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Upper-Lower Bounds For Blow-Up Time In Initial Value Boundary Problems For A Class Of Pseudo-Parabolic Equations*

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Abstract

The initial boundary value problem of a class of pseudo-parabolic equations is considered. By means of a differential inequality technique, we prove that the solutions become unbounded at a finite time T, and find an upper bound for this time with negative initial energy. Also, a lower bound for blow-up time is determined.

1 Introduction

In this paper, we consider the following pseudo-parabolic equation

$$\begin{cases} v_t - \mu\Delta v_t - \operatorname{div}(A(x,t)\,|\nabla v|^{r(x)-2}\,\nabla v) = |v|^{s(x)-2}\,v, & \text{in } \Omega\times(0,\infty), \\ v(x,t) = 0, & \text{on } \partial\Omega\times(0,\infty), \\ v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n > 1$, with smooth boundary $\partial\Omega$. The nonlinear term

$$\operatorname{div}(A\,|\nabla v|^{r(x)-2}\,\nabla v)$$

is the so-called $r(x)$-Laplace operator with the presence of a matrix $A(x,t)$. The term with a variable exponent $|v|^{s(x)-2}\,v$ plays the role of a source, and the dissipative term Δv_t is a linear strong damping term.

The matrix $A = (a_{ij}(x,t))_{i,j}$ where a_{ij} is a function of class $C^1(\overline{\Omega}\times[0,\infty[)$ and there exists a constant $a_0 > 0$ such that, for all $(x,t)\in\overline{\Omega}\times[0,\infty[$ and $\xi\in\mathbb{R}^n$, we have

$$A\xi.\xi \geq a_0\,|\xi|^2 \tag{2}$$

and

$$A'\xi.\xi \leq 0, \tag{3}$$

where $A' = \frac{\partial A}{\partial t}(.,t)$. The exponents $r(.)$ and $s(.)$ are given continuous functions defined on $\overline{\Omega}$ and satisfy

$$2 < r_- \leq r(x) \leq r_+ < s_- \leq s(x) \leq s_+ < \infty, \tag{4}$$

where

$$r_- = ess\ \inf r(x), \quad r_+ = ess\ \sup r(x),$$
$$s_- = ess\ \inf\ s(x), \quad s_+ = ess\ \sup s(x),$$

and the Zhikov–Fan conditions:

$$|r(x)-r(y)| = \frac{-a}{\log|x-y|} \text{ and } |s(x)-s(y)| = \frac{-b}{\log|x-y|} \text{ for all } x,y\in\Omega \text{ with } |x-y|<\delta, \tag{5}$$

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where $a, b > 0$ and $0 < \delta < 1$.

Problem (1) occurs in the mathematical modeling of various physical phenomena, e.g., the flows of electrorheological fluids, nonlinear viscoelasticity, fluids with temperature-dependent viscosity, processes of filtration through a porous media and image processing, and so on. See [1, 2, 10, 11]. Obviously, if $\mu = 1$, $r(x) = 2$, $A = I_n$, $s(x) = s$, then Eq (1) reduces to the following pseudo-parabolic equation

$$v_t - \Delta v_t - \Delta v = |v|^{s-2} v, \quad \text{in } \Omega \times (0, T). \tag{6}$$

In their work [3], Xu and Su proved that the solutions to the problem (6) blow up in a finite time in $H_0^1(\Omega)$-norm. In another study [4], Luo considered the same problem treated in the work of Xu and Su [3], and he obtained an upper bound and a lower bound of the blowup rate. In [5], Di et al. considered the following nonlinear equation

$$v_t - \Delta v_t - \operatorname{div}(|\nabla v|^{r(x)-2} \nabla v) = |v|^{s(x)-2} v, \quad \text{in } \Omega \times (0, T), \tag{7}$$

which is just the $A = I_n$ case of (1). By using differential inequality techniques, they obtained an upper bound and a lower bound for the blow-up time of the solution to the problem (7). On the other hand, Wang and Xu [6] considered the following nonlocal semilinear pseudo-parabolic equation

$$\begin{cases} u_t - \Delta u - \Delta u_t = |u|^{p-1} u - \int_\Omega |u|^{p-1} u dx & x \in \Omega,\ t > 0, \\ u(x, 0) = u_0(x) \neq 0 & \text{in } \Omega, \\ \int_\Omega u_0 dx = \frac{1}{|\Omega|} \int_\Omega u_0 dx = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta}(x, t) = 0 & \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{8}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open domain with smooth boundary, with $p > 1$ if $n = 1, 2$ and $1 < p < \frac{n+2}{n-2}$ if $n \geq 3$, and they proved the existence, uniqueness asymptotic behavior of the global solution and blow-up phenomena of solution with subcritical initial energy.

