+b_3)+(b_1c_{30}-b_3c_1)c_1(a_3-b_{20}-c_{30})\\
&-2(a_3b_1+b_{20}b_3+b_3c_{30})(b_1f_2-b_{20}f_1+c_1c_{30}+c_1f_3-c_{30}f_1))W\sin^2(\theta)+A_2c_1(b_{21}+c_{31})\big)\Big)=F_1(\theta,r,W), \quad (50)
\end{aligned}
$$

$$
\begin{aligned}
\dot{W} \;=\;& \frac{\varepsilon}{c_1^2A_2\sqrt{-b_1A_2}}\Big(\sqrt{-b_1A_2}\big(c_1b_1\sin(\theta)r\big(Wb_1\big(b_{20}c_1\big(a_3c_1-b_1c_{30}+b_{20}c_{30}+b_3c_1+c_{30}^2\big)-2(a_3b_1+b_{20}b_3\\
&+b_3c_{30})(b_1f_2-b_{20}f_1)\big)+b_{21}c_1\big)-\cos(\theta)\sin(\theta)r^2b_1^2\big(2b_1^2c_{30}f_2+b_{20}^2c_1^2-b_{20}\big(c_1^2+2c_{30}f_1\big)b_1\big)\big)\\
&+b_1^2c_1\cos(\theta)r\big(\big(\big(a_3c_1+b_{20}c_{30}+c_{30}^2\big)b_{20}c_1\big(b_1(b_1-b_{20}-2e_2)+2b_{20}e_1\big)\\
&-(b_1c_{30}-b_3c_1)c_1(2b_1(b_1d_2-b_{20}d_1)+b_{20}(b_{20}+c_{30})(b_1-b_{20})\big)+(a_3b_1+b_{20}b_3+b_3c_{30})\big(2b_1^2c_{30}f_2\\
&+b_{20}^2c_1^2-b_{20}\big(c_1^2+2c_{30}f_1\big)b_1\big)\big)W-b_{21}c_1(2b_{20}+c_{30})+b_{21}(b_{20}+c_{30})c_1\big)\\
&+b_1c_1^2\Big(\Big(\big(a_3c_1+b_{20}c_{30}+c_{30}^2\big)^2b_1(b_1b_{20}+b_1e_2-b_{20}e_1)+(b_1c_{30}-b_3c_1)^2b_1(b_1d_2+b_{20}(b_{20}+c_{30}-d_1))\\
&-(a_3c_1+b_{20}c_{30}+c_{30}^2)(b_1c_{30}-b_3c_1)b_1b_{20}(b_1+b_{20}+c_{30})-(a_3b_1+b_{20}b_3+b_3c_{30})b_1b_{20}c_1(a_3c_1-b_1c_{30}+b_{20}c_{30}+b_3c_1\\
&+(a_3b_1+b_{20}b_3+b_3c_{30})^2b_1(b_1f_2-b_{20}f_1)\big)W^2+b_{21}\big(\big(a_3c_1+b_{20}c_{30}+c_{30}^2\big)b_1\\
&+(b_1c_{30}-b_3c_1)b_{20}-(a_3b_1+b_{20}b_3+b_3c_{30})c_1-(b_1c_{30}-b_3c_1)(b_{20}+c_{30})\big)W\big)\\
&+r^2b_1^2\cos^2(\theta)\big(c_1^2\big(b_1^3d_2-b_{20}^3e_1\big)+b_1^3c_{30}^2f_2-b_1b_{20}\big(b_1c_1^2d_1+b_1c_{30}^2f_1-b_{20}c_1^2e_2\big)\big)\\
&-b_1^3A_2r^2(b_1f_2-b_{20}f_1)\sin^2(\theta)\Big)=F_2(\theta,r,W).
\end{aligned}
$$

We shall apply the averaging theory described in Theorem 2 to the differential system (50)–(51). Using the notation of section 2 we have $t=\theta$, $T=2\pi$, $\mathbf{x}=(r,W)^T$ *and*

$$
F(\theta,r,W)=\begin{pmatrix} F_1(,r,W)\\ F_2(,r,W)\end{pmatrix} \quad \textit{and} \quad f(r,W)=\begin{pmatrix} f_1(r,W)\\ f_2(r,W)\end{pmatrix}.
$$

It is immediate to check that system (50)–(51) satisfies all the assumptions of Theorem 2. Now we compute the integrals (5), i.e.

$$
f_1(r,W)=-\frac{b_1^2r\big(T_7W+N_7\big)}{2c_1\big(-b_1A_2\big)^{3/2}}, \qquad f_2(r,W)=\frac{b_1^2\Big(D_7W^2+R_7r^2+C_7W\Big)}{2c_1^2\big(-b_1A_2\big)^{3/2}}.
$$

