Applied Mathematics E-Notes, 24(2024), 417-423 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

On The Existence Of Hyperbolic Algebraic Limit Cycles For A Family Of Polynomial Differential Systems*

Salah Benyoucef†, Ahmed Bendjadou‡, Rima Chouader§

Received 30 April 2023

Abstract

We exhibit a family of planar polynomial differential systems of degrees greater than or equal to three by presenting polynomial curves which generally contain closed components, and by introducing a function that is a solution of a certain partial differential equation. We discuss the existence of hyperbolic algebraic limit cycles according to the parameters of these systems. Furthermore, we introduce examples that demonstrate the applicability of our results.

1 Introduction

One of the important problems in the qualitative theory of differential equations is to solve the second part of the 16th problem out of 23 problems that Hilbert presented at the International Congress of Mathematicians in Paris (1900), as discussed in [9]. The problem is to determine the upper bound for the number of limit cycles in two-dimensional polynomial vector fields of degree n of the form

$$\begin{cases} \dot{x} = \frac{dx}{dt} = P(x,y), \\ \dot{y} = \frac{dy}{dt} = Q(x,y), \end{cases} \tag{1}$$

and to investigate their relative positions, $P(x,y)$ and $Q(x,y)$ are real polynomials of the variables x and y, and $t \in \mathbb{R}$ is an independent variable. The degree of the system is the maximum of the degrees of the polynomials P and Q. These systems arise in a great variety of applications, such as ecology and population dynamics, chemical reaction and plasma physics.

Several articles have been published on the analysis of the non-existence, existence, number, and stability of limit cycles of the system (1), see for instance [1, 2, 3, 8, 10, 11, 12, 13, 15]. Colin Christopher in his paper [4] analyses the existence of limit cycle of differential system

$$\begin{cases} \dot{x} = \alpha U + DU_y, \\ \dot{y} = \beta U - DU_x, \end{cases}$$

where α and β are real numbers, $D(x,y)$ is a polynomial of first degree and $U(x,y)$ is polynomial of degree m.

Our contribution consists in analyzing the existence of hyperbolic algebraic limit cycles for a family of polynomial differential systems of degrees greater than or equal to three where we consider first degree polynomials instead of constants α, β and a second degree polynomial instead of D.

*Mathematics Subject Classifications: 34C05, 34C07, 34C25.

†Laboratory of Applied Mathematics, Department of Mathematics, Faculty of sciences, University Ferhat Abbas setif 1, Algeria

‡Laboratory of Applied Mathematics, Department of Mathematics, Faculty of sciences, University Ferhat Abbas setif 1, Algeria

§Laboratory of Applied Mathematics, Department of Mathematics, Faculty of sciences, University Ferhat Abbas setif 1, Algeria

Let us recall some useful notions. In the phase plane, a limit cycle $\Gamma = \{(x(t), y(t)), t \in [0, T]\}$ of system (1) is an isolated periodic solution in the set of all its periodic solutions. If the limit cycle is contained in the zero of an invariant algebraic curve of the plane, then it is said to be algebraic, otherwise, is said to be non-algebraic.

For $U \in \mathbb{R}[x, y]$, the algebraic curve $U = 0$ is called an invariant curve of the polynomial system (1), if for some polynomial $K \in \mathbb{R}[x, y]$ called the cofactor of the algebraic curve, we have

$$P(x, y)\frac{\partial U}{\partial x} + Q(x, y)\frac{\partial U}{\partial y} = KU.$$

Let $\mathcal{X}$ be the polynomial vector field identifies with the differential system (1). A T-periodic solution Γ is a hyperbolic limit cycle of system (1) if $\int_0^T \operatorname{div}(\mathcal{X})(\Gamma(t))\,dt \neq 0$, where $\operatorname{div}(\mathcal{X}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$.

