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On Extensions And Generalizations Of Rivlin's Inequality*

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Abstract

In 1960, T.J. Rivlin proved a well-known inequality, also known as Rivlin's inequality. This inequality states that if $P(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for $0 \le r \le 1$

$$\max_{|z|=r} |P(z)| \ge \left(\frac{1+r}{2}\right)^n \max_{|z|=1} |P(z)|.$$

In this paper, we prove some extensions and generalizations of the above inequality which also sharpen Rivlin's inequality as a special case. Some related results are also obtained and some important consequences of the results are discussed as well.

1 Introduction

If $P(z)$ is a polynomial of degree n, then for $R \ge 1$

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)| \tag{1}$$

and

$$\max_{|z|=R} |P(z)| \le R^n \max_{|z|=1} |P(z)|. \tag{2}$$

The above inequalities are the famous Bernstein inequalities [1] for polynomials. Inequality (1) is a direct consequence of Bernstein's theorem on the derivative of a trigonometric polynomial [2], and inequality (2) follows from the maximum modulus theorem (see [3, Problem 269]).

The reverse analogue of inequality (2) whenever $R \le 1$ is given by Varga [4] by proving that if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=r} |P(z)| \ge r^n \max_{|z|=1} |P(z)|, \tag{3}$$

whenever $0 \le r \le 1$. Inequality (3) attains equality whenever $P(z) = az^n$.

For the class of polynomials having no zero inside the unit circle, Rivlin [5] proved that if $P(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for $0 \le r \le 1$

$$\max_{|z|=r} |P(z)| \ge \left(\frac{1+r}{2}\right)^n \max_{|z|=1} |P(z)|. \tag{4}$$

Equality holds in inequality (4) if $P(z) = (z + a)^n$ whenever $|a| = 1$.

Aziz [6] generalized Rivlin's inequality (4) by proving that if $P(z)$ has no zero in $|z| < K, K \ge 1$, then for $0 \le r \le 1$

$$\max_{|z|=r} |P(z)| \ge \left(\frac{K+r}{K+1}\right)^n \max_{|z|=1} |P(z)|. \tag{5}$$

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The above inequality is best possible and equality holds if $P(z) = (z+a)^n$ and $|a| = K$. In inequality (5), the bound does not address the issue of how far the zeros lie outside the disc $|z| = K$. Now there arises a question naturally; is there any way to refine inequality (5) by capturing some informations on the moduli of zeros? Can we obtain a bound via two extreme coefficients of $P(z)$ which are informative about the distance of zeros from the origin? In view of the example for the equality case in inequality (5) which holds with the property $|a_0|/|a_n| = K^n$, it should be possible to improve upon the bound for polynomials $P(z) = \sum_{v=0}^{n} a_v z^v$ having no zero in $|z| < K, K \geq 1$, satisfying $|a_0|/|a_n| \neq K^n$.

As a way to this approach, Kumar and Milovanović [7] sharpened inequalities (4) and (5) significantly by proving that if $P(z) = \sum_{v=0}^{n} a_v z^v$ has no zero in $|z| < K, K \geq 1$, then for $0 \leq r \leq 1$

$$\max_{|z|=r} |P(z)| \geq \left\{ \left(\frac{K+r}{K+1}\right)^n + \frac{1}{K^{n-1}} \left(\frac{|a_0| - |a_n|K^n}{|a_0| + |a_n|}\right) \left(\frac{1-r}{K+1}\right) \right\} \max_{|z|=1} |P(z)|. \tag{6}$$

The above result is sharp and equality holds if $P(z) = (z+K)^n$ and also for $P(z) = z + a$ for any a with $|a| \geq K$.

In this paper, we prove some extensions and generalizations of inequality (6) which are sharpened forms of Rivlin's inequality.

2 Lemmas

We need the following lemmas to prove the theorems. The first lemma is due to Kumar and Milovanović [7].

Lemma 1 *For any $0 \leq r \leq 1$ and $R_k \geq K \geq 1, 1 \leq k \leq n$, then*

$$\prod_{k=1}^{n} \frac{r+R_k}{1+R_k} \geq \left(\frac{K+r}{K+1}\right)^n + \frac{1}{K^{n-1}} \left(\frac{R_1 R_2 ... R_n - K^n}{R_1 R_2 ... R_n + 1}\right) \left(\frac{1-r}{K+1}\right)^n. \tag{7}$$

Lemma 2 *The function*

$$f(x) = \frac{x - |a_n|K^n}{x + |a_n|},\ x \neq -|a_n|,$$

is a non-decreasing function for $K \geq 1$, $a_n \in C$ and n is a positive integer.