The system $f_1(r,W)=f_2(r,W)=0$ *has a unique solution* (r^*,W^*) *with* $r^*>0$, *namely*

$$
(r^*,W^*)=\left(\frac{1}{T_7}\sqrt{\frac{C_7N_7T_7-D_7N_7^2}{R_7}},-\frac{N_7}{T_7}\right),
$$

if $T_7 > 0$, *and* $R_7(C_7N_7T_7 - D_7N_7^2) > 0$, *and the Jacobian (6) at* (r^*, W^*) *takes the value* $b_1N_7\big(C_7T_7 - D_7N_7\big)/2\,T_7c_1^3A_2^3 \neq 0$ *where*

$$\begin{aligned}
T_7 = \;& \big(a_3c_1 + b_{20}c_{30} + c_{30}^2\big)^2 b_1c_1\big(b_1^2 + 2b_{20}e_1\big) - \big(b_1c_{30} - b_3c_1\big)^2 c_1\big(2b_1^2d_2 - b_{20}^3 - b_{20}^2c_{30}\big) - \big(a_3c_1 + b_{20}c_{30} \\
& + c_{30}^2\big)b_1\big((b_1c_{30} - b_3c_1)c_1\big((b_{20} + c_{30} - 2d_1)b_1 + b_{20}^2 + 2b_{20}e_2\big) + (a_3b_1 + b_{20}b_3 + b_3c_{30})b_1\big(c_1^2 + 2c_{30}f_1\big) \\
& \big) + (b_1c_{30} - b_3c_1)(a_3b_1 + b_{20}b_3 + b_3c_{30})\big(2b_1^2c_{30}f_2 + b_{20}^2c_1^2\big) + b_1A_2\big(\big(a_3c_1 + b_{20}c_{30} + c_{30}^2\big)c_1(b_{20} + b_3) \\
& + (b_1c_{30} - b_3c_1)c_1(a_3 - b_{20} - c_{30}) - 2(a_3b_1 + b_{20}b_3 + b_3c_{30})(b_1f_2 - b_{20}f_1 + c_1c_{30} + c_1f_3 - c_{30}f_1)\big),\\
D_7 = \;& 2b_1b_{20}c_1^2\big(\big(a_3c_1 + b_{20}c_{30} + c_{30}^2\big)(b_1c_{30} - b_3c_1)(b_1 + b_{20} + c_{30}) + (a_3b_1 + b_{20}b_3 + b_3c_{30})c_1\big(a_3c_1 - b_1c_{30} \\
& + b_{20}c_{30} + b_3c_1 + c_{30}^2\big)\big) - 2b_1c_1^2\Big(\big(a_3c_1 + b_{20}c_{30} + c_{30}^2\big)^2(b_1b_{20} + b_1e_2 - b_{20}e_1) + \big(b_1c_{30} - b_3c_1\big)^2(b_1d_2 \\
& + b_{20}^2 + b_{20}c_{30} - b_{20}d_1) + \big(a_3b_1 + b_{20}b_3 + b_3c_{30}\big)^2(b_1f_2 - b_{20}f_1)\Big),\\
N_7 = \;& b_{21}c_1\big(- b_1\big(a_3c_1 + 2b_{20}c_{30} + 2c_{30}^2\big) + b_{20}b_3c_1 + b_3c_1c_{30}\big) - c_1c_{31}A_2 + \big(b_1b_{20}c_{30} + b_1c_{30}^2 - b_{20}b_3c_1 \\
& - b_3c_1c_{30}\big)b_{21}c_1,\\
R_7 = \;& b_1\big(b_1b_{20}(b_1c_1^2d_1 + b_1c_{30}^2f_1 - b_{20}c_1^2e_2) - b_1^3(c_1^2d_2 + c_{30}^2f_2) + b_{20}^3c_1^2e_1 + b_1A_2(b_1f_2 - b_{20}f_1)\big),\\
C_7 = \;& -2b_{21}c_1^2\big(2b_1b_{20}c_{30} + b_1c_{30}^2 - 2b_{20}b_3c_1 - b_3c_1c_{30} - (b_{20} + c_{30})(b_1c_{30} - b_3c_1)\big).
\end{aligned}$$

Finally, we apply Theorem 2 like in the first case (i), then for $\varepsilon > 0$ *sufficiently small system (48) has a periodic solution* $(x(\theta), y(\theta), z(\theta)) = (\varepsilon X(\theta), \varepsilon Y(\theta), \varepsilon Z(\theta))$ *which tends to the origin of coordinates when* $\varepsilon \to 0$. *Therefore, it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when* $\varepsilon = 0$. *This completes the proof of Theorem 1 under the condition (vii).*

Example 7 *Consider the Kolmogorov system*

$$\begin{cases}
\dot{x} = & x\big(- 2 + y + z + 4(x - 1)^2\big),\\
\dot{y} = & y\big(- 2 + y + z - 3(x - 1)^2 + 8(y - 1)^2\big),\\
\dot{z} = & z\big(3 - 2y - z\big).
\end{cases} \tag{52}$$

This system in the new variables (X, Y, Z) *writes*

$$\begin{cases}
\dot{X} = & (X + 1)\big(4X^2 + Y + Z\big),\\
\dot{Y} = & (Y + 1)\big((X + Y + Z)\varepsilon + Y + Z - 3X^2 + 8Y^2\big),\\
\dot{Z} = & (Z + 1)\big(- Z\varepsilon - 2Y - Z\big).
\end{cases}$$

