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Higher-Order Topological Asymptotic Formula For The Elasticity Operator And Application*

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Abstract

This paper is concerned with a geometric inverse problem related to the elasticity equation. We aim to identify an unknown hole from boundary measurements of the displacement field. The Kohn-Vogelius concept is employed for formulating the inverse problem as a topology optimization one. We develop a topological sensitivity analysis based method for detecting the location, size and shape of the unknown hole. We derive a higher-order asymptotic formula describing the variation of a Kohn-Vogelius type functional with respect to the creation of an arbitrary shaped hole inside the computational domain.

1 Introduction

Let Ω be a bounded and smooth domain of $\mathbb{R}^3$, occupied by a material body of elastic nature. In this study, we assume that the total boundary $\partial\Omega$ is decomposed into two disjoint parts Γ and Σ such that $\overline{\partial\Omega} = \overline{\Gamma} \cup \overline{\Sigma}$ and $\Gamma \cap \Sigma = \emptyset$. Also, we assume that the displacement field w in Ω satisfies the following elasticity system:

$$\begin{cases} -\operatorname{div}\sigma(w) &= F \quad \text{in } \Omega, \\ \sigma(w)n &= G \quad \text{on } \Sigma, \\ u &= 0 \quad \text{on } \Gamma, \end{cases} \tag{1}$$

where F is a given body force, G is an imposed external force and n denotes the outward normal to the boundary Γ. Here, we recall that $w \mapsto \sigma(w)$ represents the stress tensor which is given by

$$\sigma_{ij}(w) = \lambda \operatorname{div}(w)\, \delta_{ij} + 2\mu\varepsilon_{ij}(w),\ 1 \leq i, j \leq 3$$

with δ_{ij} is the Kronecker symbol and $w \mapsto \varepsilon(w)$ is the strain tensor, defined as

$$\varepsilon_{ij}(w) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right), \quad 1 \leq i, j \leq 3.$$

Inside the structure domain Ω, we assume the existence of a small hole $\mathcal{O}_{z,\varepsilon} = z + \varepsilon\mathcal{O}$ that is characterized by its center z, its size ε and its shape $\mathcal{O}$, with $\mathcal{O}$ is a bounded domain of $\mathbb{R}^3$ containing the origin, whose boundary $\partial\mathcal{O}$ is connected and piecewise of class $\mathcal{C}^1$.

The aim of this work is to develop an efficient approach for identifying the unknown parameters (location, size and shape) of the hole $\mathcal{O}_{z,\varepsilon}$ from boundary measurement of the displacement field on the boundary Σ. This issue can be formulated as a geometric inverse problem having the form:

- Given a Neumann and Dirichlet data on the accessible boundary Σ: an imposed force F and a measured displacement field $\mathcal{W}_m$.

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- Determine the location z, size ε and shape $\mathcal{O}$ of the unknown hole $\mathcal{O}_{z,\varepsilon}$ such that the solution $w_{\mathcal{O}_{z,\varepsilon}}$ of the Elasticity equation satisfies the following over-determined boundary value problem

$$\begin{cases} -\operatorname{div}\sigma(w_{\mathcal{O}_{z,\varepsilon}}) & = F \quad \text{in } \Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}, \\ \sigma(w_{\mathcal{O}_{z,\varepsilon}})\mathbf{n} & = G \quad \text{on } \Sigma, \\ w_{\mathcal{O}_{z,\varepsilon}} & = \mathcal{W}_m \quad \text{on } \Sigma, \\ w_{\mathcal{O}_{z,\varepsilon}} & = 0 \quad \text{on } \Gamma, \\ \sigma(w_{\mathcal{O}_{z,\varepsilon}})\mathbf{n} & = 0 \quad \text{on } \partial\mathcal{O}_{z,\varepsilon}. \end{cases}$$

In order to examine the considered geometric inverse problem, we propose in this paper a new approach combining the advantages of the Kohn-Vogelius formulation [8, 14, 15, 32] and the topological sensitivity analysis method.

The topological sensitivity technique is an optimization method used for different applications [16, 17, 18, 19, 20, 21, 22, 23]. The main idea consists on developing of an asymptotic expansion of the objective function in relation to the domain topological perturbation. Many operators has been studied in the case of this method such as, the Laplace operator, the Stokes system, the Helmoltz equations. The majority of the existing works using topological sensitivity method are limited to the first order expansion which is sufficient in the case where the size of the domain to be detected is of infinitesimal size and not close to the boundary. However, In the case where this constraint is not ensured or if the first order term in the asymptotic expansion is equal to zero at some critical points, we need an extension of the expansion to the higher order terms. We present in this work an extension of this concept to the elasticity operator.

2 The Proposed Approach

The first step of our approach is based on the Kohn-Vogelius formulation which rephrase the considered geometric inverse problem into a topology optimization one.

2.1 Kohn-Vogelius Formulation

It leads to define for any given hole $\mathcal{O}_{z,\varepsilon}$ two forward problems: The first one which is called Neumann problem is associated to the Neumann datum F:

$$(\mathcal{P}_\varepsilon^N)\begin{cases} -\operatorname{div}\sigma(w_\varepsilon^N) & = F \quad \text{in } \Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}, \\ \sigma(w_\varepsilon^N)\mathbf{n} & = G \quad \text{on } \Sigma, \\ w_\varepsilon^N & = 0 \quad \text{on } \Gamma, \\ w_\varepsilon^N & = 0 \quad \text{on } \partial\mathcal{O}_{z,\varepsilon}. \end{cases}$$

The second one is associated to the measured displacement $\mathcal{W}_m$ which will be called as the Dirichlet problem:

$$(\mathcal{P}_\varepsilon^D)\begin{cases} -\operatorname{div}\sigma(w_\varepsilon^D) & = F \quad \text{in } \Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}, \\ w_\varepsilon^D & = \mathcal{W}_m \quad \text{on } \Sigma, \\ w_\varepsilon^D & = 0 \quad \text{on } \Gamma, \\ w_\varepsilon^D & = 0 \quad \text{on } \partial\mathcal{O}_{z,\varepsilon}. \end{cases}$$

As one can remark here, if $\mathcal{O}_{z,\varepsilon}$ coincides with the exact hole $\mathcal{O}_{z^*,\varepsilon^*} = z^* + \varepsilon^*\mathcal{O}^*$, the misfit between the solutions become zero ($w_\varepsilon^N = w_\varepsilon^D$). Starting from this remark, the inverse problem can be formulated as a topology optimization one. The unknown hole will be characterized as the minimum of the following Kohn-Vogelius type functional

$$\mathcal{K}(\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}) = \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\sigma(w_\varepsilon^N) - \sigma(w_\varepsilon^D)\right|^2 dx.$$

More precisely, the identification problem can be formulated as follow:

$$(\mathcal{P}_{opt})\left\{\ \text{find } \mathcal{O}_{z^*,\varepsilon^*} \subset \Omega, \text{ such that } \mathcal{K}(\Omega\setminus\overline{\mathcal{O}}_{z^*,\varepsilon^*}) = \min_{\mathcal{O}_{z,\varepsilon}\in\mathcal{D}_{ad}} \mathcal{K}(\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}),\right.$$

where $\mathcal{D}_{ad}$ is a given set of admissible domains.

