Applied Mathematics E-Notes, 24(2024), 97-104 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

A Note On Positive Linear Operators Involving 2-Variable Hermite-Kampé De Fériet Polynomials*

Mahvish Ali†, Pradeep Malik‡

Received 23 February 2023

Abstract

The purpose of this paper is to establish positive linear operators including the Hermite-Kampé de Fériet polynomials of two variables. The universal Korovkin-type condition is employed to validate the convergence properties of these operators, and the classical modulus of continuity is applied to compute the order of approximation. The convergence properties of these operators are also explored in weighted spaces of functions on the positive semi-axis as well as the approximation is estimated using the weighted modulus of continuity. Furthermore, numerical and graphical examples are provided to show the convergence behavior, effectiveness and accuracy of the proposed operators.

1 Introduction

So far, the notion of generalised special functions has seen a major evolution. This new development in the theory of special functions provides an analytic foundation for the vast majority of problems in mathematical physics that have been precisely solved and have broad practical applications.

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP)$H_n(x,a)$ are defined as [5]:

$$H_k(x,a) = k! \sum_{r=0}^{[\frac{k}{2}]} \frac{x^{k-2r}\ a^r}{r!\ (k-2r)!} \tag{1}$$

and

$$\exp(xt + at^2) = \sum_{k=0}^{\infty} H_k(x,a) \frac{t^k}{k!}.$$

These polynomials are solutions of the heat equation:

$$\frac{\partial}{\partial a} H_k(x,a) = \frac{\partial^2}{\partial x^2} H_k(x,a),$$

$$H_k(x,0) = x^k.$$

There exists the following relationship:

$$H_k(x,a) = (-i)^k a^{k/2} H_k\left(\frac{ix}{2\sqrt{a}}\right),$$

of the 2VHKdFP $H_k(x,a)$ with the classical Hermite polynomials $H_k(x)$. Also, it is to be noted that

$$H_k(2x,-1) = H_k(x).$$

*Mathematics Subject Classifications: 41A10, 41A25, 41A36, 33C45, 33E20.

†Department of Mathematics, Faculty of Science, Shree Guru Gobind Singh Tricentenary University, Gurugram, Haryana-122505, India

‡Department of Mathematics, Shree Guru Gobind Singh Tricentenary University, Gurugram, Haryana-122505, India

Positive approximation processes are crucial to approximation theory and frequently occur naturally in problems involving the approximation of continuous functions. In recent years, there has been a surge of interest in the study of linear positive operators based on specific polynomials see [1, 3, 2, 7, 9]. Some approximation properties of a new class of Bernstein polynomials and Stancu-Kantorovich variant of Szász-Mirakjan operators based on Bézier basis functions with shape parameter have been studied in [8, 6]. Also, in [4] the authors studied point-wise and weighted approximation properties of Stancu variant of λ-Schurer operators.

In this article, we construct the Kantorovich operators involving the 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP)$H_n(x, a)$ defined as:

$$\mathbb{K}_{n,H}(f;x) = ne^{-nx-a}\sum_{k=0}^{\infty}\frac{H_k(nx;a)}{k!}\int_{\frac{k}{n}}^{\frac{(k+1)}{n}} f(s)ds, \qquad x \in [0,\infty). \tag{2}$$

The remainder of the paper is structured as follows: In the following section some preliminary results for the operators $\mathbb{K}_{n,H}(f;x)$ are obtained. In section 3, approximation properties of the operators $\mathbb{K}_{n,H}(f;x)$ and the convergence theorem are discussed. Section 4 investigates the convergence of these operators in weighted spaces of functions and estimate the approximation with the help of weighted modulus of continuity. Some numerical examples are provided in the final part to demonstrate the validity of the observations.

2 Preliminary Results

Several noteworthy observations about the convergence of sequences of positive linear operators $(L_n(f))_{n=1}^{\infty}$ was discovered by P. P. Korovkin [13]. As an instance, if $L_n(f, x)$ converges uniformly to f in the specific cases $f = 1,\ x,\ x^2$, then it converges uniformly to f for any continuous real function f.

Let us review the definitions: $C_B[0,\infty)$ denotes the space of all bounded and continuous functions defined on $[0,\infty)$ equipped with the norm $\|f\| = \sup_{x\in[0,\infty)}|f(x)|$.