In the absence of the damped term ($\mu = 0$), Xu et al. in [7] studied the coupled parabolic systems

$$\begin{cases} u_t - \Delta u = \left(|u|^{2p} + |v|^{p+1} |u|^{p-1} \right) u, \\ v_t - \Delta v = \left(|v|^{2p} + |u|^{p+1} |v|^{p-1} \right) v, \end{cases} \tag{9}$$

with Dirichlet boundary conditions. By introducing a family of potential wells, the whole study is conducted by considering the following three cases according to initial energy: low, critical, and high initial energy cases. Under the condition $J(u_0, v_0) < d$, where d is a depth of potential well associated with the energy functional

$$J(u, v) = \frac{1}{2} \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right) - \frac{1}{2(p+1)} \|u\|_{2p+2}^{2p+2} + \|uv\|_{p+1}^{p+1} + \|v\|_{2p+2}^{2p+2},$$

they obtained the global existence and finite time blowup of the solution for the problem (9). On the other side, if $J(u_0, v_0) = d$ they proved the global solution, blow-up solution, and asymptotic behavior of the problem (9). With the high initial energy level $J(u_0, v_0) > d$, by adopting the comparison principle of the coupled parabolic systems, they gave sufficient conditions to obtain the finite time blow-up anf global existence of the solution. It is worth mentioning some other literature concerning the theory of our type equation, namely, [8, 9, 12, 13].

Based on the above-mentioned work and motivated by [5, 6], this paper aims to find an upper bound for blow-up time if the variable exponents $r(.), s(.)$, the initial data and the matrix $A(., t)$ satisfy some conditions. Also, we will give the lower bounds on blow-up time under some other conditions for the problem (1).

The outline of this paper is as follows. In Section 2, we recall the definitions of the variable exponent Lebesgue spaces $L^{p(.)}(\Omega)$, the Sobolev spaces $W^{1,p(.)}(\Omega)$, as well as some of their properties. In Section 3 and Section 4, we give a study of the blow-up of solutions to the problem under consideration.

2 Preliminaries

In this section, we present some material needed for the statement and proof of our results. In what follows, we give definitions and properties related to Lebesgue and Sobolev spaces with variable exponents. Let Ω be a domain of $\mathbb{R}^n$ and $p : \Omega \longrightarrow [1,\infty)$ be a measurable function. The Lebesgue space $L^{p(\cdot)}(\Omega)$, with variable exponent $p(.)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} | u \text{ is measurable in } \Omega \text{ and } \int_\Omega |\lambda u(x)|^{p(x)}\, dx < \infty \text{ for some } \lambda > 0 \right\}.$$

The Luxemburg-type norm is given by

$$\|u\|_{p(.)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

We notice that variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many aspects: they are Banach spaces, the Hölder inequality holds, and they are reflexive if $1 < p(x) < \infty$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \nabla u \text{ exists and } \ \nabla u \in L^{p(\cdot)}(\Omega) \right\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

The space $W_0^{1,p(\cdot)}(\Omega)$ is defined to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. The definition of the space $W_0^{1,p(\cdot)}(\Omega)$ in the constant exponent case is usually different. However, under condition (5) both definitions coincide (See [14]). The dual space $W_0^{-1,p'(\cdot)}(\Omega)$ of $W_0^{1,p(\cdot)}(\Omega)$ is defined in the same way as in the classical Sobolev spaces, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Lemma 1 (Poincare's inequality, [14]) *Suppose that $p(.)$ satisfies (5). Then,*

$$\|u\|_{p(.)} \le C \|\nabla u\|_{p(.)}, \quad u \in W_0^{1,p(\cdot)}(\Omega)$$

where $C > 0$ is a constant that depends only on $p(.)$ and Ω.