It is interesting to note that, in the absence of hole the function $\mathcal{K}$ reads

$$\mathcal{K}(\Omega) = \int_\Omega \left|\sigma(w_0^N) - \sigma(w_0^D)\right|^2 dx,$$

where w_0^N and w_0^D satisfy the Elasticity systems in the non perturbed domain Ω

$$(\mathcal{P}_0^N)\left\{\begin{aligned} -\operatorname{div}\sigma(w_0^N) &= F && \text{in } \Omega,\\ \sigma(w_0^N)\mathbf{n} &= G && \text{on } \Sigma,\\ w_0^N &= 0 && \text{on } \Gamma,\end{aligned}\right. \quad \text{and} \quad (\mathcal{P}_0^D)\left\{\begin{aligned} -\operatorname{div}\sigma(w_0^D) &= F && \text{in } \Omega,\\ w_0^D &= \mathcal{W}_m && \text{on } \Sigma,\\ w_0^D &= 0 && \text{on } \Gamma,\end{aligned}\right.$$

2.2 Sensitivity Analysis Method

To solve the topological optimization problem $(\mathcal{P}_{opt})$ and identify the unknown hole, we will propose a simplified approach based on the topological sensitivity analysis for the Kohn-Vogelius function $\mathcal{K}$.

The classical topological sensitivity analysis method is based on a first order asymptotic expansion of the form

$$\mathcal{K}(\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) = \rho(\varepsilon)\delta\mathcal{K}(z) + o\big(\rho(\varepsilon)\big),\ \forall z\in\Omega,$$

where

- $\varepsilon \mapsto \rho(\varepsilon)$ is a positive scalar function going to zero with ε;
- $z \mapsto \delta\mathcal{K}(z)$ is the first order topological gradient.

In this work, we extend the topological sensitivity notion for the high-order case. We will derive a high-order topological asymptotic expansion for the Kohn-Vogelius functional $\mathcal{K}$ with respect to the presence of the Dirichlet geometric perturbation $\mathcal{O}_{z,\varepsilon}$ inside the domain Ω. It consists in studying the variation $\mathcal{K}(\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega)$ with respect to ε and establishing an asymptotic formula on the form

$$\mathcal{K}(\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) = \sum_{q=1}^{m}\rho_q(\varepsilon)\delta\mathcal{K}^q(z,\mathcal{O}) + o(\rho_m(\varepsilon))$$

where

- $\varepsilon \mapsto \rho_q(\varepsilon)$, $1\le q\le m$ are positive scalar functions defined on $\mathbb{R}^+$ verifying $\rho_{q+1}(\varepsilon) = o(\rho_q(\varepsilon))$ and $\lim_{\varepsilon\to 0}\rho_q(\varepsilon) = 0$;
- $\delta\mathcal{K}^q$ denotes the q^{th} topological derivative of the functional $\mathcal{K}$;
- $m\in\mathbb{N}^*$ an arbitrary order of the asymptotic expansion.

To this end, we start our analysis by deriving a preliminary result showing the relationship between the quantity $\mathcal{K}(\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega)$ and the variations of the Neumann and Dirichlet perturbed solutions $w_\varepsilon^N - w_0^N$ and $w_\varepsilon^D - w_0^D$.

3 The Kohn-Vogelius Functional Variation

In this section, we discuss the sensitivity of the considered Kohn-Vogelius functional $\mathcal{K}$ with respect to the presence of a small hole $\mathcal{O}_{z,\,\varepsilon}$ inside the elastic domain. We will prove that the variation $\mathcal{K}(\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega)$ can be only described by four integral terms, involving the discrepancy between the initial and the perturbed Elasticity problem solutions.

Theorem 1 *Let $\mathcal{O}_{z,\varepsilon}$ be a small hole, strictly embedded into the elastic domain Ω. In the presence of $\mathcal{O}_{z,\varepsilon}$, the Kohn-Vogelius function $\mathcal{K}$ admits the following variation*

$$\begin{aligned}\mathcal{K}(\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) &= \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N)\mathbf{n}w_0^N\,ds - \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^D - w_0^D)\mathbf{n}w_0^D\,ds \\ &\quad + \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^D)\right|^2 dx - \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^N)\right|^2 dx.\end{aligned}$$

Proof. The variation of the function $\mathcal{K}$ is given by

$$\begin{aligned}\mathcal{K}(\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) &= \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\sigma(w_\varepsilon^N) - \sigma(w_\varepsilon^D)\right|^2\,dx - \int_\Omega \left|\sigma(w_0^N) - \sigma(w_0^D)\right|^2\,dx \\ &= \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\sigma(w_\varepsilon^N)\right|^2 dx + \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\sigma(w_\varepsilon^D)\right|^2 dx - 2\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \sigma(w_\varepsilon^D)\sigma(w_\varepsilon^N)dx \\ &\quad - \int_\Omega \left|\sigma(w_0^N)\right|^2 dx - \int_\Omega \left|\sigma(w_0^D)\right|^2 dx + 2\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \sigma(w_0^D)\sigma(w_0^N)dx.\end{aligned}$$

As one can observe $\mathcal{K}(\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega)$ can be decomposed as

$$\mathcal{K}(\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) = \mathcal{T}_N(\varepsilon) + \mathcal{T}_D(\varepsilon) - 2\mathcal{T}_M(\varepsilon),$$

with $\mathcal{T}_N$ is the Neumann term

$$\mathcal{T}_N(\varepsilon) = \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\sigma(w_\varepsilon^N)\right|^2 dx - \int_\Omega \left|\sigma(w_0^N)\right|^2 dx,$$

$\mathcal{T}_D$ is the Dirichlet term

$$\mathcal{T}_D(\varepsilon) = \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\sigma(w_\varepsilon^D)\right|^2 dx - \int_\Omega \left|\sigma(w_0^D)\right|^2 dx,$$

and $\mathcal{T}_M$ is the mixed term

$$\mathcal{T}_M(\varepsilon) = \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \sigma(w_\varepsilon^D)\sigma(w_\varepsilon^N)dx - \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \sigma(w_0^D)\sigma(w_0^N)dx.$$

Next, we will examine each term separately. We start our analysis by studying the term $\mathcal{T}_N$.