The corresponding system associated to system (50)–(51) satisfies

$$\begin{aligned}
F_1(\theta, r, W) = \;& \sin(\theta)r\cos(\theta)(6W - 3) - \sin^3(\theta)r^2 + 7\sin(\theta)r^2\cos^2(\theta) + \sin(\theta)(-3W^2 + W)\\
& - \cos(\theta)\sin^2(\theta)r^2 - \cos(\theta)(-3W^2 + W) - 5\cos^3(\theta)r^2 - r(6W - 1)\cos^2(\theta),\\
F_2(\theta, r, W) = \;& - \sin(\theta)r(-W + 1) - \cos(\theta)r(14W - 1) + 7W^2 - W - r^2\cos^2(\theta).
\end{aligned}$$

To look for the limit cycles we must solve the system given by the averaged function $f(r, W) = (f_1(r, W), f_2(r, W)) = (0, 0)$ *where*

$$f_1(r, W) = -\frac{1}{2}r\big(6W - 1\big) \quad \text{and} \quad f_2(r, W) = 7W^2 - \frac{1}{2}r^2 - W. \tag{53}$$

This system has four solutions for (r, W) *given by* $(0, 0)$, $(0, 1/7)$, $(\sqrt{2}/6, 1/6)$, $(-\sqrt{2}/6, 1/6)$. *As in Example 1 we have two good solutions, the* $(0, 1/7)$ *and the* $(\sqrt{2}/6, 1/6)$. *Since the determinants (6) at these two* *solutions are* $1/14$ *and* $-1/6$ *and thus non-zero, respectively, the Kolmogorov system (52) has two limit cycles bifurcating from the origin provided by the averaging theory of first order. We plot these two limit cycles for* $\varepsilon = 10^{-5}$ *in Figure 7.*

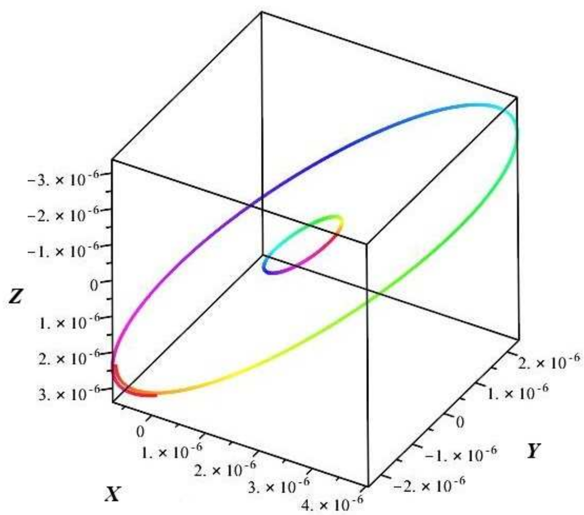

Figure 7: 1st LC: $X(0) = 8\varepsilon/49$, $Y(0) = \varepsilon/49$, $Z(0) = \varepsilon/49$. 2nd LC: $X(0) = \dfrac{(1-\sqrt{2})\varepsilon}{6}$, $Y(0) = -\dfrac{\varepsilon\sqrt{2}}{6}$, $Z(0) = \dfrac{\varepsilon\sqrt{2}}{6}$.

Since the eigenvalues of the Jacobian matrix of (f_1, f_2) *at the singular points* $(0, 1/7)$ *and* $(\sqrt{2}/6, 1/6)$ *are* $(1/14, 1)$ *and* $(4 \pm \sqrt{22})/6$*, respectively, by Theorem 2 the limit cycles are both unstable. Going back through the changes of variables as we did in the proof of Theorem 1 we obtain that the limit cycles bifurcating from the equilibrium point* $(1, 1, 1)$ *of system (52) are*

$$
\begin{aligned}
(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) &= (1 + \varepsilon/7 + O(\varepsilon^2), 1 + O(\varepsilon^2), 1 + O(\varepsilon^2)),\\
(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) &= \big(1 - \varepsilon(\sqrt{2}\cos t - 1)/6 + O(\varepsilon^2),\ 1 - \varepsilon\sqrt{2}\cos t/6 + O(\varepsilon^2)\,,\\
&\qquad 1 + \varepsilon\sqrt{2}(\cos t + \sin t)/6 + O(\varepsilon^2)\big)\,,
\end{aligned}
$$

respectively. This completes Example 7.