Lemma 2 (Embedding Proprety, [14]) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$. If $q \in C(\overline{\Omega})$ such that $q \ge 1$ and $q(x) < 2^*$ in $\overline{\Omega}$ with*

$$2^* = \begin{cases} \frac{2n}{n-2} & \text{if } n > 2, \\ \infty & \text{if } n \le 2, \end{cases}$$

then we have continuous and compact embedding $H_0^1(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$. So, there exists $C > 0$ such that

$$\|u\|_{L^{q(\cdot)}(\Omega)} \le C \|u\|_{H_0^1(\Omega)}.$$

3 Upper Bound for Blow-Up Time

We first start with the following existence and uniqueness of local solution for the problem (1), which can be obtained by using Faedo-Galerkin methods as in [1, 15, 16]. Here, the proof is thus omitted. For simplicity, we set $\mu = 1$.

Theorem 1 *Let $v_0 \in W_0^{1,r(.)}(\Omega) \cap L^{s(.)}(\Omega)$ be given. Assume that the conditions on $s(x), r(x)$, and A, given in Section 1, hold. Then, problem (1) has a unique local solution v on $[0, T_0)$*

$$v \in L^{\infty}([0,T_0]; W_0^{1,r(.)}(\Omega) \cap L^{s(.)}(\Omega)), \qquad v_t \in L^2([0,T_0]; W_0^{1,2}(\Omega))$$

for some $T_0 > 0$, satisfying

$$(v_t, w) + (\nabla v_t, \nabla w) + (A\,|\nabla v|^{r(x)-2}\,\nabla v, \nabla w) = (|v|^{s(x)-2}\,v, w), \quad \text{for all } w \in W_0^{1,r(.)}(\Omega) \cap L^{s(.)}(\Omega). \tag{10}$$

Moreover, the following alternatives hold

(i) $T_0 = +\infty$, *or*

(ii) $T_0 < +\infty$ *and* $\lim\limits_{t \longrightarrow T} \|\nabla u\|_2^2 + \|u\|_2^2 = +\infty$.

Remark 1 *It is easy to see, under the condition (4) that $|v|^{s(x)-2}\,v$, $A\,|\nabla v|^{r(x)-2}\,\nabla v \in L^2(\Omega)$; hence $(|v|^{s(x)-2}\,v, w)$ and $(A\,|\nabla v|^{r(x)-2}\,\nabla v, \nabla w)$ make sense in formula (10).*

The decay of the energy of the system (1) is given in the following lemma:

Lemma 3 *The energy functional E of the problem (1) is a decreasing function. Here*

$$E(t) = \int_\Omega \frac{1}{r(x)} A\,|\nabla v|^{r(x)-2}\,\nabla v.\nabla v dx - \int_\Omega \frac{1}{s(x)}\,|v|^{s(x)}\,dx. \tag{11}$$

Proof. It is enough to multiply the first equation in (1) by v_t and integrate over Ω, to obtain

$$\int_\Omega v_t v_t dx - \int_\Omega \Delta v_t v_t dx - \int_\Omega \operatorname{div}\left(A\,|\nabla v|^{r(x)-2}\,\nabla v\right) v_t dx = \int_\Omega |v|^{s(x)-2}\,v v_t dx.$$

Then, we use the generalized Green formula and the boundary conditions, to find

$$\int_\Omega \left(|v_t|^2 + |\nabla v_t|^2\right) dx + \int_\Omega A\,|\nabla v|^{r(x)-2}\,\nabla v.\nabla v_t dx = \frac{d}{dt}\int_\Omega \frac{1}{s(x)}\,|v|^{s(x)}\,dx.$$

This implies that

$$\begin{aligned} &\int_\Omega \left(|v_t|^2 + |\nabla v_t|^2\right) dx + \frac{d}{dt}\int_\Omega \frac{1}{r(x)} A\,|\nabla v|^{r(x)-2}\,\nabla v.\nabla v dx - \int_\Omega \frac{1}{r(x)} A'\,|\nabla v|^{r(x)-2}\,\nabla v.\nabla v dx \\ = \ &\frac{d}{dt}\int_\Omega \frac{1}{s(x)}\,|v)|^{s(x)}\,dx. \end{aligned}$$

So

$$E'(t) = -\int_\Omega \left(|v|^2 + |\nabla v_t|^2\right) dx + \int_\Omega \frac{1}{r(x)} A'\,|\nabla v|^{r(x)-2}\,\nabla v.\nabla v dx. \tag{12}$$