Variation of the Neumann term: The term $\mathcal{T}_N$ reads

$$\begin{aligned}\mathcal{T}_N(\varepsilon) &= \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\sigma(w_\varepsilon^N)\right|^2 dx - \int_\Omega \left|\sigma(w_0^N)\right|^2 dx \\ &= \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N)\sigma(w_\varepsilon^N)dx + \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N)\sigma(w_0^N)dx \\ &\quad - \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^N)\right|^2 dx.\end{aligned}$$

Using Green formula, from the Elasticity system satisfied by $w_\varepsilon^N - w_0^N$, we obtain

$$\int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N)\sigma(w_0^N)dx = \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N)\mathbf{n}w_0^N ds.$$

In addition,

$$\int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N)\sigma(w_\varepsilon^N)dx = 0.$$

Then, the term $\mathcal{T}_N$ can be written as

$$\mathcal{T}_N(\varepsilon) = \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N)\mathbf{n}w_0^N ds - \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^N)\right|^2 dx. \tag{2}$$

Variation of the Dirichlet term: We have

$$\begin{aligned}\mathcal{T}_D(\varepsilon) &= \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \left|\sigma(w_\varepsilon^D)\right|^2 dx - \int_\Omega \left|\sigma(w_0^D)\right|^2 dx\\ &= \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \left|\sigma(w_\varepsilon^D - w_0^D)\right|^2 dx + 2\int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \sigma(w_\varepsilon^D - w_0^D)\sigma(w_0^D)dx - \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^D)\right|^2 dx.\end{aligned}$$

Using Green formula, the system $(\mathcal{P}_0^D)$ implies

$$\int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \sigma(w_\varepsilon^D - w_0^D)\sigma(w_0^D)dx = \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} F(w_\varepsilon^D - w_0^D)dx - \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_0^D)\mathbf{n}w_0^D ds.$$

Then, the Dirichlet term admits the following variation

$$\begin{aligned}\mathcal{T}_D(\varepsilon) &= \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \left|\sigma(w_\varepsilon^D - w_0^D)\right|^2 dx + 2\int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} F(w_\varepsilon^D - w_0^D)dx\\ &\quad - \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^D)\right|^2 dx - 2\int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_0^D)\mathbf{n}w_0^D ds.\end{aligned}$$

Using Green formula, from the system verified by $(w_\varepsilon^D - w_0^D)$, we deduce

$$\begin{aligned}\mathcal{T}_D(\varepsilon) &= -\int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^D - w_0^D)\mathbf{n}w_0^D ds + 2\int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} F(w_\varepsilon^D - w_0^D)dx\\ &\quad -2\int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_0^D)\mathbf{n}w_0^D ds - \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^D)\right|^2 dx.\end{aligned} \tag{3}$$

Variation of the mixed term*:* We have

$$\mathcal{T}_M(\varepsilon) = \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \sigma(w_\varepsilon^N)\sigma(w_\varepsilon^D)dx\,dt - \int_\Omega \sigma(w_0^N)\sigma(w_0^D)dx.$$

From the weak formulation of problems $(\mathcal{P}_\varepsilon^N)$ and $(\mathcal{P}_0^N)$, it follows that

$$\mathcal{T}_M(\varepsilon) = \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} F(w_\varepsilon^D - w_0^D)dx - \int_{\mathcal{O}_{z,\varepsilon}} Fw_0^D dx.$$

From the fact that $-\operatorname{div}\sigma(w_0^D) = F$ in $\mathcal{O}_{z,\varepsilon}$, one can deduce

$$\int_{\mathcal{O}_{z,\varepsilon}} Fw_0^D dx = \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^D)\right|^2 dx + \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_0^D)\mathbf{n}w_0^D ds.$$

Consequently, the term $\mathcal{T}_M$ can be written as

$$\mathcal{T}_M(\varepsilon) = \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} F(w_\varepsilon^D - w_0^D)dx - \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^D)\right|^2 dx - \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_0^D)\mathbf{n}w_0^D ds. \quad (4)$$

Variation of the Kohn-Vogelius function K**:** Combining the variations (2), (3) and (4), one can deduce that the variation of $\mathcal{K}$ can be written as

$$\begin{aligned}\mathcal{K}(\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) \quad &= \quad \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N)\mathbf{n}w_0^N ds - \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^N)\right|^2 dx \\ &\quad + \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^D)\right|^2 dx - \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^D - w_0^D)\mathbf{n}w_0^D ds.\end{aligned}$$

■

4 Asymptotic Behavior of the Perturbed Solutions

In this section, we discuss the influence of the geometric perturbation $\mathcal{O}_{z,\varepsilon}$ on the solutions of the Elasticity problems $(\mathcal{P}_\varepsilon^N)$ and $(\mathcal{P}_\varepsilon^D)$. More precisely, we derive an asymptotic formula describing the variations of the displacement field with respect to the perturbation size ε. We start our analysis by the perturbed Elasticity Neumann problem.

4.1 The Neumann Perturbed Solution

This section is devoted to an asymptotic formula describing the variation of the Neumann solution $(w_\varepsilon^N - w_0^N)$ with respect ε. We begin our analysis by the following first order estimate.

Lemma 2 *Let $\mathcal{O}_{z,\varepsilon}$ be a small geometric perturbation strictly included into Ω. Then the perturbed Elasticity solution w_ε^N satisfies the behavior*

$$w_\varepsilon^N(x) - w_0^N(x) = E_0^N((x-z)/\varepsilon) + O(\varepsilon) \quad in\ \Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon},$$

where the leading term E_0^N is solution to the following Elasticity exterior problem

$$\begin{cases} -\operatorname{div}\sigma(E_0^N) &= 0 & in\ \mathbb{R}^3\setminus\overline{\mathcal{O}}, \\ E_0^N &\longrightarrow 0 & at\ \infty, \\ E_0^N &= -w_0^N(z) & on\ \partial\mathcal{O}. \end{cases}$$

Proof. The existence of E_0^N can be established with the help of a single layer potential. One can derive

$$E_0^N(y) = \int_{\partial\mathcal{O}} U(y-t)\,\eta_0^N(t)ds(t),\ \forall y\in\mathbb{R}^3\setminus\overline{\mathcal{O}}$$

where U is the fundamental solution of the Elasticity system in $\mathbb{R}^3$

$$U(y) = \frac{1}{r}\Big(\beta I + \gamma e_r e_r^T\Big),\ \beta\,\gamma\in\mathbb{R}$$

with $r = \|y\|$, $e_r = \frac{y}{r}$ and e_r^T is the transposed vector of e_r. Here η_0^N is the solution to the following boundary integral equation

$$\int_{\partial\mathcal{O}} U(y-t)\,\eta_0^N(t)ds(t) = -w_0^N(z), \forall y\in\partial\mathcal{O}.$$

Setting

$$z_\varepsilon^N(x) = w_\varepsilon^N(x) - w_0^N(x) - E_0^N((x-z)/\varepsilon).$$

As one can observe, z_ε^N satisfies the system

$$\begin{cases} -\operatorname{div}\sigma(z_\varepsilon^N) &= 0 & \text{in } \Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}, \\ \sigma(z_\varepsilon^N)\mathbf{n} &= -\frac{1}{\varepsilon}\sigma(E_0^N)((x-z)/\varepsilon)\mathbf{n} & \text{on } \Sigma, \\ w_\varepsilon^N &= -E_0^N((x-z)/\varepsilon) & \text{on } \Gamma, \\ w_\varepsilon^N &= -w_0^N + w_0^N(z) & \text{on } \partial\mathcal{O}_{z,\varepsilon}. \end{cases}$$

Using the change of variable $x = z + \varepsilon y$ and the standard energy estimate for the Elasticity problem, one can derive that there exists a constant $c > 0$, independent of ε, such that

$$\left\| z_\varepsilon^N \right\|_{H^1(\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon})} \leq c\,\varepsilon.$$

One can see ([26], Proposition 3.1) for similar proof. ■

Next, we extend this estimate to the high-order case. The obtained asymptotic behavior is illustrated by the following theorem.