Definition 1 *Let $f \in C_B[0,\infty)$ and $\delta > 0$. The modulus of continuity $w(f;\delta)$ of the function f is defined by*

$$w(f;\delta) := \sup_{\substack{x,y\in[0,\infty)\\ |x-y|\le\delta}} |f(x) - f(y)|. \tag{3}$$

Then, for any $\delta > 0$ and each $x \in [0,\infty)$, it is well known that one can write

$$|f(x) - f(y)| \le w(f;\delta)\left(\frac{|x-y|}{\delta} + 1\right), \quad \delta > 0. \tag{4}$$

If f is uniformly continuous on $[0,\infty)$, then it is necessary and sufficient that

$$\lim_{\delta\to 0} w(f,\delta) = 0.$$

Definition 2 *The second modulus of continuity of the function $f \in C_B[0,\infty)$ is defined by*

$$w_2(f;\delta) := \sup_{0<t\le\delta} \|f(.+2t) - 2f(.+t) + f(.)\|_{C_B}.$$

We denote by $C_E[0,\infty)$ the set of all continuous functions f on$[0,\infty)$ with the property that $|f(x)| \le \beta e^{\alpha x}$ for all $x \ge 0$ and some positive finite α and β.

For $h > 0$, the function

$$f_h(t) = \frac{1}{h}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} f(u)\ du$$

as well as the iteratively defined functions

$$f_{h,r}(t) = \frac{1}{h} \int\limits_{t-\frac{h}{2}}^{t+\frac{h}{2}} f_{h,r-1}(u)\ du, \quad r = 2, 3, \cdots$$

$$f_{h,1}(t) = f_h(t)$$

are known as Steklov functions.

Consider the following Lemmas:

Lemma 1 (Gavrea and Raşa [12]) *Let $g \in C^2[0,a]$ and $(K_n)_{n\geq 0}$ be a sequence of positive linear operators with the property $K_n(1;x) = 1$. Then,*

$$|K_n(g;x) - g(x)| \leq ||g'||\sqrt{K_n((s-x)^2;x)} + \frac{1}{2}||g''||\ K_n((s-x)^2;x).$$

Lemma 2 (Zhuk [14]) *Let $f \in C[a,b]$, $h \in (0, \frac{a-b}{2})$ and f_h be the second-order Steklov function attached to the function f. Then, the following inequalities are satisfied:*

$$(i)\ \ \|f_h - f\| \leq \frac{3}{4} w_2(f;h), \tag{5}$$

$$(ii)\ \ \|f_h''\| \leq \frac{3}{2h^2} w_2(f;h). \tag{6}$$

In the following lemma with the help of equation (1), one can obtain some moment estimates of operators $\mathbb{K}_{n,H}$ easily.

Lemma 3 *The operators $\mathbb{K}_{n,H}(f;x)$ satisfy*

$$\mathbb{K}_{n,H}(1;x) = 1, \tag{7}$$

$$\mathbb{K}_{n,H}(s;x) = x + \frac{1}{n}\left(2a + \frac{1}{2}\right), \tag{8}$$

$$\mathbb{K}_{n,H}(s^2;x) = x^2 + \frac{x}{n}(2+4a) + \frac{1}{n^2}\left(\frac{1}{3} + 4a^2 + 6a\right), \tag{9}$$

$$\mathbb{K}_{n,H}(s^3;x) = x^3 + \frac{x^2}{n}\left(\frac{9}{2} + 6a\right) + \frac{x}{n^2}\left(\frac{7}{2} + 12a^2 + 24a\right) + \frac{1}{n^3}\left(\frac{1}{4} + 8a^3 + 30a^2 + 16a\right),$$

$$\begin{aligned}\mathbb{K}_{n,H}(s^4;x) \quad = \quad & x^4 + \frac{x^3}{n}(8a+8) + \frac{x^2}{n^2}\left(24a^2 + 60a + 15\right) + \frac{x}{n^3}\left(32a^3 + 144a^2 + 114a + 11\right) \\ & + \frac{1}{n^4}\left(\frac{1}{5} + 16a^4 + 112a^3 + 168a^2 + 42a\right).\end{aligned}$$

By using the linearity property of the operators (2) and Lemma 3, it follows that:

$$\mathbb{K}_{n,H}((s-x);x) = \frac{1}{n}\left(\frac{1}{2} + 2a\right), \tag{10}$$

$$\mathbb{K}_{n,H}((s-x)^2;x) = \frac{x}{n} + \frac{1}{n^2}\left(\frac{1}{3} + 4a^2 + 6a\right), \tag{11}$$

$$\mathbb{K}_{n,H}((s-x)^4;x) = 3\frac{x^2}{n^2} + \frac{x}{n^3}(24a^2 + 50a + 10) + \frac{1}{n^4}\left(\frac{1}{5} + 16a^4 + 112a^3 + 168a^2 + 42a\right).$$

3 Approximation Properties of $\mathbb{K}_{n,H}$

The present section deals with the convergence properties of $\mathbb{K}_{n,H}$ operators.