Taking into account condition (3) on A', we find

$$E'(t) \le -\int_\Omega \left(|v_t|^2 + |\nabla v_t|^2\right) dx \le 0. \tag{13}$$

■

Theorem 2 *Assume that (2)–(5) hold. Let v be a solution of (1) and assume that $v_0 \in W_0^{1,r(.)}(\Omega) \cap L^{s(.)}(\Omega)$ satisfies*

$$\int_\Omega \frac{1}{s(x)} |v_0|^{s(x)} dx - \int_\Omega \frac{1}{r(x)} A(x,0) |\nabla v_0|^{r(x)-2} \nabla v_0.\nabla v_0 dx \geq 0. \tag{14}$$

Then the solution v blow up at finite time $T_{\max} > 0$ in $H_0^1(\Omega)$-norm. In addition, there exists an upper bound for the time given by

$$T_{\max} \leq \frac{2\,(G(0))^{\left(\frac{2-r_-}{2}\right)}}{(r_- - 2)\,K} \tag{15}$$

where K is a suitable positive constant and the constant $G(0) = \|v_0\|^2_{H_0^1(\Omega)}$.

Proof. Let us define the auxiliary function

$$G(t) = \|v\|^2_{H_0^1(\Omega)} = \int_\Omega v^2 dx + \int_\Omega |\nabla v|^2 dx. \tag{16}$$

Our goal is to show that G satisfies a differential inequality which leads to blow up in finite time. Multiply (1) by v and integrate over Ω to get

$$\int_\Omega v v_t dx + \int_\Omega \nabla v \nabla v_t dx = \int_\Omega |v|^{s(x)} dx - \int_\Omega A |\nabla v|^{r(x)-2} \nabla v.\nabla v dx. \tag{17}$$

Now differentiate $G(t)$ with respect to t to obtain

$$\begin{aligned} & G'(t) \\ = \;& 2\int_\Omega (v v_t dx + \nabla v \nabla v_t)\, dx = 2\int_\Omega \left(|v|^{s(x)} - A |\nabla v|^{r(x)-2} \nabla v.\nabla v\right) dx \\ = \;& 2\int_\Omega \left(s(x) \left(\frac{|v|^{s(x)}}{s(x)} - \frac{A |\nabla v|^{r(x)-2} \nabla v.\nabla v}{r(x)} \right) + s(x) \left(\frac{1}{r(x)} - \frac{1}{s(x)} \right) A |\nabla v|^{r(x)-2} \nabla v.\nabla v \right) dx. \end{aligned} \tag{18}$$

By (14) and the fact that $E(t) \leq E(0)$ ($E'(t) \leq 0$), we have

$$\begin{aligned} & \int_\Omega s(x) \left[\frac{|v|^{s(x)}}{s(x)} - \frac{A |\nabla v|^{r(x)-2} \nabla v.\nabla v}{r(x)} \right] dx \geq \int_\Omega s(x) \left[\frac{|v_0|^{s(x)}}{s(x)} - \frac{A(x,0) |\nabla v_0|^{r(x)-2} \nabla v_0.\nabla v_0}{r(x)} \right] dx \\ \geq \;& s_- \int_\Omega \left[\frac{|v_0|^{s(x)}}{s(x)} - \frac{A(x,0) |\nabla v_0|^{r(x)-2} \nabla v_0.\nabla v_0}{r(x)} \right] dx \geq 0. \end{aligned} \tag{19}$$

By (18) and (19), we see

$$G'(t) \geq 2\int_\Omega s_- \left(\frac{1}{r_+} - \frac{1}{s_-} \right) A |\nabla v|^{r(x)-2} \nabla v.\nabla v dx.$$

Using condition (2) on A, we obtain

$$G'(t) \geq a_0 C_0 \int_\Omega |\nabla v|^{r(x)} dx, \tag{20}$$

where $C_0 = 2\ s_- \left(\frac{1}{r_+} - \frac{1}{s_-} \right) > 0.$

Now we define the sets $\Omega_+ = \{x \in \Omega : |\nabla v| \geq 1\}$ and $\Omega_- = \{x \in \Omega : |\nabla v| < 1\}$. By using the fact that $\|v\|_2 \leq C\|v\|_r$ for all $r > 2$, we get

$$\begin{aligned} G^{'}(t) &\geq a_0 C_0 \left(\int_{\Omega_-} |\nabla v|^{r_+} dx + \int_{\Omega_+} |\nabla v|^{r_-} dx \right) \\ &\geq C_1 \left(\left(\int_{\Omega_-} |\nabla v|^2 dx \right)^{\frac{r_+}{2}} + \left(\int_{\Omega_+} |\nabla v|^2 dx \right)^{\frac{r_-}{2}} \right). \end{aligned}$$

