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On A Class Of Strongly Coupled Singular (p,q)-Kirchhoff Type Systems*

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Abstract

The existence of positive solutions for a singular (p,q)-Kirchhoff type system under Dirichlet boundary condition is studied. The main novelties consist in the presence of a Kirchhoff type system and in the strongly coupled reaction terms which tend to $-\infty$. Our approach relies on the method of sub- and supersolutions.

1 Introduction

The aim of this paper is to study the singular (p,q)-Kirchhoff type system

$$\begin{cases} -M_1\Big(\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = \lambda a(x)\Big(f(u,v) - \frac{1}{u^\alpha}\Big), & x\in\Omega,\\ -M_2\Big(\int_\Omega |\nabla v|^q dx\Big)\Delta_q v = \lambda b(x)\Big(g(u,v) - \frac{1}{v^\beta}\Big), & x\in\Omega,\\ u=v=0, & x\in\partial\Omega, \end{cases} \tag{1}$$

where $M_i:\mathbb{R}^+\to\mathbb{R}^+$, $i=1,2$ are two continuous and increasing functions such that $M_i(t)\geq m_i>0$ for all $t\in\mathbb{R}^+$, $\Delta_r z = div\,(|\nabla z|^{r-2}\nabla z)$, for $r>1$ denotes the r-Laplacian operator, λ is a positive parameter, Ω is a bounded domain in $\mathbb{R}^n$, $n\geq 1$ with sufficiently smooth boundary and $\alpha,\beta\in(0,1)$. Here $a(x),b(x)\in C(\overline{\Omega})$ are weight functions such that $a(x)\geq a_0>0$, $b(x)\geq b_0>0$ for all $x\in\overline{\Omega}$, $f,g\in C^1((0,\infty)\times(0,\infty))\cap C([0,\infty)\times[0,\infty))$ are nondecreasing functions for both u,v and $f(0,0),g(0,0)>0$.

Problem (1) is called nonlocal because of the presence of the term $-M(\int_\Omega|\nabla u|^r dx)$ which implies that the equation in (1) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties which makes the study of such a class of problem particularly interesting. Besides, such a problem has physical motivation. Moreover, problem (1) is related to the stationary version of Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

presented by Kirchhoff [13]. This equation extends the classical d'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in Eq. (2) have the following meanings: L is the length of the string, h is the area of cross section, E is the Youngs modulus of the material, ρ is the mass density, and P_0 is the initial tension.

When an elastic string with fixed ends is subjected to transverse vibrations, its length varies with the time: this introduces changes of the tension in the string. This induced Kirchhoff to propose a nonlinear correction of the classical D'Alembert's equation. Later on, Woinowsky-Krieger (Nash-Modeer) incorporated this correction in the classical Euler-Bernoulli equation for the beam (plate) with hinged ends. See, for example, [4, 5] and the references therein.

Moreover, nonlocal problems also appear in other fields as, for example, biological systems where (u,v) describes a process which depends on the average of itself (for instance, population density). See, for

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example, [2], [3], [10], [19] and [20] and the references therein. In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [7, 8, 12, 17, 21, 22, 23, 24] in which the authors have used different methods to get the existence of solutions.

Let $F(s,t) = [f(s,t) - \frac{1}{s^\alpha}]$ and $G(s,t) = [(g(s,t) - \frac{1}{t^\beta}]$. Then

$$\lim_{(s,t)\to(0,0)} F(s,t) = -\infty = \lim_{(s,t)\to(0,0)} G(s,t),$$

and hence we refer to (1) as an infinite semipositone problem. See [1], where the authors studied the corresponding non-singular finite system when $M_1(t) = M_2(t) \equiv 1$, and $a(x) = b(x) \equiv 1$. It is well documented that the study of positive solutions to such semipositone problems is mathematically very challenging [6], [18]. In this paper, we study the even more challenging semipositone system with

$$\lim_{(s,t)\to(0,0)} F(s,t) = -\infty = \lim_{(s,t)\to(0,0)} G(s,t).$$

We do not need the boundedness of the Kirchhoff functions M_1, M_2, as in [9]. Using the sub and supersolutions techniques, we prove the existence of positive solutions to the system (1). To our best knowledge, this is an interesting and new research topic for singular (p,q)-Kirchhoff type systems. One can refer to [14, 15, 16] for some recent existence results of infinite semipositone systems.