Proof. The result follows by the first derivative test. ■

Lemma 3 *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < K$, $K > 0$, then*

$$|P(z)| \geq m \quad \text{for} \quad |z| \leq K, \tag{8}$$

where $m = \min_{|z|=K} |P(z)|$.

The above lemma is due to Gardner et al. [9, see Lemma 2.6].

Lemma 4 *If $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zero in $|z| < K$, $K \geq 1$, then for any complex number λ with $|\lambda| < 1$ and $m = \min_{|z|=K} |P(z)|$*

$$K^n |a_n| \leq |a_0| - |\lambda| m.$$

Proof. By hypothesis, $P(z)$ has no zero in $|z| < K$. So, $P(z)$ has all its zeros in $|z| \geq K$. Then, the polynomial $S(z) = e^{-i \arg a_0} P(z)$ has the same zeros as $P(z)$. Here,

$$\begin{aligned} S(z) &= e^{-i \arg a_0} \left\{ |a_0| e^{i \arg a_0} + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \right\} \\ &= |a_0| + e^{-i \arg a_0} \left\{ a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \right\}. \end{aligned}$$

Now, on $|z| = K$ for any complex number λ with $|\lambda| < 1$ and $m = \min_{|z|=K} |P(z)| \neq 0$, we have

$$|\lambda|\, m < m \leq |S(z)|.$$

Then by Rouche's theorem, $T(z) = S(z) - |\lambda|m$ has all its zeros in $|z| \geq K$ and in case $m = 0$, $T(z) = S(z)$. Thus, in any case, $T(z)$ has all its zeros in $|z| \geq K$. Now, applying Vieta's formula to $T(z)$, we get

$$\frac{|a_0| - |\lambda|m}{|a_n|} \geq K^n,$$

i.e.

$$K^n |a_n| \leq |a_0| - |\lambda|m,$$

which completes the proof of Lemma 4. ■

3 Main Results

Our first result extends and generalizes inequality (6) which in turn sharpens and generalizes inequality (4) due to Rivlin [5]. In fact, we prove the following result.

Theorem 1 *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zero in $|z| < K, K \geq 1$ and R_k, $k = 1, 2, ..., n$, are the moduli of the zeros of $P(z) + \lambda m$, where λ is some fixed complex number with $|\lambda| < 1$, then for $0 \leq r \leq 1$*

$$\begin{aligned}\max_{|z|=r} |P(z)| \quad &\geq \quad \left\{ \left(\frac{K+r}{K+1} \right)^n + \frac{1}{K^{n-1}} \left(\frac{|a_0| - |\lambda|m - |a_n|K^n}{|a_0| - |\lambda|m + |a_n|} \right) \left(\frac{1-r}{K+1} \right)^n \right\} \max_{|z|=1} |P(z)| \\ &\quad + \left\{ 1 - \left(\prod_{k=1}^{n} \frac{r + R_k}{1 + R_k} \right) \right\} |\lambda| m,\end{aligned}$$

where $m = \min_{|z|=K} |P(z)|$.

Proof. Here, $m = \min_{|z|=K} |P(z)|$ and if $P(z) = \sum_{v=0}^{n} a_v z^v$ has a zero on $|z| = K$, $K \geq 1$, then $m = 0$. Henceforth, we assume that $P(z)$ has no zero on $|z| = K$. Therefore, for $|z| = K$

$$m \leq |P(z)|. \tag{9}$$

If λ is any real or complex number with $|\lambda| < 1$, we have on $|z| = K$

$$|\lambda|m < m \leq |P(z)|.$$

By Rouche's theorem, it follows that the polynomial $F(z) = P(z) + \lambda m$ does not vanish in $|z| < K$ for every real or complex number λ with $|\lambda| < 1$. If $R_k, k = 1, 2, ..., n$, are the moduli of the zeros of $F(z)$, then $R_k \geq K$, $K \geq 1$. Now, for any $0 \leq r \leq 1$ and $0 \leq \phi < 2\pi$,