This implies that

$$\left(G^{'}(t)\right)^{\frac{2}{r_+}} \geq C_2 \left(\int_{\Omega_-} |\nabla v|^2 dx \right) \quad \text{and} \quad \left(G^{'}(t)\right)^{\frac{2}{r_-}} \geq C_3 \left(\int_{\Omega_+} |\nabla v|^2 dx \right). \tag{21}$$

The Poincare inequality gives $\|\nabla v\|_2^2 \geq \lambda \|v\|_2^2$, where λ is the first eigenvalue of $(-\Delta)$. Therefore, we get

$$\|\nabla v\|_2^2 = \frac{1}{1+\lambda} \|\nabla v\|_2^2 + \frac{\lambda}{1+\lambda} \|\nabla v\|_2^2 \geq \frac{\lambda}{1+\lambda} \|v\|_2^2 + \frac{\lambda}{1+\lambda} \|\nabla v\|_2^2 = \frac{\lambda}{1+\lambda} \|v\|_{H_0^1(\Omega)}^2. \tag{22}$$

It follows from (21) and (22) that

$$\left(G^{'}(t)\right)^{\frac{2}{r_+}} + \left(G^{'}(t)\right)^{\frac{2}{r_-}} \geq (C_2 + C_3) \|\nabla v\|_2^2 \geq \frac{(C_2 + C_3)\lambda}{1+\lambda} \|v\|_{H_0^1(\Omega)}^2 = C_4 G(t). \tag{23}$$

Since we have $G(t) \geq G(0) > 0$ (because $G^{'}(t) \geq 0$), and from (23), we get

$$\left(G^{'}(t)\right)^{\frac{2}{r_+}} \geq \frac{C_4}{2} G(t) \geq \frac{C_4}{2} G(0) \quad \text{or} \quad \left(G^{'}(t)\right)^{\frac{2}{r_-}} \geq \frac{C_4}{2} G(t) \geq \frac{C_4}{2} G(0). \tag{24}$$

This implies that

$$G^{'}(t) \geq C_5 (G(0))^{\frac{r_+}{2}} \quad \text{or} \quad G^{'}(t) \geq C_5 (G(0))^{\frac{r_-}{2}}.$$

Now put $\beta = \min\left\{ C_5 (G(0))^{\frac{r_+}{2}},\ C_5 (G(0))^{\frac{r_-}{2}} \right\}$, then we get

$$G^{'}(t) \geq \beta. \tag{25}$$

(23) implies that

$$\left(G^{'}(t)\right)^{\frac{2}{r_-}} \left(1 + \left(G^{'}(t)\right)^{2\left(\frac{1}{r_+} - \frac{1}{r_-}\right)} \right) \geq C_4 G(t). \tag{26}$$

From (4), we observe that $2\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \leq 0$. Making use of (25), we get

$$G^{'}(t) \geq K (G(t))^{\frac{r_-}{2}} \tag{27}$$

where $K = \left(\frac{C_4}{1 + \beta^{2\left(\frac{1}{r_+} - \frac{1}{r_-}\right)}} \right)^{\frac{r_-}{2}}$ is a positive constant. Integrating (27) from 0 to t we get

$$G(t) \geq \frac{1}{\left((G(0))^{1-\frac{r_-}{2}} + \frac{(2-r_-)Kt}{2} \right)^{\frac{2}{r_- - 2}}}$$

which implies that $G(t) \longrightarrow \infty$ as $t \longrightarrow T_{\max}$ in $H_0^1(\Omega)$, where

$$T_{\max} \leq \frac{2\,(G(0))^{\left(\frac{2-r_-}{2}\right)}}{(r_- - 2)\,K}.$$

Consequently, the solution to the problem (1) blows up in finite time. Hence the proof is completed. ■

4 Lower Bound for Blow-Up Time

In this section, we determine a lower bound for the blow-up time of the problem (1).