To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -\Delta_r \phi = \lambda\, |\phi|^{r-2}\, \phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

Let $\phi_{1,r}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,r}$ of (3) such that $\phi_{1,r}(x) > 0$ in Ω, and $\|\phi_{1,r}\|_\infty = 1$ for $r = p, q$. Let $m, \sigma, \delta > 0$ be such that

$$\sigma \le \phi_{1,r}^{\frac{r}{r-1+s}} \le 1, \quad x \in \Omega - \overline{\Omega_\delta},$$

$$|\nabla \phi_{1,r}|^r \ge m, \quad x \in \overline{\Omega_\delta},$$

for $r = p, q$, and $s = \alpha, \beta$, where $\overline{\Omega_\delta} := \{x \in \Omega | d(x, \partial\Omega) \le \delta\}$. (This is possible since $|\nabla \phi_{1,r}|^r \ne 0$ on $\partial\Omega$ while $\phi_{1,r} = 0$ on $\partial\Omega$ for $r = p, q$. We will also consider the unique solution $\zeta_r \in W_0^{1,r}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta_r \zeta_r = 1, & x \in \Omega, \\ \zeta_r = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result, it is known that $\zeta_r > 0$ in Ω and $\frac{\partial \zeta_r}{\partial n} < 0$ on $\partial\Omega$.

2 Existence Result

In this section, we shall establish our existence result via the method of sub-supersolution (see [1]). By a solution pair of (1), we mean a pair of functions of the form $(u,v) \in W^{1,p} \cap C(\overline{\Omega}) \times W^{1,q} \cap C(\overline{\Omega})$ such that $(u,v) = (0,0)$ on $\partial\Omega$,

$$M_1\Big(\int_\Omega |\nabla u|^p dx\Big) \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla w\, dx \le \lambda \int_\Omega a(x)\Big(f(u,v) - \frac{1}{u^\alpha}\Big) w\, dx,$$

and

$$M_2\Big(\int_\Omega |\nabla v|^q dx\Big) \int_\Omega |\nabla v|^{q-2}\, \nabla v \cdot \nabla w\, dx \le \lambda \int_\Omega b(x)\Big(g(u,v) - \frac{1}{v^\beta}\Big) w\, dx.$$

By a sub-solution pair of (1), we mean a solution pair of the form $(\psi_1, \psi_2) \in W^{1,p} \cap C(\overline{\Omega}) \times W^{1,q} \cap C(\overline{\Omega})$ such that $(\psi_1, \psi_2) \le (0,0)$ on $\partial\Omega$,

$$M_1\Big(\int_\Omega |\nabla \psi_1|^p dx\Big) \int_\Omega |\nabla \psi_1|^{p-2}\, \nabla \psi_1 \cdot \nabla w\, dx \le \lambda \int_\Omega a(x)\Big(f(\psi_1, \psi_2) - \frac{1}{\psi_1^\alpha}\Big) w\, dx,$$

and

$$M_2\Big(\int_\Omega |\nabla\psi_2|^q dx\Big)\int_\Omega |\nabla\psi_2|^{q-2}\,\nabla\psi_2\cdot\nabla w\,dx \le \lambda\int_\Omega b(x)\Big(g(\psi_1,\psi_2)-\frac{1}{\psi_2^\beta}\Big)w\,dx.$$

By a super-solution pair of (1), we mean a solution pair of the form $(z_1,z_2)\in W^{1,p}\cap C(\overline{\Omega})\times W^{1,q}\cap C(\overline{\Omega})$ such that $(z_1,z_2)\ge(0,0)$ on $\partial\Omega$,

$$M_1\Big(\int_\Omega |\nabla z_1|^p dx\Big)\int_\Omega |\nabla z_1|^{p-2}\,\nabla z_1\cdot\nabla w\,dx \ge \lambda\int_\Omega a(x)\Big(f(z_1,z_2)-\frac{1}{z_1^\alpha}\Big)w\,dx,$$

and

$$M_2\Big(\int_\Omega |\nabla z_2|^q dx\Big)\int_\Omega |\nabla z_2|^{q-2}\,\nabla z_2\cdot\nabla w\,dx \ge \lambda\int_\Omega b(x)\Big(g(z_1,z_2)-\frac{1}{z_2^\beta}\Big)w\,dx,$$

for all $w\in W=\{w\in C_0^\infty(\Omega)|w\ge 0, x\in\Omega\}$. We remark that, a solution (u_λ,v_λ) of (1) is large if $u_\lambda\to\infty$ and $v_\lambda\to\infty$ as $\lambda\to\infty$. Then the following result holds:

Lemma 1 (See [11, 15, 7]) *Suppose there exist sub and supersolutions (ψ_1,ψ_2) and (z_1,z_2) respectively of (1) such that $(\psi_1,\psi_2)\le(z_1,z_2)$. Then (1) has a solution (u,v) such that $(u,v)\in[(\psi_1,\psi_2),(z_1,z_2)]$.*

We make the following hypotheses:

(H1) $f,g\in C^1((0,\infty)\times(0,\infty))\cap C([0,\infty)\times[0,\infty))$ are nondecreasing functions for both u,v such that $f(0,0)$, $g(0,0)>0$.

$$\lim_{s\to+\infty} f(s,s)=\lim_{s\to+\infty} g(s,s)=+\infty,$$

and

$$\lim_{s\to+\infty}\frac{g(s,s)}{s^{q-1}}=0.$$

(H2) $\lim_{s\to\infty}\frac{f(Ag(s,s)^{\frac{1}{q-1}})}{s^{p-1}}=0$, for all $A>0$.

We establish:

Theorem 2 *Let (H1) and (H2) hold. Then (1) has a large positive solution (u,v) for $\lambda\gg 1$.*

Proof. For fixed $r_1\in(\frac{1}{p-1+\alpha},\frac{1}{p-1})$ and $r_2\in(\frac{1}{q-1+\beta},\frac{1}{q-1})$, we shall verify that

$$(\psi_1,\psi_2)=\Big(\lambda^{r_1}\frac{(p-1+\eta)}{p}\,(\frac{a_0}{m_1})^{\frac{1}{p-1}}\,\phi_{1,p}^{\frac{p}{p-1+\alpha}},\lambda^{r_2}\frac{(q-1+\beta)}{q}\,(\frac{b_0}{m_2})^{\frac{1}{q-1}}\,\phi_{1,q}^{\frac{q}{q-1+\beta}}\Big),$$

is a sub-solution of (1). Let $w\in W$. Then a calculation shows that

$$\nabla\psi_1=\lambda^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}\,\phi_{1,p}^{\frac{1-\alpha}{p-1+\alpha}}\,\nabla\phi_{1,p},$$

and we have

$$\begin{aligned}
& M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\int_\Omega |\nabla\psi_1|^{p-2}\,\nabla\psi_1\cdot\,\nabla w\,dx\\
= \;& \frac{a_0\lambda^{(p-1)r_1}}{m_1}M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\int_\Omega \phi_{1,p}^{1-\frac{\alpha p}{p-1+\alpha}}|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\nabla w dx\\
= \;& \frac{a_0\lambda^{(p-1)r_1}}{m_1}M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\int_\Omega |\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\Big\{\nabla(\phi_{1,p}^{1-\frac{\alpha p}{p-1+\alpha}}w)-w\nabla(\phi_{1,p}^{1-\frac{\alpha p}{p-1+\alpha}})\Big\}\,dx\\
= \;& \frac{a_0\lambda^{(p-1)r_1}}{m_1}M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\Big\{\int_\Omega \big[\lambda_{1,p}\,\phi_{1,p}^{p-\frac{\alpha p}{p-1+\alpha}}-|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}\nabla(\phi_{1,p}^{1-\frac{\alpha p}{p-1+\alpha}})\big]wdx\Big\}\\
= \;& \frac{a_0\lambda^{(p-1)r_1}}{m_1}M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\Big\{\int_\Omega \big[\lambda_{1,p}\,\phi_{1,p}^{p-\frac{\alpha p}{p-1+\alpha}}-|\nabla\phi_{1,p}|^{p}\,(1-\frac{\alpha p}{p-1+\alpha})\phi_{1,p}^{-\frac{\alpha p}{p-1+\alpha}}\big]wdx\Big\}\\
\le \;& a_0\lambda^{(p-1)r_1}\left\{\int_\Omega \big[\lambda_{1,p}\,\phi_{1,p}^{\frac{p(p-1)}{p-1+\alpha}}-\frac{(1-\alpha)(p-1)}{p-1+\alpha}\,\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\alpha p}{p-1+\alpha}}}\big]wdx\right\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& M_2\Big(\int_\Omega |\nabla\psi_2|^q dx\Big)\int_\Omega |\nabla\psi_2|^{q-2}\,\nabla\psi_2\cdot\,\nabla w\,dx\\
\le \;& b_0\lambda^{(q-1)r_2}\left\{\int_\Omega \big[\lambda_{1,q}\,\phi_{1,q}^{\frac{q(q-1)}{q-1+\beta}}-\frac{(1-\beta)(q-1)}{q-1+\beta}\,\frac{|\nabla\phi_{1,q}|^q}{\phi_{1,q}^{\frac{\beta q}{q-1+\beta}}}\big]wdx\right\}.
\end{aligned}$$