The curve $\Gamma = \{(x, y) \in \mathbb{R}^2, U(x, y) = 0\}$ is non-singular of system (1), i.e. it must not contain any singular point that satisfy

$$\begin{cases} P(x, y) = 0, \\ Q(x, y) = 0. \end{cases}$$

System (1) is said to be integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non-constant continuously differentiable function $H : \Omega \to \mathbb{R}$ called a first integral of this system on Ω which is constant on the trajectories of the polynomial system (1) contained in Ω, i.e. if

$$\frac{dH(x, y)}{dt} = \frac{\partial H(x, y)}{\partial x}P(x, y) + \frac{\partial H(x, y)}{\partial y}Q(x, y) = 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is arbitrary constant.

It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait and if the first integral is rational, the system (1) cannot possess a limit cycle.

2 The Mean Result

Let $\Gamma = \{(x, y) \in \mathbb{R}^2, U(x, y) = 0\}$, be a curve of degree n with $n \geq 2$. We consider a polynomial differential system of degree m with $m \geq 3$

$$\begin{cases} \dot{x} = R(x, y)U + \Phi(x, y)U_y, \\ \dot{y} = S(x, y)U - \Phi(x, y)U_x, \end{cases} \tag{2}$$

where U_x and U_y denote, respectively, the partial derivatives of U with respect to the variables x and y. $R(x, y)$ and $S(x, y)$ are polynomials of first degree and $\Phi(x, y)$ is a polynomial of second degree chosen so that they satisfy the partial differential equation

$$(R_x + S_y)\Phi - S\Phi_y - R\Phi_x = C, \tag{3}$$

where C is a real constant.

Our contribution consists of analyzing the existence of hyperbolic algebraic limit cycles according to parameters of system (2). Consider $R(x, y) = \alpha x + \sigma y$, $S(x, y) = \lambda x + \beta y$ and $\Phi(x, y) = ax^2 + by^2 + cxy + d$.

The polynomials $R(x, y)$, $S(x, y)$, $\Phi(x, y)$ satisfying the partial differential equation (3) are given by

$$\begin{aligned} R(x, y) &= \alpha x + \sigma y, \\ S(x, y) &= \lambda x + \beta y, \\ \Phi(x, y) &= -c\frac{\lambda}{\alpha - \beta}x^2 + c\frac{\sigma}{\alpha - \beta}y^2 + cxy + d, \end{aligned}$$

or

$$\begin{aligned} R(x,y) &= c\alpha x + b(\alpha-\beta)y, \\ S(x,y) &= a(\beta-\alpha)x + c\beta y, \\ \Phi(x,y) &= ax^2+by^2+cxy+d. \end{aligned}$$

Theorem 1 *Consider a differential system of degree m with $m \geq 3$*

$$\begin{cases} \dot{x} = \left(ax^2+by^2+cxy+d\right)U_y + (c\alpha x + b(\alpha-\beta)y)U, \\ \dot{y} = -\left(ax^2+by^2+cxy+d\right)U_x + (c\beta y + a(\beta-\alpha)x)U, \end{cases} \tag{4}$$

where $U = U(x,y)$ is a polynomial function of degree n with $n \geq 2$, $U_x = \frac{\partial U}{\partial x}, U_y = \frac{\partial U}{\partial y}$. The coefficients a, b, c are real numbers, α, β, d are non-zero real numbers. Choose the curve $ax^2+by^2+cxy+d=0$ that lies outside all the non-singular bounded components of curve $\Gamma = \left\{(x,y) \in \mathbb{R}^2, U(x,y)=0\right\}$ of the differential system (4). If $\alpha+\beta \neq 0$ and $c \neq 0$, then the system (4) admits all bounded components of Γ as hyperbolic algebraic limit cycles.

Proof. We assume that all bounded components of Γ are non-singular of system (4). To show that all the bounded components of Γ are hyperbolic algebraic limit cycles of system (4), we will prove that Γ is an invariant curve of the system (4), and $\int_0^T \operatorname{div}(\mathcal{X})(\Gamma(t))\,dt \neq 0$. (See for instance [14]).