Theorem 3 *Let $\mathcal{O}_{z,\varepsilon} = z + \mathcal{O}$ be a small geometric perturbation, strictly embedded in the elastic domain Ω. Then the displacement fild variation satisfis the following asymptotic behavior*

$$w_\varepsilon^N(x) - w_0^N(x) = E_0^N((x-z)/\varepsilon) + \sum_{k=1}^{m} \varepsilon^k [V_k^N(x) + E_k^N((x-z)/\varepsilon))] + o(\varepsilon^m),$$

where $(V_k^N)_{0\leq k\leq m}$ is a set of smooth functions satisfying the Elasticity system in Ω and $(E_k^N)_{0\leq k\leq m}$ is a set of smooth functions verifying the exterior Elasticity problem in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$.

Proof. The terms of the derived asymptotic expansion are built iteratively.

Initialization: We start our construction process by the terms associated with $k = 0$. The two sequences $(V_k^N)_{0\leq k\leq m}$ and $(E_k^N)_{0\leq k\leq m}$ are initialized as follows:

− $V_0^N = w_0^N$ which is the solution to the Elasticity problem $(\mathcal{P}_0^N)$, defined in the non perturbed domain Ω.

− E_0^N is the solution to the exterior Elasticity problem (5), defined in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$.

The k^{th} term: Let $k \in \{1, ..., m\}$. Assume that we have already derived the terms V_i^N and E_i^N for all $0 \leq i \leq k-1$, and we want to derive the terms V_k^N and E_k^N.

In order to define the desired terms, we need to establish a preliminary calculus. It concerns the asymptotic behavior of the functions E_i^N with respect to ε. Recalling that E_i^N is constructed as a solution to an exterior Elasticity problem defined in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$. Then, due to a single layer potential, E_i^N can be written as

$$E_i^N(y) = \int_{\partial\mathcal{O}} U(y-t)\,\eta_i^N(t)ds(t), \;\; \forall y \in \mathbb{R}^3 \setminus \overline{\mathcal{O}},$$

where η_i^N is the solution to a boundary integral equation defined on $\partial\mathcal{O}$. From the fact that $U(y/\varepsilon) = \varepsilon U(y)$ it follows that for each $x \in \mathbb{R}^3 \setminus \overline{\mathcal{O}_{z,\varepsilon}}$ we have

$$\begin{aligned} E_i^N((x-z)/\varepsilon) &= \int_{\partial\mathcal{O}} U((x-z)/\varepsilon - t)\,\eta_i^N(t)ds(t) \\ &= \varepsilon \int_{\partial\mathcal{O}} U((x-z) - \varepsilon\, t)\,\eta_i^N(t)ds(t). \end{aligned}$$

From the fact that $\mathcal{O}_{z,\varepsilon}$ is not close to the boundary $\partial\Omega$, one can remark that for all $t \in \partial\mathcal{O}$, the function $\varepsilon \longmapsto U_{x-z,t}(\varepsilon) = \varepsilon\, U((x-z)-\varepsilon\, t)$ is smooth with respect to ε and admits the following asymptotic expansion

$$U_{x-z,t}(\varepsilon) = \sum_{p=1}^{m} \frac{\varepsilon^p}{(p-1)!} U_{x-z,t}^{(p-1)}(0) + o(\varepsilon^m),$$

where $U^{(p)}_{x-z,t}(0)$ is the p^{th} derivative of $U_{x-z,t}$ at $\varepsilon = 0$. It depends on the p^{th} derivative of the function U at the point $x - z$. Consequently, the function $\varepsilon \longmapsto E^N_i((x-z)/\varepsilon)$ satisfies the following asymptotic behavior

$$E^N_i((x-z)/\varepsilon) = \sum_{p=1}^{m} \varepsilon^p \, E^{(p)}_{N,i}(x-z) + o(\varepsilon^m), \tag{5}$$

with $E^{(p)}_{N,i}$ is the smooth function defined in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$ by

$$E^{(p)}_{N,i}(y) = \frac{1}{(p-1)!} \int_{\partial\mathcal{O}} U^{(p-1)}_{y,t}(0)\eta^N_i(t)ds(t), \;\; \forall y \in \mathbb{R}^3 \setminus \overline{\mathcal{O}}. \tag{6}$$

−**Determining the term:** V^N_k. It is constructed using the functions $E^N_i, \;\; 0 \le i \le k-1$. It is defined as the solution to the following Elasticity system

$$\begin{cases} -\operatorname{div}\sigma(V^N_k) & = 0 & \text{in } \Omega, \\ \sigma(V^N_k)\mathbf{n} & = -\sum_{p=1}^{k} \sigma(E^{(p)}_{N,k-p})(x-z)\mathbf{n} & \text{on } \Sigma, \\ V^N_k & = -\sum_{p=1}^{k} E^{(p)}_{N,k-p}(x-z) & \text{on } \Gamma, \end{cases} \tag{7}$$

where the functions $E^{(p)}_{N,j}$ is defined by (6).

−**Determining the term:** E^N_k. It is constructed using the functions $V^N_i, \;\; 0 \le i \le k$. This term is defined as a solution to the following exterior problem

$$\begin{cases} -\operatorname{div}\sigma(E^N_k) & = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\ E^N_k & \longrightarrow 0 & \text{at } \infty, \\ E^N_k & = -V^N_k(z) - \sum_{p=1}^{k} \frac{1}{p!} D^p V^N_{k-p}(z)(y^p) & \text{on } \partial\mathcal{O}, \end{cases} \tag{8}$$

where $D^p V^N_{k-p}(z)$ is the p^{th} derivative of the function V^N_{k-p} at the point z and $y^p = (y, ..., y) \in (\mathbb{R}^3)^p$.