Thus (ψ_1,ψ_2) is a sub-solution if

$$a_0\lambda^{r_1}\big\{\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\alpha}}-\frac{(1-\alpha)(p-1)}{p-1+\alpha}\,\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\alpha p}{p-1+\alpha}}}\big\}\le\lambda a(x)\Big(f(\psi_1,\psi_2)-\frac{1}{\psi_1^\alpha}\Big)$$

and

$$b_0\lambda^{r_2}\big\{\lambda_{1,q}\phi_{1,q}^{\frac{q(q-1)}{q-1+\beta}}-\frac{(1-\beta)(q-1)}{q-1+\beta}\,\frac{|\nabla\phi_{1,q}|^q}{\phi_{1,q}^{\frac{\beta q}{q-1+\beta}}}\big\}\le\lambda\,b(x)\Big(g(\psi_1,\psi_2)-\frac{1}{\psi_2^\beta}\Big).$$

First we consider the case when $x\in\overline{\Omega_\delta}$. Since $1-(p-1)r_1-r_1\alpha<0$, for $\lambda\gg1$, we have

$$-\lambda^{(p-1)r_1}\,\frac{(1-\alpha)(p-1)}{p-1+\alpha}\,\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{p\alpha}{p-1+\alpha}}}\le\lambda\Big[-\frac{1}{\Big(\lambda^{r_1}\,(\frac{a_0}{m_1})^{\frac{1}{p-1}}\,(\frac{p-1+\alpha}{p})\,\phi_{1,p}^{\frac{p}{p-1+\alpha}}\Big)^\alpha}\Big].$$

Also in $\bar\Omega_\delta$ (in fact in Ω), since $(p-1)r_1<1$, if $\lambda\gg1$,

$$\begin{aligned}
\lambda^{(p-1)r_1}\,\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\alpha}} \;&\le\; \lambda\,f(0,0)\\
&\le\; \lambda f\Big(\lambda^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}\,(\frac{p-1+\alpha}{p})\,\phi_{1,p}^{\frac{p}{p-1+\alpha}},\lambda^{r_2}(\frac{b_0}{m_2})^{\frac{1}{q-1}}\,(\frac{q-1+\beta}{q})\,\phi_{1,q}^{\frac{q}{q-1+\beta}}\Big).
\end{aligned}$$

It follows that in $\overline{\Omega_\delta}$ for $\lambda \gg 1$, we have

$$\begin{aligned}
& a_0\lambda^{r_1}\Big[\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\alpha}} - \frac{(1-\alpha)(p-1)}{p-1+\alpha}\,\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\alpha p}{p-1+\alpha}}}\Big] \\
= \;& a_0\Big[\lambda^{r_1}\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\alpha}} - \lambda^{r_1}\frac{(1-\alpha)(p-1)}{p-1+\alpha}\,\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\alpha p}{p-1+\alpha}}}\Big] \\
\leq \;& \lambda a(x) f\Big(\lambda^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}\,(\frac{p-1+\alpha}{p})\,\phi_{1,p}^{\frac{p}{p-1+\alpha}}, \lambda^{r_2}(\frac{b_0}{m_2})^{\frac{1}{q-1}}\,(\frac{q-1+\beta}{q})\,\phi_{1,q}^{\frac{q}{q-1+\beta}}\Big) \\
& -\frac{\lambda}{\Big(\lambda^{r_1}\,(\frac{a_0}{m_1})^{\frac{1}{p-1}}\,(\frac{p-1+\alpha}{p})\,\phi_{1,p}^{\frac{p}{p-1+\alpha}}\Big)^\alpha} \\
= \;& \lambda a(x)\Big(f(\psi_1,\psi_2) - \frac{1}{\psi_1^\alpha}\Big).
\end{aligned}$$