$$\begin{aligned}\left| \frac{F(re^{i\phi})}{F(e^{i\phi})} \right| &= \prod_{k=1}^{n} \left| \frac{re^{i\phi} - R_k e^{i\phi_k}}{e^{i\phi} - R_k e^{i\phi_k}} \right| \\ &= \prod_{k=1}^{n} \left| \frac{re^{i(\phi-\phi_k)} - R_k}{e^{i(\phi-\phi_k)} - R_k} \right| \\ &= \prod_{k=1}^{n} \left\{ \frac{r^2 + R_k^2 - 2rR_k cos(\phi - \phi_k)}{1 + R_k^2 - 2R_k cos(\phi - \phi_k)} \right\}^{1/2} \\ &\geq \prod_{k=1}^{n} \frac{r + R_k}{1 + R_k},\end{aligned}$$

which is equivalent to

$$\left|F(re^{i\phi})\right| \geq \prod_{k=1}^{n} \frac{r+R_k}{1+R_k} |F(e^{i\phi})|,$$

which gives

$$\left|P(re^{i\phi})+\lambda m\right| \geq \prod_{k=1}^{n} \frac{r+R_k}{1+R_k} |P(e^{i\phi})+\lambda m|. \tag{10}$$

By Lemma 3, we have

$$|P(e^{i\phi})+\lambda m| \geq |P(e^{i\phi})| - |\lambda| m. \tag{11}$$

Using inequality (11) on the right hand side of inequality (10), we get

$$|P(re^{i\phi})+\lambda m| \geq \prod_{k=1}^{n} \frac{r+R_k}{1+R_k} \left\{|P(e^{i\phi})| - |\lambda| m\right\} \geq 0. \tag{12}$$

Let ϕ_0 be such that $\max_{0\leq\phi<2\pi} |P(e^{i\phi})| = |P(e^{i\phi_0})|$. Then, in particular, inequality (12) becomes

$$|P(re^{i\phi_0})+\lambda m| \geq \prod_{k=1}^{n} \frac{r+R_k}{1+R_k} \left\{|P(e^{i\phi_0})| - |\lambda| m\right\}. \tag{13}$$

We choose the argument of λ suitably on the left hand side of inequality (13) such that

$$|P(re^{i\phi_0})+\lambda m| = |P(re^{i\phi_0})| - |\lambda| m. \tag{14}$$

Using (14), inequality (13) becomes

$$|P(re^{i\phi_0})| - |\lambda| m \geq \prod_{k=1}^{n} \frac{r+R_k}{1+R_k} \left\{|P(e^{i\phi_0})| - |\lambda| m\right\},$$

or equivalently

$$|P(re^{i\phi_0})| \geq \left(\prod_{k=1}^{n} \frac{r+R_k}{1+R_k}\right) |P(e^{i\phi_0})| + \left(1 - \prod_{k=1}^{n} \frac{r+R_k}{1+R_k}\right) |\lambda| m. \tag{15}$$

Using inequality (7) to the first term in the right hand side of inequality (15), we get

$$\begin{aligned} |P(re^{i\phi_0})| \quad \geq \quad & \left\{\left(\frac{K+r}{K+1}\right)^n + \frac{1}{K^{n-1}}\left(\frac{R_1R_2...R_n - K^n}{R_1R_2...R_n+1}\right)\left(\frac{1-r}{K+1}\right)^n\right\} |P(e^{i\phi_0})| \\ & + \left(1 - \prod_{k=1}^{n} \frac{r+R_k}{1+R_k}\right) |\lambda| m, \end{aligned}$$

which is also equivalent to

$$\begin{aligned} |P(re^{i\phi_0})| \quad \geq \quad & \left\{\left(\frac{K+r}{K+1}\right)^n + \frac{1}{K^{n-1}}\left(\frac{|a_0+\lambda m| - |a_n|K^n}{|a_0+\lambda m|+|a_n|}\right)\left(\frac{1-r}{K+1}\right)^n\right\} |P(e^{i\phi_0})| \\ & + \left(1 - \prod_{k=1}^{n} \frac{r+R_k}{1+R_k}\right) |\lambda| m. \end{aligned} \tag{16}$$