Theorem 1 *Let $f \in C_B[0,\infty)$. Then*

$$\lim_{n\to\infty} \mathbb{K}_{n,H}(f) = f \tag{12}$$

uniformly on each compact subset of $[0,\infty)$.

Proof. In view of Lemma 3, it follows that

$$\lim_{n\to\infty} \mathbb{K}_{n,H}(s^i, x) = x^i, \quad i = 0,1,2,3,4 \tag{13}$$

uniformly on each compact subset of $[0,\infty)$. Using (13) and Korovkin theorem, (12) follows. ■

Next, we obtain the order of approximation of the operators $\mathbb{K}_{n,H}$.

Theorem 2 *Let $f \in C_B[0,\infty)$. Then the operators $\mathbb{K}_{n,H}$ satisfy the following inequality:*

$$|\mathbb{K}_{n,H}(f;x) - f(x)| \leqslant \left(1 + \sqrt{x + \frac{1}{n}\left(\frac{1}{3} + 4a^2 + 6a\right)}\right) w\left(f; \frac{1}{\sqrt{n}}\right). \tag{14}$$

Proof. In view of the modulus of continuity (4), we have

$$\begin{aligned} |\mathbb{K}_{n,H}(f;x) - f(x)| &\leqslant ne^{-nx-a} \sum_{k=0}^{\infty} \frac{H_k(nx;a)}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(s) - f(x)|\, ds \\ &\leqslant w(f;\delta) \left\{1 + \frac{1}{\delta} ne^{-nx-a} \sum_{k=0}^{\infty} \frac{H_k(nx;a)}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |s - x|\, ds \right\}. \end{aligned} \tag{15}$$

By Cauchy-Schwarz inequality, the above inequality becomes

$$\begin{aligned} |\mathbb{K}_{n,H}(f;x) - f(x)| &\leqslant w(f;\delta) \left\{1 + \frac{1}{\delta}\left(ne^{-nx-a} \sum_{k=0}^{\infty} \frac{H_k(nx;a)}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (s-x)^2 \right)^{\frac{1}{2}} \right\} \\ &= w(f;\delta) \left\{1 + \frac{1}{\delta}\sqrt{\mathbb{K}_{n,H}((s-x)^2;x)}\right\}. \end{aligned}$$

On choosing $\delta = \frac{1}{\sqrt{n}}$, assertion (14) follows. ■

Theorem 3 *The following inequality holds true:*

$$|\mathbb{K}_{n,H}(f;x) - f(x)| \leq \frac{2}{\alpha}||f||\; l^2 + \frac{3}{4}(\alpha + 2 + l^2) w_2(f;l), \tag{16}$$

where

$$f \in C[0,\alpha], \quad l := l_n(x) = \sqrt[4]{\mathbb{K}_{n,H}((s-x)^2;x)}$$

and the second-order modulus of continuity is given by $w_2(f;l)$ with the norm $\|f\| = \max_{x\in[a,b]} |f(x)|$.

Proof. Denote the second-order Steklov function attached to the function f by f_l. Using equation (7), we get

$$\begin{aligned} |\mathbb{K}_{n,H}(f;x) - f(x)| &\leq |\mathbb{K}_{n,H}(f - f_l;x)| + |\mathbb{K}_{n,H}(f_l;x) - f_l(x)| + |f_l(x) - f(x)| \\ &\leq 2\,\|f_l - f\| + |\mathbb{K}_{n,H}(f_l;x) - f_l(x)|\,, \end{aligned}$$

which on using inequality (5) becomes

$$|\mathbb{K}_{n,H}(f;x) - f(x)| \leq \frac{3}{2}w_2(f;l) + |\mathbb{K}_{n,H}(f_l;x) - f_l(x)|. \tag{17}$$