Theorem 3 *Suppose that the conditions on $s(x)$, $r(x)$, and A, given in Section 1, hold. Furthermore assume that $2 < s_+ < \infty$ if $n \leq 2$, $2 < s_+ < \frac{2n}{n-2}$ if $n > 2$, $v_0 \in W_0^{1,r(.)}(\Omega) \cap L^{s(.)}(\Omega)$ and v be a blow-up solution of problem (1), then a lower bound for blow-up time $T_{\min}$ can be estimated in the form*

$$T_{\min} \geq \int_{G(0)}^{\infty} \frac{d\xi}{2\max(C_-^{s_+}, C_+^{s_-})\left(\xi^{\frac{s_+}{2}} + \xi^{\frac{s_-}{2}}\right)}, \tag{28}$$

where C_-, C_+ are the optimal constants satisfying the Sobolev embedding inequalities

$$\|u\|_{L^{s_-}} \leq C_- \|\nabla u\|_2 \quad \text{and} \quad \|u\|_{L^{s_+}} \leq C_+ \|\nabla u\|_2\,, \quad \text{respectively.}$$

Proof. Consider $G(t)$ as in (16)

$$G(t) = \|v\|^2_{H_0^1(\Omega)}\,.$$

Multiply (1) by v and integrate over Ω to get

$$\int_\Omega v v_t dx + \int_\Omega \nabla v \nabla v_t dx = \int_\Omega |v|^{s(x)}\,dx - \int_\Omega A\,|\nabla v|^{r(x)-2}\,\nabla v.\nabla v dx.$$

A direct differentiation of $G(t)$ yields

$$G'(t) = 2\int_\Omega (v v_t + \nabla v \nabla v_t)\,dx,$$

then

$$G'(t) = 2\left[\int_\Omega |v|^{s(x)}\,dx - \int_\Omega A\,|\nabla v|^{r(x)-2}\,\nabla v.\nabla v dx\right].$$

Taking into account condition (2) on A, we find

$$G'(t) \leq 2\int_\Omega |v|^{s(x)}\,dx. \tag{29}$$

Defining the sets

$$\Omega_+ = \{x \in \Omega : |v| \geq 1\} \quad \text{and } \Omega_- = \{x \in \Omega : |v| < 1\}\,.$$

Thus, we have

$$\begin{aligned}
\int_\Omega |v|^{s(x)}\,dx &= \int_{\Omega_+} |v|^{s(x)}\,dx + \int_{\Omega_-} |v|^{s(x)}\,dx \\
&\leq \int_{\Omega_+} |v|^{s_+}\,dx + \int_{\Omega_-} |v|^{s_-}\,dx \\
&\leq \int_\Omega |v|^{s_+}\,dx + \int_\Omega |v|^{s_-}\,dx.
\end{aligned}$$

By the Sobolev embeddings (Lemma 2), we have

$$\begin{aligned}\int_\Omega |v|^{s(x)}\,dx &\leq C_+^{s_+}\left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{s_+}{2}} + C_-^{s_-}\left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{s_-}{2}} \\ &\leq \max(C_-^{s_+}, C_+^{s_-})\left(\left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{s_+}{2}} + \left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{s_-}{2}}\right) \\ &\leq \max(C_-^{s_+}, C_+^{s_-})\left((G(t))^{\frac{s_+}{2}} + (G(t))^{\frac{s_-}{2}}\right) \qquad (30)\end{aligned}$$

where C_- and C_+ are the corresponding embedding constants. Therefore, (29) becomes

$$G^{'}(t) \leq 2\max(C_-^{s_+}, C_+^{s_-})\left((G(t))^{\frac{s_+}{2}} + (G(t))^{\frac{s_-}{2}}\right). \qquad (31)$$

By integrating both sides of the last inequality over $(0,T)$, we obtain

$$\int_{G(0)}^{G(t)} \frac{d\xi}{2\max(C_-^{s_+}, C_+^{s_-})\left(\xi^{\frac{s_+}{2}} + \xi^{\frac{s_-}{2}}\right)} \leq T.$$

If v blow-up in H_0^1-norm, then we establish a lower bound for $T_{\min}$ by the form

$$T_{\min} \geq \int_{G(0)}^{\infty} \frac{d\xi}{2\max(C_-^{s_+}, C_+^{s_-})\left(\xi^{\frac{s_+}{2}} + \xi^{\frac{s_-}{2}}\right)},$$

which is the desired result. ∎

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