Now we perturb the Kolmogorov system (1) with the parameters given in statement (viii) of Proposition 1, as it is indicated in (2). We translate the equilibrium point $(1, 1, 1)$ *to the origin of coordinates doing the change of variables* $x = X + 1$, $y = Y + 1$, $z = Z + 1$. *Then system (1) becomes*

$$
\begin{aligned}
\dot{X} &= (X+1)\big(-2c_{31}X\varepsilon + (-b_2 - c_{30})X + b_1Y + c_1Z + d_1X^2 + e_1Y^2 + f_1Z^2\big),\\
\dot{Y} &= (Y+1)\big(X^2d_2 + Y^2e_2 + Z^2f_2 + Xa_2 + Yb_2 + Zc_2\big),\\
\dot{Z} &= (Z+1)\,(((a_2b_1c_{31} + (b_2 + c_{30})b_2c_{31} + c_{31}(b_2c_{30} - b_3c_2)X)/(b_1c_2 - b_2c_1) + c_{31}Z)\,\varepsilon + (a_2(b_1c_{30} - b_3c_1)\\
&\quad + (b_2 + c_{30})(b_2c_{30} - b_3c_2))X/(b_1c_2 - b_2c_1) + b_3Y + c_{30}Z + d_3X^2 + e_3Y^2 + f_3Z^2\Big).
\end{aligned}
\tag{54}
$$

In order to facilitate the application of the averaging theory for computing the zero-Hopf bifurcation we write the linear part of system (54) with $\varepsilon = 0$ *at the equilibrium point* $(0, 0, 0)$ *in its real Jordan normal form, i.e.*

into the form

$$\begin{pmatrix} 0 & -\dfrac{\sqrt{(b_1c_2-b_2c_1)A}}{b_1c_2-b_2c_1} & 0 \\ \dfrac{\sqrt{(b_1c_2-b_2c_1)A}}{b_1c_2-b_2c_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} A &= -(-b_2^3c_1 + b_1b_2^2c_2 + b_2(-2b_3c_1c_2 + b_1c_2c_3) + c_2(b_1b_3c_2 - b_3c_1c_3 + b_1c_3^2) \\ &\quad + a_2(b_1^2c_2 - b_3c_1^2 + b_1c_1(c_3 - b_2))). \end{aligned}$$

Then doing the change of variables

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \dfrac{FL}{EA} & -\dfrac{FM}{EA} & -\dfrac{F}{A} \\ -\dfrac{K\sqrt{FA}}{EA} & \dfrac{F\sqrt{FA}}{EA} & 0 \\ -\dfrac{G}{A} & \dfrac{H}{A} & -\dfrac{F}{A} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

whose inverse is

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} F & -\dfrac{c_1A}{\sqrt{AF}} & -F \\ K & -\dfrac{c_2A}{\sqrt{AF}} & -K \\ \dfrac{S}{F} & -\dfrac{c_{30}A}{\sqrt{AF}} & P \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where

$$E = a_2c_1^2 - b_1c_2^2 + 2b_2c_1c_2 + c_1c_2c_{30}, \quad K = a_2c_1 + b_2c_2 + c_2c_{30}, \quad M = a_2b_1c_1 + b_1c_2c_{30} + b_2^2c_1,$$

$$P = a_2b_1 + b_2^2 + b_2c_{30}, \quad F = b_1c_2 - b_2c_1, \quad G = b_2c_{30} - b_3c_2,$$

$$L = a_2b_1c_2 + a_2c_1c_{30} + b_2^2c_2 + 2b_2c_2c_{30} + c_2c_{30}^2, \quad H = b_1c_{30} - b_3c_1,$$

$$S = a_2b_1c_1c_{30} - a_2b_3c_1^2 + b_1b_3c_2^2 + b_1c_2c_{30}^2 + b_2^2c_1c_{30} - 2b_2b_3c_1c_2 - b_3c_1c_2c_{30}.$$

And by following the same steps as the first case (i) indicated in (2), we get the system

$$\begin{aligned} \dot{r} &= \frac{1}{FAE\sqrt{FA}}\Big(\big(\sin(\theta)\sqrt{FA}\big((F^3d_2 + F^2K(a_2 + b_2 + c_{30} - d_1) - F\big(K^2(b_1 - b_2 - e_2) - P((c_1 - c_2)K \\ &\quad + Pf_2)\big) - K^3e_1 - KP^2f_1\big)W^2 - 2WFKc_{31}\big) - (\cos(\theta)\big(\big(FE(F^2d_3 + K^2e_3 - P(-Kb_3 + Pc_{30}) - KPb_3 \\ &\quad + P^2(c_{30} + f_3)\big) + F^2(F(L(b_2 + c_{30} - d_1) + Md_2) - K(Lb_1 - Ma_2) + LPc_1) - F\big(K^2(Le_1 - Mb_2 - Me_2) \\ &\quad + KMPc_2 + P^2(Lf_1 - Mf_2)\big)\big)W^2 - c_{31}(EFG + 2F^2L)W)\big)\big)F^2 + \cos^2(\theta)\sin(\theta)\sqrt{FA}r^2\big(F^5d_2 \\ &\quad + F^4K(a_2 + b_2 + c_{30} - d_1) + F^2(S + Fc_{30}(c_{30} + 2f_3))) - 2FK^3e_1 - FK(L(b_1c_1 + 2c_2e_1) - M(a_2c_1 \\ &\quad + (c_{30} + 2e_2 + 2b_2)c_2)) - F(S((c_1 - c_2)K + 2Pf_2) - P\big(\big(c_1^2 + 2c_{30}f_1\big)L - M\big(c_2^2 + 2c_{30}f_2\big)\big)\big)\big) \\ &\quad + F^2(E(2Fc_1d_3 - Kb_3c_{30} + Pc_{30}^2) + F(KP(c_1 - c_2) - 2K^2(b_1 - b_2 - e_2) - L(b_1c_2 - c_1(2b_2 + c_{30} - 2d_1)) \\ &\quad + M(a_2c_2 + 2c_1d_2))\big) + 2F^4(Fd_2 + K(a_2 + b_2 + c_{30} - d_1)) + 2FKPSf_1)W - Fc_{31}\big(E(Hc_2 + Pc_1) \\ &\quad + 2F^2K + 2FLc_1)\big)\big) - FA\big(\cos(\theta)\sin^2(\theta)r^2(F^2K(c_1(a_2 + 2b_2 + c_{30} - 2d_1) - c_2(b_1 - 2b_2 - c_{30} - 2e_2)) \\ &\quad + F^3(a_2c_2 + 2c_1d_2) + E(F(c_1^2d_3 + c_2^2e_3 + c_{30}^2f_3) + Sc_{30}) - F\big(K^2(b_1c_1 + 2c_2e_1) + L(c_1^2d_1 + c_2^2e_1 + c_{30}^2f_1 \end{aligned}$$