By Lemma 3, we have for $|z| \leq K, K \geq 1$ and $|\lambda| < 1$

$$|P(z)| \geq m > |\lambda| m. \tag{17}$$

If we put $z = 0$ in inequality (17), then

$$|P(0)| > |\lambda|m,$$

which gives

$$|a_0| > |\lambda|m. \tag{18}$$

By inequality (18), we have

$$|a_0 + \lambda m| \geq |a_0| - |\lambda|m. \tag{19}$$

Therefore by Lemma 2, we have

$$\frac{|a_0 + \lambda m| - |a_n|K^n}{|a_0 + \lambda m| + |a_n|} \geq \frac{|a_0| - |\lambda|m - |a_n|K^n}{|a_0| - |\lambda|m + |a_n|}. \tag{20}$$

It is worth to note from Lemma 4 that the right hand side of inequality (20) is always non-negative.

Using inequality (20), inequality (16) gives

$$\begin{aligned} |P(re^{i\phi_0})| \quad &\geq \quad \left\{ \left(\frac{K+r}{K+1} \right)^n + \frac{1}{K^{n-1}} \left(\frac{|a_0| - |\lambda|m - |a_n|K^n}{|a_0| - |\lambda|m + |a_n|} \right) \left(\frac{1-r}{K+1} \right)^n \right\} |P(e^{i\phi_0})| \\ &\quad + \left(1 - \prod_{k=1}^{n} \frac{r + R_k}{1 + R_k} \right) |\lambda|m. \end{aligned} \tag{21}$$

Since

$$\max_{|z|=r} |P(re^{i\phi})| \geq |P(re^{i\phi_0})| \quad \text{and} \quad \max_{|z|=1} |P(e^{i\phi})| = |P(e^{i\phi_0})|,$$

we get the desired result from inequality (21). ■

Remark 1 *When $\lambda = 0$, Theorem 1 reduces to inequality (6).*

Remark 2 *When $\lambda = 0$ and $K = 1$, Theorem 1 reduces to the following improvement of Rivlin's inequality due to Kumar [8].*

Corollary 1 *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zero in $|z| < 1$, then for $0 \leq r \leq 1$,*

$$\max_{|z|=r} |P(z)| \geq \left\{ \left(\frac{1+r}{2} \right)^n + \left(\frac{|a_0| - |a_n|}{|a_0| + |a_n|} \right) \left(\frac{1-r}{2} \right)^n \right\} \max_{|z|=1} |P(z)|. \tag{22}$$

Equality holds in inequality (22) if $P(z) = (z + a)^n$ whenever $|a| = 1$ and also for $P(z) = z + a$ for any a with $|a| \geq 1$. As an interesting consequence of Theorem 1, we get an inequality for the class of polynomials having all its zeros in $|z| \leq K$, $K \leq 1$. To elaborate it, if $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \leq K$, $K \leq 1$, then its reciprocal polynomial $Q(z) = z^n P(1/z)$ has no zero in $|z| < 1/K$, $1/K \geq 1$. If R_k, $k = 1, 2, ..., n$, are the moduli of the zeros of $P(z) + z^n \lambda m / K^n$, then $1/R_k$, $k = 1, 2, ..., n$, are the moduli of the zeros of $Q(z) + \lambda m_0$, where $m_0 = \min_{|z|=1/K} |Q(z)|$. Applying Theorem 1 to the polynomial $Q(z)$ with $r = 1/R$, $R \geq 1$, we get

$$\begin{aligned} \max_{|z|=1/R} |Q(z)| \quad &\geq \quad \left\{ \left(\frac{1/K + 1/R}{1/K + 1} \right)^n + K^{n-1} \left(\frac{|a_n| - |\lambda|m_0 - |a_0|/K^n}{|a_n| - |\lambda|m_0 + |a_0|} \right) \left(\frac{1 - 1/R}{1/K + 1} \right)^n \right\} \\ &\quad \times \max_{|z|=1} |Q(z)| + \left(1 - \prod_{k=1}^{n} \frac{r + 1/R_k}{1 + 1/R_k} \right) |\lambda|m_0. \end{aligned} \tag{23}$$

Now,

$$m_0 = \min_{|z|=1/K} |Q(z)| = \frac{1}{K^n} \max_{|z|=K} |P(z)| = \frac{m}{K^n}. \tag{24}$$

Using equality (24) in inequality (23) and simplifying, the following corollary is obtained.