1) Γ is an invariant curve of system (4):

$$\begin{aligned} \frac{dU}{dt} &= U_x\left(\left(ax^2+xy+by^2+d\right)U_y + (\alpha x + b(\alpha-\beta)y)U\right) \\ &\quad + U_y\left(-\left(ax^2+xy+by^2+d\right)U_x + (\beta y - a(\alpha-\beta)x)U\right) \\ &= U\left(x\alpha U_x + y\beta U_y - ax\alpha U_y + ax\beta U_y + by\alpha U_x - by\beta U_x\right). \end{aligned}$$

The cofactor is $K(x,y) = U_x(x\alpha + by\alpha - by\beta) + U_y(y\beta - ax\alpha + ax\beta)$.

2) $\int_0^T \operatorname{div}(\mathcal{X})(\Gamma(t))\,dt \neq 0$:

Note that

$$\int_0^T \operatorname{div}(\mathcal{X})(\Gamma(t))\,dt = \int_0^T K(x(t),y(t))dt,$$

see for instance [6, Theorem 2].

$$\begin{aligned} \int_0^T K(x(t),y(t))dt &= \oint_\Gamma \frac{(by(\alpha-\beta)+c\alpha x)U_x}{-(ax^2+by^2+cxy+d)U_x}dy + \oint_\Gamma \frac{(ax(\beta-\alpha)+c\beta y)U_y}{(ax^2+by^2+cxy+d)U_y}dx \\ &= -\oint_\Gamma \frac{(by(\alpha-\beta)+c\alpha x)}{(ax^2+by^2+cxy+d)}dy + \oint_\Gamma \frac{(ax(\beta-\alpha)+c\beta y)}{(ax^2+by^2+cxy+d)}dx. \end{aligned}$$

By applying Green's formula, we obtain

$$\begin{aligned} & \oint_\Gamma \frac{(ax(\beta-\alpha)+c\beta y)}{(ax^2+by^2+cxy+d)}dx - \oint_\Gamma \frac{(by(\alpha-\beta)+c\alpha x)}{(ax^2+by^2+cxy+d)}dy \\ =& \int\int_{int(\Gamma)} \left(\frac{\partial\left(\frac{(ax(\beta-\alpha)+c\beta y)}{(ax^2+by^2+cxy+d)}\right)}{\partial y} + \frac{\partial\left(\frac{(by(\alpha-\beta)+c\alpha x)}{(ax^2+by^2+cxy+d)}\right)}{\partial x} \right) dxdy \\ =& \; cd\int\int_{int(\Gamma)} \frac{\alpha+\beta}{(ax^2+cxy+by^2+d)^2}dxdy. \end{aligned}$$

As $c \neq 0$, $d \neq 0$, $\alpha+\beta \neq 0$ and the curve $ax^2+cxy+by^2+d=0$ lies outside Γ, the integral $\int_0^T K(x(t),y(t))dt$ is non-zero. ■

Remark 1 *If $c^2 - 4ab \leq 0$ and $bd > 0$, the set $\left\{(x,y) \in \mathbb{R}^2 : ax^2 + cxy + by^2 + d = 0\right\}$ is empty and the condition that the curve $ax^2+cxy+by^2+d = 0$ must lie outside Γ can be removed. So, we have the particular case, if $c \neq 0$, $\alpha + \beta \neq 0$, $c^2 - 4ab \leq 0$ and $bd > 0$ the system (4) admits all bounded components of Γ as hyperbolic algebraic limit cycles.*