$$
\begin{aligned}
&+Fc_1) - M\big((b_2+c_{30}+e_2)c_2^2 + a_2c_1c_2 + c_1^2d_2 + c_{30}^2f_2) - S(c_2^2+2c_{30}f_2)\big) - KS(c_1^2+2c_{30}f_1)\big)\\
&-\big(\sin^3(\theta)\sqrt{FA}r^2\big(F\big((b_2+c_{30}+e_2)c_2^2+a_2c_1c_2+c_1^2d_2+c_{30}^2f_2\big) - K(c_1^2d_1+c_2^2e_1+c_{30}^2f_1+Fc_1)\big)\big)\Big)\\
&+F\Big(r\Big(-2AFKc_1c_{31} + \big(\big(2F^4(Ed_3+Md_2+L(b_2+c_{30}-d_1)) + EF^2\big(2K^2e_3 - P(-Kb_3+Pc_{30})\\
&-KPb_3\big) - F^3(2K(Lb_1-Ma_2) - LPc_1) - 2F^2K^2(Le_1 - M(b_2+e_2)) - F^2KMPc_2 + S\big(-PEF(c_{30}+2f_3)\\
&-F^2Lc_1 + F(KMc_2 + 2P(Lf_1 - Mf_2))\big)\big)W - Fc_{31}(EF(P+G) + 2F^2L - 2KAc_1 + ES)\big)\cos^2(\theta)\\
&+FW(FK(K-F+c_1(b_2+c_{30}-2d_1) + c_2(b_2+2e_2)) - FP(c_2^2+2c_{30}f_2) + F^2(a_2c_2+2c_1d_2)\\
&-\big((b_1c_1+2c_2e_1)K - P(c_1^2+2c_{30}f_1)\big)K\big)A\sin(\theta)^2\big) - \big(\cos^3(\theta)r^2\big(F(KS(Eb_3+Mc_2) + F(K^2(Ee_3\\
&-Le_1 + M(b_2+e_2)) - LSc_1) + ES(-Kb_3+Pc_{30})\big) + F^4(Ed_3+Md_2+L(b_2+c_{30}-d_1))\\
&-F^3K\big(Lb_1 - Ma_2\big) + S^2(Ec_{30}+Ef_3-Lf_1+Mf_2)\big)\big)\Big)\Big) = F_1(\theta,r,W), \qquad (55)
\end{aligned}
$$