Justification of the asymptotic formulas: Here we will prove that the constructed sequences $(V^N_k)_{0 \le k \le m}$ and $(E^N_k)_{0 \le k \le m}$ permit us to derive the expected asymptotic formulas. Posing

$$R^N_{m,\varepsilon}(x) = w^N_0(x) + E^N_0((x-z)/\varepsilon)) + \sum_{k=1}^{m} \varepsilon^k [V^N_k(x) + E^N_k((x-z)/\varepsilon))] - w^N_\varepsilon.$$

One can easily verify that $R^N_{m,\varepsilon}$ solves the Elasticity system in $\Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon}$

$$-\operatorname{div}\sigma(R^N_{m,\varepsilon}) = 0 \quad \text{in}\Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon},$$

and satisfies the following boundaries conditions:

On $\partial\mathcal{O}_{z,\varepsilon}$: Using the systems (7)–(8), the multi-linearity of $D^p V^N_{k-p}(z)$, Taylor's Theorem and the fact that $\|x - z\| = O(\varepsilon)$ on $\partial\mathcal{O}_{z,\varepsilon}$, one can derive

$$R^N_{m,\varepsilon}(x) = \sum_{k=0}^{m} \varepsilon^k \Big[V^N_k(x) - \sum_{p=0}^{m-k} \frac{1}{p!} D^p V^N_k(z)((x-z)^p) \Big] = o(\varepsilon^m).$$

On Γ: the Dirichlet boundary condition in (7) and the asymptotic expansions (5) imply

$$\begin{aligned} R^N_{m,\varepsilon}(x) &= \sum_{k=0}^{m} \varepsilon^k E^N_k((x-z)/\varepsilon)) + \sum_{k=1}^{m} \varepsilon^k V^N_k(x) \\ &= \sum_{k=0}^{m} \varepsilon^k \Big[\sum_{p=1}^{m} \varepsilon^p \, E^{(p)}_{N,k}(x-z) \Big] - \sum_{k=1}^{m} \varepsilon^k \Big[\sum_{p=1}^{k} E^{(p)}_{N,k-p}(x-z) \Big] + o(\varepsilon^m) \\ &= o(\varepsilon^m). \end{aligned}$$

On Σ: by the change of variable $x = z + \varepsilon y$, we have

$$\sigma(R_{m,\varepsilon}) = \frac{1}{\varepsilon}\sum_{k=0}^{m}\varepsilon^k \sigma_y(E_k^N)((x-z)/\varepsilon) + \sum_{k=1}^{m}\varepsilon^k\sigma(V_k^N)(x).$$

Using again the change of variable, from (5) one can deduce

$$\sigma_y(E_k^N)((x-z)/\varepsilon) = \varepsilon\sum_{p=1}^{m}\varepsilon^p\sigma(E_{N,k}^{(p)})(x-z) + o(\varepsilon^m).$$

The two last relations combined with the Neumann condition used in (7) imply

$$\sigma(R_{m,\varepsilon})\mathbf{n} = o(\varepsilon^m) \quad \text{on } \Sigma.$$

■

4.2 The Dirichlet Perturbed Solution

This section is concerned with brief analysis of the Dirichlet case. Here, we consider $(V_k^D)_{0\le k\le m}$ a set of smooth functions satisfying the Elasticity system in Ω and $(E_k^D)_{0\le k\le m}$ a set of smooth functions verifying the exterior Elasticity problem in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$. The two considered sequences $(V_k^D)_{0\le k\le m}$ and $(E_k^D)_{0\le k\le m}$ are initialized as follows:

- $V_0^D = w_0^D$ which is the solution to the Elasticity problem $(\mathcal{P}_0^D)$.
- E_0^D is defined as the solution to the following exterior Elasticity problem

$$\begin{cases} -\operatorname{div}\sigma(E_0^D) &= 0 & \text{in } \mathbb{R}^3\setminus\overline{\mathcal{O}}, \\ E_0^D &\longrightarrow 0 & \text{at } \infty, \\ E_0^D &= -w_0^D(z) & \text{on } \partial\mathcal{O}, \end{cases}$$

The term V_k^D: It is defined as the unique solution to the following Elasticity system

$$\begin{cases} -\operatorname{div}\sigma(V_k^D) &= 0 & \text{in } \Omega, \\ V_k^D &= -\sum_{p=1}^{k} E_{D,k-p}^{(p)}(x-z) & \text{on } \Gamma\cup\Sigma, \end{cases} \tag{9}$$

where the functions $E_{D,j}^{(p)}, 0 \le j \le k$ are defined by

$$E_{D,j}^{(p)}(y) = \frac{1}{(p-1)!}\int_{\partial\mathcal{O}} U_{y,t}^{(p-1)}(0)\eta_j^D(t)ds(t), \quad \forall y\in\mathbb{R}^3\setminus\overline{\mathcal{O}}. \tag{10}$$

The term E_k^D: It is constructed using the functions $V_j^D,\ 0\le j\le k$. This term is defined as a solution to the following exterior problem

$$\begin{cases} -\operatorname{div}\sigma(E_k^D) &= 0 & \text{in } \mathbb{R}^3\setminus\overline{\mathcal{O}}, \\ E_k^D &\longrightarrow 0 & \text{at } \infty, \\ E_k^D &= -V_k^D(z) - \sum_{p=1}^{k}\frac{1}{p!}D^pV_{k-p}^D(z)(y^p) & \text{on } \partial\mathcal{O}, \end{cases} \tag{11}$$

Theorem 4 *In the presence of a small geometric perturbation $\mathcal{O}_{z,\varepsilon} = z + \mathcal{O}$ inside the elastic domain Ω, the solution w_ε^D of the perturbed Dirichlet Elasticity problem admits the asymptotic behavior*

$$w_\varepsilon^D(x) - w_0^D(x) = E_0^D((x-z)/\varepsilon) + \sum_{k=1}^{m}\varepsilon^k[V_k^D(x) + E_k^D((x-z)/\varepsilon))] + o(\varepsilon^m).$$

5 High-Order Topological Asymptotic Expansion

In this section we extend the topological derivative notion for the high-order case. We derive a high-order term in the topological asymptotic expansion for the Elasticity operator. We will derive an asymptotic formula describing the variation of the Kohn-Vogelius functional $\mathcal{K}$ with respect to the insertion of a small hole inside the elastic domain Ω.

From Theorem 1, the variation caused by the presence of the geometric perturbation $\mathcal{O}_{z,\varepsilon} = z + \varepsilon\mathcal{O}$ can be decomposed as

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) = \mathcal{J}^N(\varepsilon) - \mathcal{J}^D(\varepsilon),$$

where the Neumann and Dirichlet terms are defined by

$$\mathcal{J}^N(\varepsilon) = \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N)\mathbf{n} w_0^N ds - \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^N)\right|^2 dx, \tag{12}$$

$$\mathcal{J}^D(\varepsilon) = \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^D - w_0^D)\mathbf{n} w_0^D ds - \int_{\mathcal{O}_{z,\varepsilon}} \left|\sigma(w_0^D)\right|^2 dx. \tag{13}$$

To derive the expected high-order asymptotic expansion for Kohn-Vogelius functional $\mathcal{K}$ we will examine the terms $\mathcal{J}^N$ and $\mathcal{J}^D$ separately.