Now, we will analyze a general form with more parameters. Consider

$$\begin{aligned} R(x,y) &= \alpha x + \beta y + \gamma, \\ S(x,y) &= \lambda x + \sigma y + \mu, \\ \Phi(x,y) &= \left(-\alpha \tfrac{\lambda}{\alpha-\sigma}\right) x^2 + \alpha xy + \left(-\tfrac{\alpha}{(\alpha\sigma-\beta\lambda)(\alpha-\sigma)} \left(\alpha\sigma\mu - \alpha^2\mu + \alpha\lambda\gamma + \sigma\lambda\gamma - 2\beta\lambda\mu\right)\right) x \\ &\quad + \alpha \tfrac{\beta}{\alpha-\sigma} y^2 + \tfrac{\alpha}{(\alpha\sigma-\beta\lambda)(\alpha-\sigma)} \left(\alpha\sigma\gamma - \sigma^2\gamma + \alpha\beta\mu + \sigma\beta\mu - 2\beta\lambda\gamma\right) y + h. \end{aligned} \tag{5}$$

The polynomials $R(x,y)$, $S(x,y)$, $\Phi(x,y)$ satisfying the partial differential equation (3).

Theorem 2 *Consider a differential system of degree m with $m \geq 3$,*

$$\begin{cases} \dot{x} = \Phi U_y + RU, \\ \dot{y} = -\Phi U_x + SU, \end{cases} \tag{6}$$

where $U = U(x,y)$ is a polynomial function of degree n with $n \geq 2$, $U_x = \frac{\partial U}{\partial x}$, $U_y = \frac{\partial U}{\partial y}$, R, S, Φ as defined in (5), where α, β, γ, λ, σ, μ, h are non-zero real numbers, $\alpha - \sigma \neq 0$ and $\alpha\sigma - \beta\lambda \neq 0$. Choose the curve $\Phi(x,y) = 0$ that lies outside all the non singular bounded components of curve $\Gamma = \left\{(x,y) \in \mathbb{R}^2, U(x,y) = 0\right\}$ of the differential system (6). If $\alpha+\sigma \neq 0$ and $h \neq \alpha \frac{\beta\mu^2 - \lambda\gamma^2 + \alpha\gamma\mu - \sigma\gamma\mu}{(\alpha-\sigma)(\alpha\sigma-\beta\lambda)}$ then the system (6) admits all bounded components of Γ as hyperbolic algebraic limit cycles.

Proof. Assume that all bounded components of Γ are non-singular of system (6). It easy to see that Γ is an invariant curve of system (6),

$$\frac{dU}{dt} = U_x\left(\Phi U_y + RU\right) + U_y\left(-\Phi U_x + SU\right) = U\left(RU_x + SU_y\right).$$

The cofactor is $K(x(t),y(t)) = RU_x + SU_y$ and

$$\begin{aligned} \int_0^T K(x(t),y(t))dt &= \oint_\Gamma \frac{RU_x}{-\Phi U_x} dy + \oint_\Gamma \frac{SU_y}{\Phi U_y} dx \\ &= \oint_\Gamma \frac{S}{\Phi} dx - \oint_\Gamma \frac{R}{\Phi} dy \\ &= \int\int_{int(\Gamma)} \left(\frac{S_y\Phi - S\Phi_y}{\Phi^2} + \frac{R_x\Phi - R\Phi_x}{\Phi^2}\right) dxdy \\ &= \int\int_{int(\Gamma)} \left(\frac{\left(R_x + S_y\right)\Phi - S\Phi_y - R\Phi_x}{\Phi^2}\right) dxdy. \end{aligned}$$

$$\left(R_x + S_y\right)\Phi - S\Phi_y - R\Phi_x = -\frac{(\alpha+\sigma)}{(\alpha\sigma-\beta\lambda)(\alpha-\sigma)}\left(-(\alpha-\sigma)(\alpha\sigma-\beta\lambda)h + \alpha\left(\beta\mu^2 - \lambda\gamma^2 + \alpha\gamma\mu - \sigma\gamma\mu\right)\right).$$

As $\alpha-\sigma \neq 0$, $(\alpha\sigma - \beta\lambda) \neq 0$, $\alpha+\sigma \neq 0$, $h \neq \alpha \frac{\beta\mu^2 - \lambda\gamma^2 + \alpha\gamma\mu - \sigma\gamma\mu}{(\alpha-\sigma)(\alpha\sigma-\beta\lambda)}$ and the curve $\Phi(x,y) = 0$ lies outside Γ, the integral $\int_0^T K(x(t),y(t))dt$ is non-zero. So the system admits all bounded components of Γ as hyperbolic algebraic limit cycles. ■