$$
\begin{aligned}
\dot W \;=\; &-\frac{1}{FA\sqrt{FA}}\Big(\sin(\theta)\sqrt{FA}r\big(W\big(F^2\big(2Fc_1d_3 - Kb_3c_{30} + Pc_{30}^2\big) + F\big(GF\big(F - c_1(b_2+c_{30}-2d_1)\big)\\
&-F\big(H(a_2c_2+2c_1d_2) - K(b_3c_{30}+2c_2e_3)\big) - P\big(b_1c_{30}(a_2c_1+2c_2c_{30}) + b_2c_1c_{30}(b_2-c_{30}) + 2Fc_{30}f_3\\
&-b_3\big(a_2c_1^2 - b_1c_2^2 + c_1c_2(2b_2+c_{30})\big)\big)\big) + GF\big(K(b_1c_1+2c_2e_1) - P(c_1^2+2c_{30}f_1)\big)\\
&-HF\big(K(a_2c_1+(2b_2+c_{30}+2e_2)c_2) - P\big(c_2^2+2c_{30}f_2\big)\big)\big) - c_{31}(F(Hc_2+Pc_1) - 2Fc_1G)\big)\\
&+F\big(\big(F^4d_3 - F^2\big(G\big(F(b_2+c_{30}-d_1) - Kb_1 + Pc_1\big) + H(Fd_2+Ka_2) + P(-Kb_3+Pc_{30})\\
&-K(Ke_3-Pb_3) - P^2(c_{30}+f_3)\big) + F\big(G\big(K^2e_1+P^2f_1\big) - HK^2(b_2+e_2) + HP(Kc_2-Pf_2)\big)\big)W^2\\
&+WGF^2c_{31}\big) + r^2\big(\big(F^2(KF(Gb_1-Ha_2+Ke_3) + S(-Kb_3+Pc_{30})) + F(K^2F(Ge_1-Hb_2-He_2)\\
&+SF(Gc_1+Kb_3) - HKSc_2 + S^2(c_{30}+f_3)\big) - F^4((b_2+c_{30}-d_1)G + Hd_2 - Fd_3) + S^2(Gf_1-Hf_2)\big)\cos^2(\theta)\\
&+FA\big(G\big(c_1^2d_1+c_2^2e_1+c_{30}^2f_1+Fc_1\big) - H(c_1^2d_2+c_2^2e_2+c_{30}^2f_2+Kc_2) + F(c_1^2d_3+c_2^2e_3+c_{30}^2f_3)\\
&+c_{30}^2(Fc_{30}+Pc_1) - b_3c_{30}(-Fc_2+Kc_1)\big)\sin^2(\theta)\big) - (\cos(\theta)r(2F^5Wd_3 - F^3(2F(GW(b_2+c_{30}-d_1)\\
&+HWd_2) + PW(-Kb_3+Pc_{30}) + c_{31}(P+G)) + F^2\big(W\big(FG(2Kb_1-Pc_1) - 2FHKa_2 + 2K^2(Fe_3+Ge_1\\
&-H(b_2+e_2)\big) + KP(-Fb_3+Hc_2) + S(Gc_1 - P(c_{30}+2f_3))\big) + c_{31}(2FG-S)\big)\\
&-SWF(2GPf_1 + H(Kc_2-2Pf_2)))) - (\cos(\theta)\sin(\theta)\sqrt{FA}r^2(F^2(2Fc_1d_3+2Kc_2e_3+Pc_{30}^2\\
&+G(F-c_1(b_2+c_{30}-2d_1)) - H(a_2c_2+2c_1d_2)) + KF(G(b_1c_1+2c_2e_1) - H(a_2c_1+(2b_2+c_{30}+2e_2)c_2))\\
&+S\big(2Fc_{30}f_3 - b_3(a_2c_1^2 - b_1c_2^2 + c_1c_2(2b_2+c_{30})\big) + c_{30}(b_1(a_2c_1+2c_2c_{30}) + b_2c_1(b_2-c_{30})))\\
&+S\big(\big(c_1^2+2c_{30}f_1\big)G - H(c_2^2+2c_{30}f_2)\big)\big)\big)\big)\Big) = F_2(\theta,r,W). \qquad (56)
\end{aligned}
$$

We shall apply the averaging theory described in Theorem 2 to the differential system (55)–(56). Using the notation of section 2 we have $t=\theta$, $T=2\pi$, $\mathbf{x}=(r,W)^T$ and

$$
F(\theta,r,W) = \begin{pmatrix} F_1(,r,W) \\ F_2(,r,W) \end{pmatrix} \quad \textit{and} \quad f(r,W) = \begin{pmatrix} f_1(r,W) \\ f_2(r,W) \end{pmatrix}.
$$

It is immediate to check that system (55)–(56) satisfies all the assumptions of Theorem 2. Now we compute the integrals (5), i.e.

$$
f_1(\theta,r,W) = \frac{rF\big(T_8W - N_8\big)}{2AE\sqrt{FA}}, \quad f_2(r,W) = -\frac{\big(D_8W^2 + R_8r^2 + C_8W\big)}{2FA\sqrt{FA}}.
$$

The system $f_1(r,W) = f_2(r,W) = 0$ has a unique solution (r^*,W^*) with $r^*>0$, namely

$$
(r^*,W^*) = \left(\frac{1}{T_8}\sqrt{\frac{D_8N_8^2 - C_8N_8T_8}{R_8}},\ \frac{N_8}{T_8}\right),
$$

if $T_8 > 0$, *and* $R_8(D_8N_8^2 - C_8N_8T_8) > 0$, *and the Jacobian (6) at* (r^*, W^*) *takes the value* $-N_8(C_8T_8 + D_8N_8)/2\,T_8A^3FE \neq 0$ *where*