Taking into account that $f_l \in C^2[0,\alpha]$, from Lemma 1, it follows that

$$|\mathbb{K}_{n,H}(f_l;x) - f_l(x)| \leq ||f_l'||\sqrt{\mathbb{K}_{n,H}((s-x)^2;x)} + \frac{1}{2}\ ||f_l''||\ \mathbb{K}_{n,H}((s-x)^2;x),$$

which in view of inequality (6) becomes

$$|\mathbb{K}_{n,H}(f_l;x) - f_l(x)| \leq ||f_l'||\sqrt{\mathbb{K}_{n,H}((s-x)^2;x)} + \frac{3}{4l^2}w_2(f;l)\mathbb{K}_{n,H}((s-x)^2;x). \tag{18}$$

Further, the Landau inequality

$$||f_l'|| \leq \frac{2}{\alpha}||f_l|| + \frac{\alpha}{2}||f_l''||,$$

and Lemma 2 gives

$$||f_l'|| \leq \frac{2}{\alpha}||f|| + \frac{3\alpha}{4l^2}w_2(f;l). \tag{19}$$

Using inequality (19) in inequality (18) and taking $l = \sqrt[4]{\mathbb{K}_{n,H}((s-x)^2);x}$, we find

$$|\mathbb{K}_{n,H}(f_l;x) - f_l(x)| \leq \frac{2}{\alpha}||f||\ l^2 + \frac{3}{4}(\alpha + l^2)w_2(f;l). \tag{20}$$

Use of the inequality (20) in (17), then establishes the assertion (16). ■

4 Weighted Approximation

In this section, the approximation properties of the operators $\mathbb{K}_{n,H}(f;x)$ on the weighted spaces of continuous functions are given with exponential growth on $\mathbb{R}_0^+ = [0,\infty)$ with the help of the weighted Korovkin type theorem established in [10, 11]. First, we consider the following weighted spaces of functions which are defined on the $[0,\infty)$.

Let $\rho(x)$ be the weight function and M_f be a positive constant, then we define

$$\begin{aligned} B_\rho(\mathbb{R}_0^+) &= \{f : \mathbb{R}_0^+ \to \mathbb{R} : |f(x)| < M_f\rho(x)\}, \\ C_\rho(\mathbb{R}_0^+) &= \{f \in B_\rho(\mathbb{R}_0^+) : f \text{ is continuous}\}, \\ C_\rho^k(\mathbb{R}_0^+) &= \left\{f \in C_\rho(\mathbb{R}_0^+) : \lim_{n\to\infty}\frac{f(x)}{\rho(x)} = K_f < \infty\right\}. \end{aligned}$$

It should be noted that $C_\rho^k(\mathbb{R}_0^+) \subset C_\rho(\mathbb{R}_0^+) \subset B_\rho(\mathbb{R}_0^+)$. The space $B_\rho(\mathbb{R}_0^+)$ is a linear normed space with the following norm:

$$||f||_\rho = \sup_{x\in\mathbb{R}_0^+}\frac{|f(x)|}{\rho(x)}.$$

The following results on the sequence of positive linear operators in these spaces are given [10, 11].

Lemma 4 ([10, 11]) *The sequence of positive linear operators $(L_n)_{n\geq 1}$ which act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$ if and only if there exists a positive constant k such that*

$$L_n(\rho, x) \leq k\rho(x), \text{ i.e.,}$$

$$||L_n(\rho, x)||_\rho \leq k.$$

Theorem 4 ([10, 11]) *Let $(L_n)_{n\geq 1}$ be the sequence of positive linear operators that act from $C_\rho(\mathbb{R}_0^+)$ to $B_\rho(\mathbb{R}_0^+)$ such that*

$$\lim_{n\to\infty} ||L_n(t^i, x) - x^i||_\rho = 0, \quad i \in \{0, 1, 2\}.$$

Then for any $f \in C_\rho^k(\mathbb{R}_0^+)$

$$||L_n f - f||_\rho = 0.$$

Now, we will prove the following lemma:

Lemma 5 *Let $\rho(x) = 1 + x^2$ be a weight function. If $f \in C_\rho^k(\mathbb{R}_0^+)$, there exists a positive constant M such that $\|\mathbb{K}_{n,H}(\rho; x)\|_\rho \leq 1 + M$.*