5.1 Estimate of the Neumann Terms

Here, we derive a sensitivity analysis for each term in $\mathcal{J}^N(\varepsilon)$ with respect to the parameter ε. We will establish a high-order asymptotic expansion for each term. Our mathematical analysis is based on the asymptotic behavior of the perturbed solution w_ε^N.

Lemma 5 *The first term in (13) admits the estimate*

$$\begin{aligned}\int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N)\mathbf{n}.w_0^N ds \quad &= \quad \sum_{k=0}^{m-1} \varepsilon^{k+1} \int_{\partial\mathcal{O}} \sigma_y(E_k^N)(y)\mathbf{n}(y).w_0^N(z+\varepsilon y)ds(y) \\ &\quad + \sum_{k=1}^{m-2} \varepsilon^{k+2} \int_{\partial\mathcal{O}} \sigma(V_k^N)(z+\varepsilon y)\mathbf{n}.w_0^N(z+\varepsilon y)ds(y) + o(\varepsilon^m).\end{aligned} \tag{14}$$

Proof. Using the previous relation and the decomposition presented in Theorem 3, one can derive

$$\sigma(w_\varepsilon^N - w_0^N) = \frac{1}{\varepsilon}\sigma_y(E_0^N)((x-z)/\varepsilon)) + \sum_{k=1}^{m} \varepsilon^k \sigma(V_k^N)(x) + \sum_{k=1}^{m+1} \varepsilon^{k-1}\sigma_y(E_k^N)((x-z)/\varepsilon) + o(\varepsilon^m).$$

Then, the first term in (13) satisfies the estimate

$$\begin{aligned}\int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N)\mathbf{n}.w_0^N ds \quad &= \quad \frac{1}{\varepsilon}\int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma_y(E_0^N)((x-z)/\varepsilon))\mathbf{n}.w_0^N ds \\ &\quad + \sum_{k=1}^{m+1} \varepsilon^{k-1} \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma_y(E_k^N)((x-z)/\varepsilon)\mathbf{n}.w_0^N ds \\ &\quad + \sum_{k=1}^{m} \varepsilon^k \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(V_k^N)(x)\mathbf{n}.w_0^N ds + o(\varepsilon^m).\end{aligned}$$

Making use of the change of variable $x = z + \varepsilon y$, one can deduce

$$\int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma_y(E_k^N)((x-z)/\varepsilon)\mathbf{n}.w_0^N ds = \varepsilon^2 \int_{\partial\mathcal{O}} \sigma_y(E_k^N)(y)\mathbf{n}(y).w_0^N(z+\varepsilon y)ds(y),$$

$$\int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(V_k^N)(x)\mathbf{n}.w_0^N ds = \varepsilon^2 \int_{\partial \mathcal{O}} \sigma(V_k^N)(z+\varepsilon y)\mathbf{n}(y).w_0^N(z+\varepsilon y)ds(y).$$

Consequently, we obtain

$$\begin{aligned}\int_{\partial \mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N)\mathbf{n}.w_0^N ds &= \sum_{k=0}^{m-1} \varepsilon^{k+1} \int_{\partial \mathcal{O}} \sigma_y(E_k^N)(y)\mathbf{n}(y).w_0^N(z+\varepsilon y)ds(y) \\ &+ \sum_{k=1}^{m-2} \varepsilon^{k+2} \int_{\partial \mathcal{O}} \sigma(V_k^N)(z+\varepsilon y)\mathbf{n}.w_0^N(z+\varepsilon y)ds(y) + o(\varepsilon^m).\end{aligned}$$

■

Next, we will examine the two integral terms in the right hand side of (14).

Lemma 6 *We have*

$$\sum_{k=0}^{m-1} \varepsilon^{k+1} \int_{\partial \mathcal{O}} \sigma_y(E_k^N)(y)\mathbf{n}(y).w_0^N(z+\varepsilon y)ds(y) = \sum_{q=0}^{m-1} \varepsilon^{q+1} \mathcal{K}_q^{1,N}(z,\mathcal{O}) \; + \; o(\varepsilon^m),$$

where the functions $z \longmapsto \mathcal{K}_q^{1,N}(z,\mathcal{O})$, $0 \leq q \leq m$ *are defined in* Ω *by*

$$\mathcal{K}_q^{1,N}(z,\mathcal{O}) = \sum_{p=0}^{q} \frac{1}{p!} \int_{\partial \mathcal{O}} \sigma_y(E_{q-p}^N)(y)\mathbf{n}(y).[D^{(p)}w_0^N(z)(y^p)]ds(y)$$

with $D^{(p)}w_0^N(z)$ *denotes the* p^{th} *derivative of the function* w_0^N *at the point* $z \in \Omega$ *and* $y^p = (y, ..., y) \in (\mathbb{R}^3)^p$.

Proof. Due to the smoothness of the velocity field w_0^N, by Taylor's theorem one can derive

$$w_0^N(z+\varepsilon y) = \sum_{p=0}^{m} \frac{\varepsilon^p}{p!} D^{(p)}w_0^N(z)(y^p) + o(\varepsilon^m). \tag{15}$$

It follows

$$\begin{aligned}&\sum_{k=0}^{m-1} \varepsilon^{k+1} \int_{\partial \mathcal{O}} \sigma_y(E_k^N)(y)\mathbf{n}(y).w_0^N(z+\varepsilon y)ds(y) \\ =& \sum_{k=0}^{m-1} \varepsilon^{k+1} \Big(\sum_{p=0}^{m} \frac{\varepsilon^p}{p!} \int_{\partial \mathcal{O}} \sigma_y(E_k^N)(y)\mathbf{n}(y).[D^{(p)}w_0^N(z)(y^p)]ds(y) \Big) + o(\varepsilon^m) \\ =& \sum_{q=0}^{m-1} \varepsilon^{q+1} \Big(\sum_{p=0}^{q} \frac{1}{p!} \int_{\partial \mathcal{O}} \sigma_y(E_{q-p}^N)(y)\mathbf{n}(y).[D^{(p)}w_0^N(z)(y^p)]ds(y) \Big) + o(\varepsilon^m)\end{aligned}$$

■

Lemma 7 *The second integral term in the right hand side of (14) satisfies*

$$\sum_{k=1}^{m-2} \varepsilon^{k+2} \int_{\partial \mathcal{O}} \sigma(V_k^N)(z+\varepsilon y)\mathbf{n}.w_0^N(z+\varepsilon y)ds(y) = \sum_{q=1}^{m-2} \varepsilon^{q+2} \mathcal{K}_q^{2,N}(z,\mathcal{O}) \; + \; o(\varepsilon^m),$$

where the functions $z \longmapsto \mathcal{K}_q^{2,N}(z,\mathcal{O})$, $1 \leq q \leq m-2$ *are defined in* Ω *by*

$$\mathcal{K}_q^{2,N}(z,\mathcal{O}) = \sum_{p=0}^{q-1} \sum_{l=0}^{p} \frac{1}{l!(p-l)!} \int_{\partial \mathcal{O}} \sigma^{(l)}(V_{q-p}^N)(z)\mathbf{n}.D^{(p-l)}w_0^N(z)(y^{p-l})ds(y).$$