On the other hand, on $\Omega - \overline{\Omega_\delta}$, since $\sigma \leq \phi_{1,r}^{\frac{r}{r-1+s}} \leq 1$, for $r = p, q$ and $s = \alpha, \beta$,

$$\begin{aligned}
& f\Big(\lambda^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}\,(\frac{p-1+\alpha}{p})\,\sigma, \lambda^{r_2}(\frac{b_0}{m_2})^{\frac{1}{q-1}}\,(\frac{q-1+\beta}{q})\,\sigma\Big) - \frac{1}{\Big(\lambda^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}\,(\frac{p-1+\alpha}{p})\,\phi_{1,p}^{\frac{p}{p-1+\alpha}}\Big)^\alpha} \\
\leq \;& f(\psi_1,\psi_2) - \frac{1}{\psi_1^\alpha}.
\end{aligned}$$

Also, since $(p-1)r_1 < 0$, for $\lambda \gg 1$,

$$\begin{aligned}
& a_0\lambda^{r_1}\Big[\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\alpha}} - \frac{(1-\alpha)(p-1)}{p-1+\alpha}\,\frac{|\nabla\phi_{1,p}|^p}{\phi_{1,p}^{\frac{\alpha p}{p-1+\alpha}}}\Big] \\
\leq \;& a_0\lambda^{r_1}\lambda_{1,p}\phi_{1,p}^{\frac{p(p-1)}{p-1+\alpha}} \\
\leq \;& \lambda a_0\Big[f\Big(\lambda^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}\,(\frac{p-1+\alpha}{p})\,\sigma, \lambda^{r_2}(\frac{b_0}{m_2})^{\frac{1}{q-1}}\,(\frac{q-1+\beta}{q})\,\sigma\Big) - \frac{1}{\Big(\lambda^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}\,(\frac{p-1+\alpha}{p})\,\phi_{1,p}^{\frac{p}{p-1+\alpha}}\Big)^\alpha}\Big] \\
\leq \;& \lambda a(x)\Big[f\Big(\lambda^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}\,(\frac{p-1+\alpha}{p})\,\phi_{1,p}^{\frac{p}{p-1+\alpha}}, \lambda^{r_2}(\frac{b_0}{m_2})^{\frac{1}{q-1}}\,(\frac{q-1+\beta}{q})\,\phi_{1,q}^{\frac{q}{q-1+\beta}}\Big) \\
& -\frac{1}{\Big(\lambda^{r_1}(\frac{a_0}{m_1})^{\frac{1}{p-1}}\,(\frac{p-1+\alpha}{p})\,\phi_{1,p}^{\frac{p}{p-1+\alpha}}\Big)^\alpha}\Big] \\
= \;& \lambda a(x)\Big(f(\psi_1,\psi_2) - \frac{1}{\psi_1^\alpha}\Big).
\end{aligned}$$

Hence, if $\lambda \gg 1$, we see that

$$M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\int_\Omega |\nabla\psi_1|^{p-2}\,\nabla\psi_1\cdot\nabla w\,dx \leq \lambda\int_\Omega a(x)\Big(f(\psi_1,\psi_2) - \frac{1}{\psi_1^\alpha}\Big)w\,dx.$$

Similarly, for $\lambda \gg 1$, we get

$$M_2\Big(\int_\Omega |\nabla\psi_2|^q dx\Big)\int_\Omega |\nabla\psi_2|^{q-2}\,\nabla\psi_2\cdot\nabla w\,dx \leq \lambda\int_\Omega b(x)\Big(g(\psi_1,\psi_2) - \frac{1}{\psi_2^\beta}\Big)w\,dx.$$

Thus, (ψ_1,ψ_2) is a positive subsolution of (1).