$$\begin{aligned}
T_8 = {}& 2F^3\big(Ed_3 + (b_2 + c_{30} - d_1)L + Md_2\big) - F^2\big(2K(Lb_1 - Ma_2) - LPc_1 + A(K - a_2c_2 - 2c_1d_2)\big) + F\big(\\
& E\big(2K^2e_3 - P^2c_{30}\big) - K^2(2Le_1 - 2Mb_2 - 2Me_2 - A) - K(MPc_2 - A((b_2 + c_{30} - 2d_1)c_1 + c_2(b_2+\\
& 2e_2))) - PA\big(c_2^2 + 2c_{30}f_2\big) - LSc_1\big) - KA\big(K(b_1c_1 + 2c_2e_1) - P\big(c_1^2 + 2c_{30}f_1\big)\big) - S(EP(c_{30} + 2f_3)-\\
& KMc_2 - 2P(Lf_1 - Mf_2)),\\
R_8 = {}& F^3K\big(Gb_1 - Ha_2 + Ke_3\big) - F\big(HKSc_2 - S^2(c_{30} + f_3) - A\big(G\big(c_1^2d_1 + c_2^2e_1 + c_{30}^2f_1\big) - H\big(c_1^2d_2 + c_2^2e_2\\
& +c_{30}^2f_2 + Kc_2\big) - c_1c_{30}(Kb_3 - Pc_{30})\big)\big) + F^2\big(K^2(Ge_1 - Hb_2 - He_2) + \big(b_3c_2c_{30} + c_{30}^3 + Gc_1 + c_1^2d_3\\
& +c_2^2e_3 + c_{30}^2f_3\big)A + S(Gc_1 + Pc_{30})\big) + F^5d_3 + S^2(Gf_1 - Hf_2) - F^4\big((b_2 + c_{30} - d_1)G + Hd_2\big),\\
D_8 = {}& 2F^5d_3 - 2F^4\big((b_2 + c_{30} - d_1)G + Hd_2\big) + 2F^3\big(K\big(Gb_1 - Ha_2 + Ke_3\big) - P\big(Gc_1 - Pf_3\big)\big) + 2F^2\big(G\big(\\
& K^2e_1 + P^2f_1\big) - HK^2(b_2 + e_2) + HP(Kc_2 - Pf_2)\big),\\
N_8 = {}& c_{31}\big(EF(G + P) + 2F^2L + 2KAc_1 + ES\big),\\
C_8 = {}& 2F^3Gc_{31}.
\end{aligned}$$

Finally, we apply Theorem 2 like in the first case (i), then for $\varepsilon > 0$ *sufficiently small system (54) has a periodic solution* $(x(\theta), y(\theta), z(\theta)) = (\varepsilon X(\theta), \varepsilon Y(\theta), \varepsilon Z(\theta))$ *which tends to the origin of coordinates when* $\varepsilon \to 0$. *Therefore, it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when* $\varepsilon = 0$. *This completes the proof of Theorem 1 under the condition (viii).*

Example 8 *Consider the Kolmogorov system*

$$\left\{\begin{aligned}
\dot{x} = {}& x\big(x - 10 + 6y + 3z\big),\\
\dot{y} = {}& y\left(-\frac{1}{3}x + \frac{4}{3} - 2y + z - 2(z-1)^2\right),\\
\dot{z} = {}& z\big(x - 1 - y + z - (x-1)^2 - 3(y-1)^2 - (z-1)^2\big).
\end{aligned}\right. \tag{57}$$

This system in the new variables (X, Y, Z) *writes*

$$\left\{\begin{aligned}
\dot{X} = {}& (X+1)\big(-2X\varepsilon + X + 6Y + 3Z\big),\\
\dot{Y} = {}& (Y+1)\left(-2Z^2 - \frac{1}{3}X - 2Y + Z\right),\\
\dot{Z} = {}& (Z+1)\left(\left(-\frac{1}{12}X + Z\right)\varepsilon - \frac{1}{6}X - Y + Z - X^2 - 3Y^2 - Z^2\right).
\end{aligned}\right.$$

The corresponding system associated to system (55)–(56) satisfies

$$\begin{aligned}
F_1(\theta, r, W) = {}& -\frac{913}{6}\cos^2(\theta)\sin(\theta)r^2 + \frac{464}{3}\sin(\theta)\cos(\theta)rW + \frac{824}{3}\sqrt{2}\cos^3(\theta)r^2 + \frac{49}{6}\sin(\theta)\cos(\theta)r\\
& -\frac{4}{3}r^2\sin(\theta) - \frac{1808}{3}\sqrt{2}\cos^2(\theta)Wr - \frac{16}{3}\sin(\theta)W - \frac{5}{3}\sqrt{2}\cos^2(\theta)r + 312\sqrt{2}\cos(\theta)W^2\\
& +34\sqrt{2}\cos(\theta)r^2 + \frac{10}{3}\sqrt{2}\cos(\theta)W - \frac{56}{3}\sqrt{2}Wr - \frac{2}{3}\sqrt{2}r,\\
F_2(\theta, r, W) = {}& -\frac{1}{864}\big(6\sin(\theta)\sqrt{2}r(-10800W - 180) - 269568W^2 - 1728W + r^2(-267624\cos^2(\theta)\\
& -11664\sin^2(\theta)) - \cos(\theta)r(-537840W - 864) + 64800\sin(\theta)\cos(\theta)\sqrt{2}r^2\big)\sqrt{2}.
\end{aligned}$$