Corollary 2 *If $P(z)=\sum_{v=0}^{n}a_vz^v$ is a polynomial of degree n having all its zeros in $|z|\le K$, $K\le 1$ and R_k, $k=1,2,...,n$, are the moduli of the zeros of $P(z)+z^n\lambda m/K^n$, where λ is some fixed complex number with $|\lambda|<1$, then for $r=1/R$, $R\ge 1$,*

$$\begin{aligned}\max_{|z|=R}|P(z)| \quad &\ge \quad \left\{\left(\frac{K+R}{K+1}\right)^n+K^{n-1}\left(\frac{|a_n|K^n-|\lambda|m-|a_0|}{|a_n|-|\lambda|m/K^n+|a_0|}\right)\left(\frac{R-1}{K+1}\right)^n\right\}\max_{|z|=1}|P(z)|\\ &\quad +\left(\frac{R}{K}\right)^n\left(1-\prod_{k=1}^{n}\frac{rR_k+1}{R_k+1}\right)|\lambda|m,\end{aligned}$$

where $m=\min_{|z|=K}|P(z)|$.

Govil [10] generalized inequality (4) by studying the relative growth of a polynomial $P(z)$ having no zero in the open disk, with respect to two circles $|z|=r$ and $|z|=R$ whenever $0\le r<R\le 1$. In particular, he proved that if $P(z)$ is a polynomial of degree n having no zero in $|z|<1$, then for $0\le r<R\le 1$

$$\max_{|z|=r}|P(z)|\ge\left(\frac{1+r}{1+R}\right)^n\max_{|z|=R}|P(z)|. \tag{25}$$

Our next result sharpens inequality (25) and it also extends and generalizes some results as special cases.

Theorem 2 *If $P(z)=\sum_{v=0}^{n}a_vz^v$ has no zero in $|z|<K$, $K>0$ and R_k, $k=1,2,...,n$, are the moduli of the zeros of $P(z)+\lambda m$, where λ is some fixed complex number with $|\lambda|<1$, then for $0\le r<R\le K$,*

$$\begin{aligned}\max_{|z|=r}|P(z)| \quad &\ge \quad \left\{\left(\frac{K+r}{K+R}\right)^n+\left(\frac{R}{K}\right)^{n-1}\left(\frac{|a_0|-|\lambda|m-|a_n|K^n}{|a_0|-|\lambda|m+|a_n|R^n}\right)\left(\frac{R-r}{K+R}\right)^n\right\}\max_{|z|=R}|P(z)|\\ &\quad +\left(1-\prod_{k=1}^{n}\frac{r+R_k}{R+R_k}\right)|\lambda|m,\end{aligned} \tag{26}$$

where $m=\min_{|z|=K}|P(z)|$.

Proof. If $P(z)$ has no zero in $|z|<K$, then the polynomial $P(Rz)$ has no zero in $|z|<K/R$, where $K/R\ge 1$. Then the polynomial $P(Rz)$ satisfies the hypothesis of Theorem 1 and applying Theorem 1 to $P(Rz)$, we have

$$\begin{aligned}\max_{|z|=r/R}|P(Rz)| \quad &\ge \quad \left\{\left(\frac{K/R+r/R}{K/R+1}\right)^n+\left(\frac{R}{K}\right)^{n-1}\left(\frac{|a_0|-|\lambda|m'-|a_nR^n|\,(K/R)^n}{|a_0|-|\lambda|m'+|a_nR^n|}\right)\right.\\ &\quad \times\left.\left(\frac{1-r/R}{K/R+1}\right)^n\right\}\max_{|z|=1}|P(Rz)|+\left(1-\prod_{k=1}^{n}\frac{r/R+R_k/R}{1+R_k/R}\right)|\lambda|m',\end{aligned} \tag{27}$$

where $m'=\min_{|z|=K/R}|P(Rz)|$. Now,

$$m'=\min_{|z|=K/R}|P(Rz)|=\min_{|z|=K}|P(z)|=m.$$

Using this equality in inequality (27) and simplifying, we get

$$\begin{aligned}\max_{|z|=r}|P(z)| \quad &\ge \quad \left\{\left(\frac{K+r}{K+R}\right)^n+\left(\frac{R}{K}\right)^{n-1}\left(\frac{|a_0|-|\lambda|m-|a_n|K^n}{|a_0|-|\lambda|m+|a_n|R^n}\right)\left(\frac{R-r}{K+R}\right)^n\right\}\max_{|z|=R}|P(z)|\\ &\quad +\left(1-\prod_{k=1}^{n}\frac{r+R_k}{R+R_k}\right)|\lambda|m.\end{aligned}$$