Now, we construct a supersolution $(z_1,z_2)\geq(\psi_1,\psi_2)$. By (H1) and (H2) we can choose $C\gg 1$ so that

$$\frac{m_1}{\|a\|_\infty}\geq\frac{\lambda f\Big(C\|\zeta_p\|_\infty,\Big[\frac{\|b\|_\infty\lambda}{m_2}\Big]^{\frac{1}{q-1}}[g(C\|\zeta_p\|_\infty,C\|\zeta_p\|_\infty)]^{\frac{1}{q-1}}\|\zeta_q\|_\infty\Big)}{C^{p-1}}.$$

Let

$$(z_1,z_2)=(C\zeta_p\,,\,\Big[\frac{\|b\|_\infty\lambda}{m_2}\Big]^{\frac{1}{q-1}}[g(C\|\zeta_p\|_\infty),C\|\zeta_p\|_\infty)]^{\frac{1}{q-1}}\zeta_q).$$

We shall show that (z_1,z_2) is a supersolution of (1). Then

$$\begin{aligned}
&M_1\Big(\int_\Omega|\nabla z_1|^p dx\Big)\int_\Omega|\nabla z_1|^{p-2}\,\nabla z_1\cdot\nabla w\,dx\\
=\;&C^{p-1}M_1\Big(\int_\Omega|\nabla z_1|^p dx\Big)\int_\Omega|\nabla\zeta_p|^{p-2}\,\nabla\zeta_p\cdot\nabla w\,dx\\
=\;&C^{p-1}M_1\Big(\int_\Omega|\nabla z_1|^p dx\Big)\int_\Omega w\,dx\\
\geq\;&m_1C^{p-1}\int_\Omega w\,dx\\
\geq\;&\lambda\|a\|_\infty\int_\Omega f\Big(C\|\zeta_p\|_\infty,\Big[\frac{\|b\|_\infty\lambda}{m_2}\Big]^{\frac{1}{q-1}}[g(C\|\zeta_p\|_\infty,C\|\zeta_p\|_\infty)]^{\frac{1}{q-1}}\|\zeta_q\|_\infty\Big)wdx\\
\geq\;&\lambda\int_\Omega a(x)\Big(f(z_1,z_2)-\frac{1}{z_1^\alpha}\Big)w\,dx.
\end{aligned}$$

Again by (H1) for C large enough we have

$$\frac{1}{\Big(\frac{\lambda\|b\|_\infty}{m_2}\Big)^{\frac{1}{q-1}}\|\zeta_q\|_\infty}\geq\frac{\Big[g\Big(C\|\zeta_p\|_\infty,C\|\zeta_p\|_\infty\Big)\Big]^{\frac{1}{q-1}}}{C\|\zeta_p\|_\infty}.$$

Hence

$$\begin{aligned}
&M_2\Big(\int_\Omega|\nabla z_2|^q dx\Big)\int_\Omega|\nabla z_2|^{q-2}\,\nabla z_2\cdot\nabla w\,dx\\
=\;&\frac{\lambda\|b\|_\infty}{m_2}g(C\|\zeta_p\|_\infty,C\|\zeta_p\|_\infty)M_2\Big(\int_\Omega|\nabla z_2|^q dx\Big)\int_\Omega|\nabla\zeta_q|^{q-2}\,\nabla\zeta_q\cdot\nabla w\,dx\\
=\;&\frac{\lambda\|b\|_\infty}{m_2}g(C\|\zeta_p\|_\infty,C\|\zeta_p\|_\infty)M_2\Big(\int_\Omega|\nabla z_2|^q dx\Big)\int_\Omega w\,dx\\
\geq\;&\lambda\|b\|_\infty g(C\|\zeta_p\|_\infty,C\|\zeta_p\|_\infty)\int_\Omega w\,dx\\
\geq\;&\lambda\|b\|_\infty\int_\Omega g\Big(C\|\zeta_p\|_\infty,[\frac{\|b\|_\infty\lambda}{m_2}]^{\frac{1}{q-1}}[g(C\|\zeta_p\|_\infty,C\|\zeta_p\|_\infty))]^{\frac{1}{q-1}}\|\zeta_q\|_\infty\Big)w\,dx\\
\geq\;&\lambda\int_\Omega b(x)g(z_1,z_2)w\,dx\\
\geq\;&\lambda\int_\Omega b(x)\Big(g(z_1,z_2)-\frac{1}{z_2^\beta}\Big)w\,dx.
\end{aligned}$$

i.e., (z_1,z_2) is a supersolution of (1). Furthermore, C can be chosen large enough so that $(z_1,z_2)\geq(\psi_1,\psi_2)$. Thus, there exist a positive solution (u,v) of (1) such that $(\psi_1,\psi_2)\leq(u,v)\leq(z_1,z_2)$. This completes the proof of Theorem 2. ∎

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