To look for the limit cycles we must solve the system given by the averaged function $f(r,W) = (f_1(r,W), f_2(r,W)) = (0,0)$ *where*

$$f_1(r,W) = -\frac{1}{216}r\big(69120W + 324\big)\sqrt{2}, \quad \textit{and} \quad f_2(r,W) = -\frac{1}{1728}\big(-539136W^2 - 279288r^2 - 3456W\big)\sqrt{2}. \tag{58}$$

This system has four solutions for (r,W) *given by*

$$(0,0),\ \left(0, -\frac{1}{156}\right),\ \left(\frac{\sqrt{18533}}{34480}, \frac{-3}{640}\right),\ \left(-\frac{\sqrt{18533}}{34480}, \frac{-3}{640}\right).$$

As in Example 1, we have two good solutions, the $(0,-1/156)$ *and the* $(\sqrt{18533}/34480, -3/640)$*. Since the determinants (6) at these two solutions are* $-86/39$ *and* $129/40$ *and thus non-zero, respectively, the Kolmogorov system (57) has two limit cycles bifurcating from the origin provided by the averaging theory of first order. We plot these two limit cycles for* $\varepsilon = 10^{-5}$ *in Figure 8.*

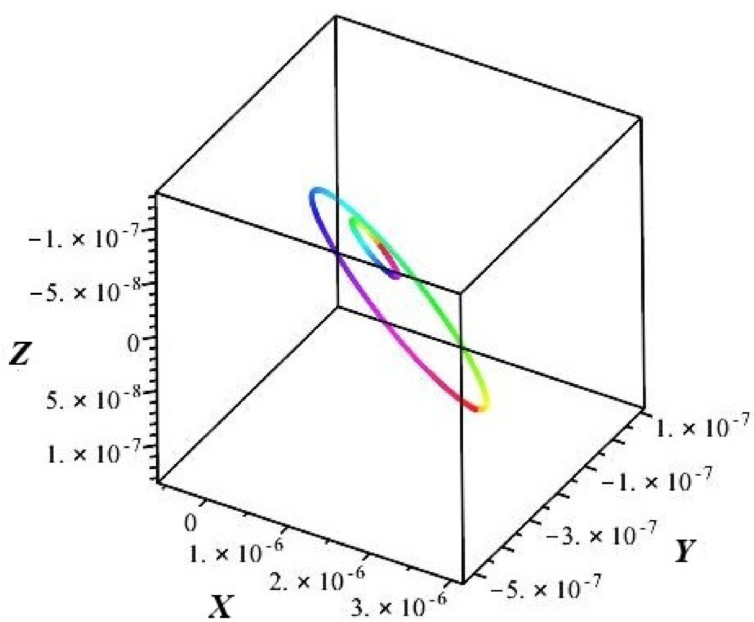

Figure 8: 1st LC: $X(0) = 7\varepsilon/78,\ Y(0) = 0,\ Z(0) = \varepsilon/78$. 2nd LC: $X(0) = \dfrac{(24\sqrt{18533} + 3879)\varepsilon}{68960},\ Y(0) = -\dfrac{(8\sqrt{18533} + 1293)\varepsilon}{137920},\ Z(0) = -\dfrac{\varepsilon\sqrt{18533}}{68960}$.

Since the eigenvalues of the Jacobian matrix of (f_1, f_2) *at the singular points*

$$(0,-1/156) \ \ \textit{and} \ \ (\frac{\sqrt{18533}}{34480}, -3/640)$$

are $(-2\sqrt{2}, 43\sqrt{2}/78)$ *and* $(-37\sqrt{2} \pm i\sqrt{17902})/80$*, respectively, by Theorem 2 the limit cycles are unstable and stable. Going back through the changes of variables as we did in the proof of Theorem 1 we obtain that* *the limit cycles bifurcating from the equilibrium point* $(1,1,1)$ *of system (57) are*

$$(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon)) = (1+\varepsilon/13+O(\varepsilon^2), 1-\varepsilon/78+O(\varepsilon^2), 1+O(\varepsilon^2)),$$

$$\begin{aligned}(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon)) = \Bigg(&1-\frac{3\left(\sqrt{37066}\sin(\sqrt{2}t/2)-8\sqrt{18533}\cos(\sqrt{2}t/2)-1293\right)\varepsilon}{68960}+O(\varepsilon^2),\\ &1-\frac{\varepsilon(2\sqrt{37066}\sin(\sqrt{2}t/2)+8\sqrt{18533}\cos(\sqrt{2}t/2)+1293)}{137920}+O(\varepsilon^2),\\ &1-\frac{\sqrt{18533}(\sqrt{2}\sin(\sqrt{2}t/2)+\cos(\sqrt{2}t/2))\varepsilon}{68960}+O(\varepsilon^2)\Bigg),\end{aligned}$$

respectively. This completes Example 8.

Acknowledgment. The second author is partially supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00, the H2020 European Research Council grant MSCA-RISE-2017-777911, AGAUR (Generalitat de Catalunya) grant 2021SGR00113, and by the Acadèmia de Ciències i Arts de Barcelona.

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