This completes the proof of the theorem. ■

Remark 3 *When $\lambda = 0$ and $K = 1$, Theorem 2 reduces to the following result due to Kumar and Milovanović [7] which is an improvement and generalization of inequality (25).*

Corollary 3 *If $P(z) = \sum_{v=0}^{n} a_v z^v$ has no zero in $|z| < 1$, then for $0 \leq r < R \leq 1$,*

$$\max_{|z|=r} |P(z)| \geq \left\{ \left(\frac{1+r}{1+R} \right)^n + R^{n-1} \left(\frac{|a_0| - |a_n|}{|a_0| + |a_n| R^n} \right) \left(\frac{R-r}{1+R} \right)^n \right\} \max_{|z|=R} |P(z)|. \tag{28}$$

The result is best possible and equality holds in inequality (28) if $P(z) = (z+a)^n$ where $|a| = 1$ and also for $P(z) = z + a$ for any a with $|a| \geq 1$.

Remark 4 *When $\lambda = 0$, $K = 1$ and $R = 1$, Theorem 2 reduces to Corollary 1, which is an improved version of Rivlin's inequality.*

If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \leq K$, $K \geq 1$, then the reciprocal polynomial $Q(z) = z^n P(1/z)$ has all its zeros in $|z| \geq 1/K$. Now if R_k, $k = 1, 2, ..., n$, are the moduli of the zeros of $P(z) + z^n \lambda m / K^n$, then $1/R_k$, $k = 1, 2, ..., n$, are the moduli of the zeros of $Q(z) + \lambda m_0$ where $m_0 = \min_{|z|=1/K} |Q(z)|$. Also if $1 \leq K \leq R < r$, then $0 \leq 1/r < 1/R \leq 1/K$. Applying Theorem 2 to the polynomial $Q(z)$, we get for some fixed complex number λ with $|\lambda| < 1$

$$\begin{aligned} \max_{|z|=1/r} |Q(z)| \quad \geq \quad & \left\{ \left(\frac{1/K + 1/r}{1/K + 1/R} \right)^n + \left(\frac{K}{R} \right)^{n-1} \left(\frac{|a_n| - |\lambda| m_0 - |a_0|/K^n}{|a_n| - |\lambda| m_0 + |a_0|/R^n} \right) \left(\frac{1/R - 1/r}{1/R + 1/K} \right)^n \right\} \\ & \times \max_{|z|=1/R} |Q(z)| + \left(1 - \prod_{k=1}^{n} \frac{1/r + 1/R_k}{1/R + 1/R_k} \right) |\lambda| m_0. \end{aligned} \tag{29}$$

Using equality (24) in inequality (29) and simplifying, the following corollary is obtained.

Corollary 4 *If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z| \leq K$, $K \geq 1$ and R_k, $k = 1, 2, ..., n$, are the moduli of the zeros of $P(z) + z^n \lambda m / K^n$, where λ is some fixed complex number with $|\lambda| < 1$, then for $1 \leq K \leq R < r$,*

$$\begin{aligned} \max_{|z|=r} |P(z)| \quad \geq \quad & \left\{ \left(\frac{K+r}{K+R} \right)^n + \left(\frac{K}{R} \right)^{n-1} \left(\frac{|a_n| K^n - |\lambda| m - |a_0|}{|a_n| R^n - |\lambda| m \left(R/K \right)^n + |a_0|} \right) \left(\frac{r-R}{K+R} \right)^n \right\} \\ & \times \max_{|z|=R} |P(z)| + \frac{1}{K^n} \left(r^n - R^n \prod_{k=1}^{n} \frac{r + R_k}{R + R_k} \right) |\lambda| m, \end{aligned} \tag{30}$$

where $m = \min_{|z|=K} |P(z)|$.

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