Proof. From Lemma 3, it follows that

$$\begin{aligned} ||\mathbb{K}_{n,H}(\rho; x)||_\rho &= \sup_{x\geq 0}\left\{\frac{1}{1+x^2}\left(1 + x^2 + \frac{x}{n}(2+4a) + \frac{1}{n^2}\left(\frac{1}{3} + 4a^2 + 6a\right)\right)\right\} \\ &\leqslant 1 + \frac{x}{n}(2+4a) + \frac{1}{n^2}\left(\frac{1}{3} + 4a^2 + 6a\right). \end{aligned}$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n^2} = 0$, there exists a positive constant M such that

$$\mathbb{K}_{n,H}(\rho; x) \leqslant 1 + M.$$

This completes the proof. ■

Theorem 5 *Let $\rho(x) = 1 + x^2$. Then, for each $f \in C_\rho^k(\mathbb{R}_0^+)$,*

$$\lim_{n\to\infty} \|\mathbb{K}_{n,H}(f; x) - f(x)\|_\rho = 0.$$

Proof. It is sufficient to show the conditions of the weighted form of Korovkin type approximation theorem proved in Theorem 4, are satisfied. From (7), it is immediately seen that

$$\lim_{n\to\infty} \|\mathbb{K}_{n,H}(e_0; x) - e_0\|_\rho = 0. \tag{21}$$

Using (8), we have

$$||\mathbb{K}_{n,H}(e_1; x) - e_1||_\rho = \frac{1}{n}\left(2a + \frac{1}{2}\right),$$

which implies that

$$\lim_{n\to\infty} \|\mathbb{K}_{n,H}(e_1; x) - e_1(x)\|_\rho = 0. \tag{22}$$

By means of (9), we get

$$\|\mathbb{K}_{n,H}(e_2; x) - e_2(x)\|_\rho \leqslant \frac{1}{n^2}\left(\frac{1}{3} + 4a^2 + 6a\right).$$

Therefore,

$$\lim_{n\to\infty} \|\mathbb{K}_{n,H}(e_2; x) - e_2(x)\|_\rho = 0. \tag{23}$$

From equations (21)–(23), it follows that

$$\lim_{n\to\infty} ||\mathbb{K}_{n,H}(e_\nu; x) - e_\nu(x)||_\rho = 0, \quad \nu \in \{0, 1, 2\}.$$

In view of Theorem 4, we obtain the desired result. ■

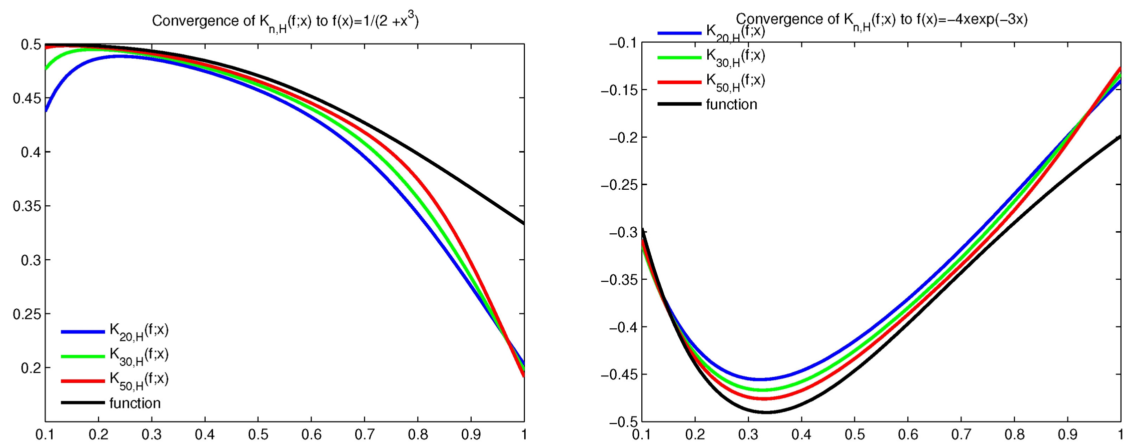

Figure 1: The convergence of operator (2) to $f(x)$.