Proof. Here we exploit the smoothness of the functions V_q^N. This follows from the fact that V_q^N is solution to the Elasticity problem (7), defined in a smooth domain Ω and verifying smooth boundary data on $\partial\Omega$. Using Taylor's formula, one can derive

$$\frac{\partial (V_k^N)_i}{\partial x_j}(z+\varepsilon y) = \sum_{p=0}^{m} \frac{\varepsilon^p}{p!} D^{(p)} \left(\frac{\partial (V_k^N)_i}{\partial x_j} \right)(z)(y^p) + o(\varepsilon^m), \;\; 1 \leq i,j \leq 3.$$

Recalling that

$$\sigma(V_k^N)_{i,j} = \mu\left(\frac{\partial (V_k^N)_i}{\partial x_j} + \frac{\partial (V_k^N)_j}{\partial x_i}\right) + \lambda\, div\, (V_k^N)\delta_{i,j}$$

where $\delta_{i,j}$ is the Kronecker symbol. Then, for each $1 \leq i,j \leq 3$, one can derive

$$\sigma(V_k^N)_{i,j}(z+\varepsilon y) = \sum_{p=0}^{m} \frac{\varepsilon^p}{p!} \sigma_{i,j}^{(p)}(V_k^N)(z) + o(\varepsilon^m), \tag{16}$$

with $\sigma_{i,j}^{(p)}(V_k^N)(z) = D^{(p)}\left(\sigma(V_k^N)_{i,j}\right)(z)$ which is the p^{th} derivative of the function

$$\mu\left(\frac{\partial (V_k^N)_i}{\partial x_j} + \frac{\partial (V_k^N)_j}{\partial x_i}\right) + \lambda\, div\, (V_k^N)\delta_{i,j}$$

at the point $z \in \Omega$.

Due to the Cauchy product formula, the relations (15) and (16) imply

$$\begin{aligned} &\int_{\partial\mathcal{O}} \sigma(V_k^N)(z+\varepsilon y)\mathbf{n}.w_0^N(z+\varepsilon y)ds(y) \\ = \;& \sum_{p=0}^{m} \varepsilon^p \Big(\sum_{l=0}^{p} \frac{1}{l!(p-l)!} \int_{\partial\mathcal{O}} \sigma^{(l)}(V_k^N)(z)\mathbf{n}.D^{(p-l)}w_0^N(z)(y^{p-l})ds(y) \Big) + o(\varepsilon^m). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} &\sum_{k=1}^{m-2} \varepsilon^{k+2} \int_{\partial\mathcal{O}} \sigma(V_k^N)(z+\varepsilon y)\mathbf{n}.w_0^N(z+\varepsilon y)ds(y) \\ = \;& \sum_{q=1}^{m-2} \varepsilon^{q+2} \Big(\sum_{p=0}^{q-1}\sum_{l=0}^{p} \frac{1}{l!(p-l)!} \int_{\partial\mathcal{O}} \sigma^{(l)}(V_{q-p}^N)(z)\mathbf{n}.D^{(p-l)}w_0^N(z)(y^{p-l})ds(y) \Big) + o(\varepsilon^m). \end{aligned}$$

■

Lemma 8 *The second term in (13) admits the estimate*

$$\int_{\mathcal{O}_{z,\varepsilon}} |\sigma(w_0^N)|^2 dx = \sum_{q=0}^{m-3} \varepsilon^{q+3} \mathcal{K}_q^{3,N}(z,\mathcal{O}) \; + \; o(\varepsilon^m),$$

where $\{z \longmapsto \mathcal{K}_q^{3,N}(z,\mathcal{O}),\, 0 \leq q \leq m-3\}$ *is a set of functions, defined in* Ω *by*

$$\mathcal{K}_q^{3,N}(z,\mathcal{O}) = \sum_{p=0}^{q} \frac{1}{p!(q-p)!} \int_{\mathcal{O}} D^{(p+1)}w_0^N(z)(y^p) : D^{(q-p+1)}w_0^N(z)(y^{q-p})dy.$$

Proof. The change of variable $x = z + \varepsilon y$, implies

$$\int_{\mathcal{O}_{z,\varepsilon}} |\sigma(w_0^N)|^2 dx = \varepsilon^3 \int_{\mathcal{O}} |\sigma(w_0^N)(z+\varepsilon y)|^2 dy.$$

Since w_0^N is smooth near z, by Taylor's theorem it follows

$$\sigma(w_0^N)(z+\varepsilon y) = \sum_{p=0}^{m} \frac{1}{p!} D^{(p+1)} w_0^N(z)(y^p) + o(\varepsilon^m).$$

By Cauchy product formula, we obtain

$$|\sigma(w_0^N)|^2(z+\varepsilon y) = \sum_{q=0}^{m} \varepsilon^q \left(\sum_{p=0}^{q} \frac{1}{p!(q-p)!} D^{(p+1)} w_0^N(z)(y^p) : D^{(q-p+1)} w_0^N(z)(y^{q-p}) \right) + o(\varepsilon^m).$$

■

5.2 Estimate of the Dirichlet Terms

In this section, we examine the Dirichlet terms involved in the Kohn-Vogelius functional variation. Based on the asymptotic behavior of the perturbed solution w_ε^D with respect to the parameter ε, we establish a high-order asymptotic formula for each term of the function $\mathcal{J}^D(\varepsilon)$. Using Theorem 4, the function $\mathcal{J}^D(\varepsilon)$ admits the following estimate

$$\begin{aligned} \mathcal{J}^D(\varepsilon) \quad = \quad & \sum_{k=0}^{m-1} \varepsilon^{k+1} \int_{\partial\mathcal{O}} \sigma_y(E_k^D)(y)\mathbf{n}(y).w_0^D(z+\varepsilon y) ds(y) \\ & + \sum_{k=1}^{m-2} \varepsilon^{k+2} \int_{\partial\mathcal{O}} \sigma(V_k^D)(z+\varepsilon y)\mathbf{n}.w_0^D(z+\varepsilon y) ds(y) \\ & -\varepsilon^3 \int_{\mathcal{O}} |\sigma(w_0^D)(z+\varepsilon y)|^2 dy. + o(\varepsilon^m). \end{aligned}$$

Similar to the Naumann case, we derive the following preliminary lemmas estimating the integral terms in the last equality.