Remark 2 *If $\alpha+\sigma=0$ the system (6) becomes*

$$\begin{cases} \dot{x}=\Phi_y U-\Phi U_y, \\ \dot{y}=-\Phi_x U+\Phi U_x, \end{cases} \tag{7}$$

where

$$\Phi\left(x,y\right)=-\frac{1}{2}\lambda x^2+\frac{1}{2}\beta y^2+\alpha xy-\mu x+\gamma y+h.$$

The system (7) is Hamiltonian with rational first integral

$$H(x,y)=\frac{\Phi\left(x,y\right)}{U(x,y)},$$

and it has no limit cycle.

3 Examples

Example 1 *Let $\alpha=\beta=a=b=c=d=1$ in system (4), $U(x,y)=\left(x^2+y^2-5\right)^2-xy^2+x^2y+1$. The quintic differential system*

$$\begin{cases} \dot{x}=\left(x^2+y^2+xy+1\right)\left(4x^2y+x^2-2xy+4y^3-20y\right)+x\left(\left(x^2+y^2-5\right)^2-xy^2+x^2y+1\right), \\ \dot{y}=-\left(x^2+y^2+xy+1\right)\left(4x^3+4xy^2+2xy-20x-y^2\right)+y\left(\left(x^2+y^2-5\right)^2-xy^2+x^2y+1\right), \end{cases} \tag{8}$$

admits three hyperbolic algebraic limit cycles represented by the curve

$$\Gamma_1=\left\{(x,y)\in\mathbb{R}^2:\left(x^2+y^2-5\right)^2-xy^2+x^2y+1=0\right\}.$$

Each bounded component of the curve Γ_1 which is a limit cycle, surrounds a stable focus. See Figure 1.

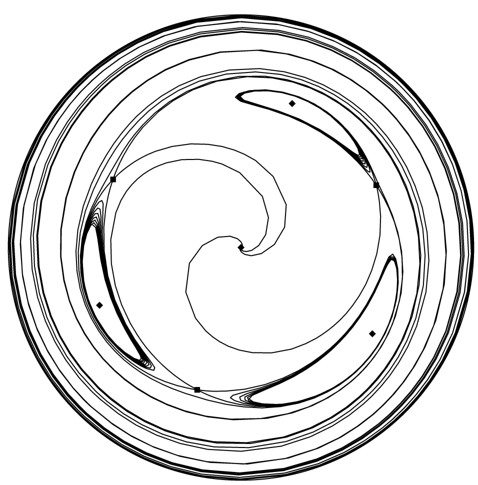

Figure 1: The phase portrait with limit cycles of the polynomial differential system (8) in the Poincaré disk.

Example 2 *Let $\alpha = \beta = \gamma = \lambda = \mu = 1, \sigma = h = 2$ in system (6), $U(x,y) = 2x^2 + y^2 - 8x - 4y + 11$. The cubic differential system*

$$\begin{cases} \dot{x} = 2x^3 + 4x^2y - 10x^2 + 3xy^2 - 12xy - 5x - y^3 + 3y^2 + 7y + 3, \\ \dot{y} = -2x^3 - 6x^2 + 5xy^2 - 16xy + 11x + 2y^3 - 15y^2 + 26y + 27, \end{cases} \tag{9}$$

admits a limit cycle represented by the curve $\Gamma_2 = \left\{(x,y) \in \mathbb{R}^2 / \ 2x^2 + y^2 - 8x - 4y + 11 = 0\right\}$. See Figure 2.