5 Numerical Results

The numerical outcomes are presented in this section. Making use of the expression (11) in Lemma 1, we find that

$$
\begin{aligned}
|\mathbb{K}_{n,H}(f;x) - f(x)| \quad &\leq \quad ||f'|| \sqrt{\frac{x}{n} + \frac{1}{n^2}\left(\frac{1}{3} + 4a^2 + 6a\right)} \\
&+ \frac{1}{2}||f''|| \left(\frac{x}{n} + \frac{1}{n^2}\left(\frac{1}{3} + 4a^2 + 6a\right)\right). \qquad (24)
\end{aligned}
$$

Taking $a = 0.1$, the absolute error $|\mathbb{K}_{n,H}(f;x) - f(x)|$ of the operators (2) with the function $f(x) = \frac{1}{2+x^3}$ and $f(x) = -4xe^{-3x}$ are represented in Table 1 and Table 2, respectively.

n	error bound at $x = 0.2$	error bound at $x = 0.4$	error bound at $x = 0.6$	error bound at $x = 0.8$
20	0.0087	0.0275	0.0552	0.0830
30	0.0061	0.0205	0.0422	0.0642
50	0.0040	0.0144	0.0306	0.0470

Table 1

n	error bound at $x = 0.2$	error bound at $x = 0.4$	error bound at $x = 0.6$	error bound at $x = 0.8$
20	0.1141	0.0496	0.1107	0.1252
30	0.0856	0.0372	0.0870	0.0982
50	0.0620	0.0266	0.0650	0.0731

Table 2

For $n = 20, 30, 50$ and $a = 0.1$, the convergence of operator (2) to the functions

$$f(x) = \frac{1}{2 + x^3} \quad \text{and} \quad f(x) = -4xe^{-3x}$$

are demonstrated in Figure 1, respectively.

6 Conclusions

In this study, the positive linear operators involving the 2-variable Hermite-Kampé de Fériet polynomials are introduced. The convergence theorem and rate of convergence of these operators are also established. The impact of different choices of n on the absolute error of approximation are presented. Both numerical and pictorial findings show that the suggested operators have good convergence behaviour as n increases in value.

Acknowledgment. The authors are thankful to the reviewer(s) for several useful comments and suggestions towards the improvement of this paper.

References

[1] T. Acar, A. Aral and S. A. Mohiuddine, Approximation by bivariate (p, q)-Bernstein-Kantorovich operators, Iran. J. Sci. Technol. Trans. Sci., 42(2018), 655–662.

[2] M. Ali and U. Kadak, Approximation by Szász-Chlodowsky type operators associating 2D-Appell polynomials, Z. Angew. Math. Mech., 102(2022), 17 pp.

[3] M. Ali and R. B. Paris, Generalization of Szász operators involving multiple Sheffer polynomials, J. Anal., 31(2022), 1–19.

[4] K. J. Ansari, F. Özger and Z. Ö. Özger, Numerical and theoretical approximation results for Schurer-Stancu operators with shape parameter λ, Comp. Appl. Math., 41(2022), 18 pp.

[5] P. Appell, J. Kampé de Fériet, Fonctions Hypergéométriques et Hypersphériques: Polynômes d' Hermite, Gauthier-Villars, Paris, 1926.

[6] R. Aslan, On a Stancu form Szász-Mirakjan-Kantorovich operators based on shape parameter λ, Adv. Studies: Euro-Tbilisi Math. J., 15(2022), 151–166.

[7] R. Aslan, Approximation properties of univariate and bivariate new class λ-Bernstein-Kantorovich operators and its associated GBS operators, Comput. Appl. Math., 42(2023), 22 pp.

[8] R. Aslan and M. Mursaleen, Some approximation results on a class of new type λ-Bernstein polynomials, J. Math. Inequalities, 16(2022), 445–462.

[9] N. L. Braha and U. Kadak, Approximation properties of the generalized Szász operators by multiple Appell polynomials via power summability method, Math. Methods Appl. Sci., 43(2020), 2337–2356.

[10] A. D. Gadjiev, The convergence problem for a sequence of positive linear operators on bounded sets and theorems analogous to that of P. P. Korovkin, Dokl. Akad. Nauk SSSR, 218(1974); Transl. in Soviet Math. Dokl., 15(1974), 1433–1436.

[11] A. D. Gadjiev, On P. P. Korovkin type theorems, Mat. Zametki, 20(1976), 781–786; Transl. in Math. Notes, (5-6)(1978), 995–998.

[12] I. Gavrea and I. Raşa, Remarks on some quantitative Korovkin-type results, Rev. Anal. Numér. Théor. Approx., 22(1993), 173–176.

[13] P. P. Korovkin, On convergence of linear positive operators in the space of continous functions, Dokl. Akad. Nauk SSSR, 90(1953), 961–964.

[14] V. V. Zhuk, Functions of the Lip 1 class and S. N. Bernstein's polynomials, Vestnik Leningrad. Univ. Mat. Mekh. Astronom., 1(1989), 25–30.