Lemma 9 *We have*

$$\sum_{k=0}^{m-1} \varepsilon^{k+1} \int_{\partial\mathcal{O}} \sigma_y(E_k^D)(y)\mathbf{n}(y).w_0^D(z+\varepsilon y) ds(y) = \sum_{q=0}^{m-1} \varepsilon^{q+1} \mathcal{K}_q^{1,D}(z,\mathcal{O}) \; + \; o(\varepsilon^m),$$

where the functions $z \longmapsto \mathcal{K}_q^{1,D}(z,\mathcal{O})$, $0 \le q \le m$ are defined in Ω by

$$\mathcal{K}_q^{1,D}(z,\mathcal{O}) = \sum_{p=0}^{q} \frac{1}{p!} \int_{\partial\mathcal{O}} \sigma_y(E_{q-p}^D)(y)\mathbf{n}(y).[\sigma^{(p)} w_0^D(z)(y^p)] ds(y).$$

Lemma 10 *The second integral term in the right hand side of (14) satisfies*

$$\sum_{k=1}^{m-2} \varepsilon^{k+2} \int_{\partial\mathcal{O}} \sigma(V_k^D)(z+\varepsilon y)\mathbf{n}.w_0^D(z+\varepsilon y) ds(y) = \sum_{q=1}^{m-2} \varepsilon^{q+2} \mathcal{K}_q^{2,D}(z,\mathcal{O}) \; + \; o(\varepsilon^m),$$

where the functions $z \longmapsto \mathcal{K}_q^{2,D}(z,\mathcal{O})$, $1 \le q \le m-2$ are defined in Ω by

$$\mathcal{K}_q^{2,D}(z,\mathcal{O}) = \sum_{p=0}^{q-1} \sum_{l=0}^{p} \frac{1}{l!(p-l)!} \int_{\partial\mathcal{O}} \sigma^{(l)}(V_{q-p}^D)(z)\mathbf{n}.\sigma^{(p-l)} w_0^D(z)(y^{p-l}) ds(y)$$

Lemma 11 *The second term in (13) admits the estimate*

$$\int_{\mathcal{O}} |\sigma(w_0^D)(z+\varepsilon y)|^2 dy = \sum_{q=0}^{m} \varepsilon^q \, \mathcal{K}_q^{3,D}(z,\mathcal{O}) \, + \, o(\varepsilon^m),$$

where $\{z \longmapsto \mathcal{K}_q^{3,D}(z,\mathcal{O}),\, 0 \leq q \leq m\}$ *is a set of functions, defined in* Ω *by*

$$\mathcal{K}_q^{3,D}(z,\mathcal{O}) = \sum_{p=0}^{q} \frac{1}{p!(q-p)!} \int_{\mathcal{O}} \sigma^{(p+1)} w_0^D(z)(y^p) : \sigma^{(q-p+1)} w_0^D(z)(y^{q-p}) dy.$$

5.3 Topological Asymptotic Formula

In this section, we derive a high-order topological asymptotic expansion for the Kohn-Vogelius functional $\mathcal{K}$. The main result of this section is described by Theorem 12.

Theorem 12 *Let* $\mathcal{O}_{z,\varepsilon} = z + \varepsilon\mathcal{O}$ *be a small hole inserted in the elastic domain* Ω. *In the presence of* $\mathcal{O}_{z,\varepsilon}$, *the Kohn-Vogelius functional* $\mathcal{K}$ *satisfies the following high-order asymptotic expansion*

$$\mathcal{K}(\Omega \backslash \overline{\mathcal{O}_{z,\varepsilon}}) = \mathcal{K}(\Omega) + \sum_{q=1}^{m} \varepsilon^q \delta^q \mathcal{K}(z,\mathcal{O}) + o(\varepsilon^m),$$

where $\delta^q \mathcal{K}$ *is the* q^{th} *topological derivative order, defined in* Ω *by*

$$\delta^q \mathcal{K}(z,\mathcal{O}) = \left[\mathcal{K}_{q-1}^{1,N} - \mathcal{K}_{q-1}^{1,D}\right](z,\mathcal{O}) \quad \textit{if } q = 1,2,$$

and

$$\delta^q \mathcal{K}(z,\mathcal{O}) = \left[\mathcal{K}_{q-1}^{1,N} - \mathcal{K}_{q-1}^{1,D}\right](z,\mathcal{O}) + \left[\mathcal{K}_{q-2}^{2,N} - \mathcal{K}_{q-2}^{2,D}\right](z,\mathcal{O}) - \left[\mathcal{K}_{q-3}^{3,N} - \mathcal{K}_{q-3}^{3,D}\right](z,\mathcal{O}) \quad \textit{if } 3 \leq q \leq m.$$

Proof. According to Lemmas 5–8, the Neuman term $\mathcal{J}^N(\varepsilon)$ satisfies the estimate

$$\mathcal{J}^N(\varepsilon) = \sum_{q=0}^{m-1} \varepsilon^{q+1} \mathcal{K}_q^{1,N}(z,\mathcal{O}) + \sum_{q=1}^{m-2} \varepsilon^{q+2} \mathcal{K}_q^{2,N}(z,\mathcal{O}) - \sum_{q=0}^{m-3} \varepsilon^{q+3} \, \mathcal{K}_q^{3,N}(z,\mathcal{O}) \, + \, o(\varepsilon^m).$$

Based on Lemmas 9–11, the Dirichlet term $\mathcal{J}^D(\varepsilon)$ can be estimated as

$$\mathcal{J}^D(\varepsilon) = \sum_{q=0}^{m-1} \varepsilon^{q+1} \mathcal{K}_q^{1,D}(z,\mathcal{O}) + \sum_{q=1}^{m-2} \varepsilon^{q+2} \mathcal{K}_q^{2,D}(z,\mathcal{O}) - \sum_{q=0}^{m-3} \varepsilon^{q+3} \mathcal{K}_q^{3,D}(z,\mathcal{O}) \, + \, o(\varepsilon^m).$$

Combining the two previous estimates and using the fact that

$$\mathcal{K}(\Omega \backslash \overline{\mathcal{O}_{z,\varepsilon}}) - \mathcal{K}(\Omega) = J^N(\varepsilon) - \mathcal{J}^D(\varepsilon),$$

one can derive the desired asymptotic formula. ■

6 Conclusion

In this paper, we have derived a high-order topological asymptotic formula describing the variation of the Kohn-Vogelius functional with respect to the presence of a small hole immersed in the elastic domain. The obtained formula can serve as very useful tools for the numerical identification of the location "z", the size"ε" and the shape "$\mathcal{O}$" of the unknown geometric perturbation.

From the asymptotic formula in Theorem 12, it now follows that, up to terms of smaller order, the unknown parameters z, ε and $\mathcal{O}$ can be characterized as the solution of a parameters estimate problem minimizing the nonlinear scalar function

$$\mathcal{F}(\varepsilon, z, \mathcal{O}) = \sum_{q=1}^{m} \varepsilon^q \delta^q \mathcal{K}(z, \mathcal{O}).$$

A first task of the identification process, is then the determination of the location "z" (center of the geometric perturbation) and the size"ε". A second task would be (as well as possible) the determination of other informations about the created hole, such as its shape "$\mathcal{O}$".

A detailed account of this work and some numerical investigations will be the subject of a forthcoming paper.

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