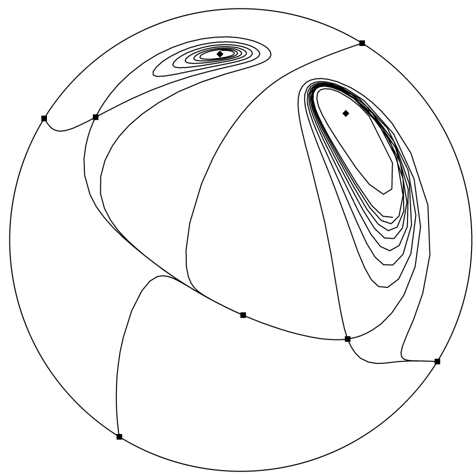

Figure 2: The phase portrait with limit cycle of the polynomial differential system (9) in the Poincaré disk.

4 Conclusion

We have proposed a family of planar polynomial differential systems of degree m with $m \geq 3$. We may check certain conditions on the parameters of the system to conclude directly the existence of hyperbolic algebraic limit cycles.

Note: All figures are plotted on the Poincaré disk by using a program for polynomial planar phase portraits, such as the one described in [5, Pages 233-257].

Acknowledgment. The authors would like to express their gratitude to the referee for his valuable remarks and directions. This work has been realized thanks to the: Directorate-General for Scientific Research and Technological Development (DGRSDT) and the research project under code : PRFU COOLO3UN190120180007, both based in Algeria's Ministry of Higher Education and scientific research (MESRS).

References

[1] S. Benyoucef, Polynomial differential systems with hyperbolic algebraic limit cycles, Electron. J. Qual. Theory Differ. Equ., 34(2020), 7 pp.

[2] T. R. Blows and N. G. Lloyd, The number of limit cycles of certain polynomial differential equations, Proc. Roy. Soc. Edinburgh Sect. A, 98(1984), 215–239.

[3] J. Chavarriga, H. Giacomini and J. Giné, On a new type bifurcation of limit cycles for a planar cubic systems, Nonlinear Anal., 36(1999), Ser. A: Theory Methods, 139–149.

[4] C. Christopher, Polynomial vector fields with prescribed algebraic limit cycles, Geom. Dedicata, 88(2001), 255–258.

[5] F. Dumortier, J. Llibre and J. Artés, Qualitative Theory of Planar Differential Systems, (Universitex) Berlin. Springer. 2006.

[6] H. Giacomini and M. Grau, On the stability of limit cycles for planar differential systems, J. Differential Equations, 213(2005), 368–388.

[7] H. Giacomini, J. Llibre and M. Viano, On the nonexistence, existence, and uniqueness of limit cycles, Nonlinearity, 9(1996), 501–516.

[8] J. Gine, Non-existence of limit cycles for planar vector fields, Electron. J. Differential Equations, 75(2014), 8 pp.

[9] D. Hilbert, Mathematical problems, Bull. Amer. Math. Soc., 8(1902), 437–479.

[10] J. Llibre, D. D. Novaes and I. O. Zeli, Limit cycles of piecewise polynomial perturbations of higher dimensional linear differential systems, Rev. Mat. Iberoam., 36(2020), 291–318.

[11] J. Llibre and C. Valls, Normal forms and hyperbolic algebraic limit cycles for a class of polynomial differential systems, Electron. J. Differential Equations, 83(2018), 7 pp.

[12] J. Llibre and X. Zhang, The non-existence, existence and uniqueness of limit cycles for quadratic polynomial differential systems, Proc. Roy. Soc. Edinburgh Sect. A, 149(2019), 1–14.

[13] Llibre J, Zhao Y. Algebraic Limit Cycles in Polynomial Systems of Differential Equations. Journal of Physics A Mathematical and Theoretical 2007; 40 (47): 14207-14222.

[14] L. Perko, Differential Equations and Dynamical Systems, Third edition. Texts in applied mathematics. 7. Springer-Verlag. New York. 2006.

[15] X. Zhang, The 16th Hilbert problem on algebraic limit cycles, J. Differential Equations, 251(2011), 1778–1789.