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Fixed-Circle Theorems On G-Metric Spaces*

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Abstract

In this paper, we introduce the concept of fixed-circle on $\mathbb{G}$-metric spaces. Using this new concept, we prove some existence and uniqueness theorems for fixed-circles of self-mappings with a geometric interpretation. We also give some examples that verify the conditions of our results. Finally, we present an application to the Rectified linear unit (ReLU) activation functions to show the importance of our obtained results.

1 Introduction

The concept of a metric space is one of the most useful and fundamental tools in topology, analysis and nonlinear analysis. Its vast area provides a powerful tool for studying variational inequalities, optimization and approximation theory and so many. In recent years, several generalizations of metric spaces have appeared. In 1963, Gahler [1] introduced the concept of 2-metric spaces that he claims to be a generalization of the metric spaces. But, in 1988, Ha et al. [2] proved that there is no relation between these two functions. Then, in 1992, Dhage [3] introduced a new class of generalized metrics called D-metrics. However, in 2003, Mustafa and Sims demonstrated in [4] that most of the claims concerning the fundamental topological properties of D-metric spaces are incorrect. Subsequently, in 2006, the same authors [5] introduced the concept of $\mathbb{G}$-metric space as a generalization of a metric space. In 2012, Sedghi et al. [6] introduced a new generalized metric called an $\mathbb{S}$-metric space. There are also different generalizations, such as $\mathbb{S}$-normed spaces [7] and $\mathbb{A}$-normed spaces [8].

The fixed-point theory is one of the most exciting subjects studied in metric spaces and generalized metric spaces. Banach [9] gave the most important and notable result in this direction in 1922, popularly known as the "Banach contraction principle". This principle has been generalized using various techniques. One of them is the investigation of geometric properties of the fixed-point set when the number of fixed points is more than one. In this case, the fixed-circle problem is given on metric spaces in [10] by giving a different perspective to these theorems. Then, new solutions have been studied in both a metric space [10]–[14] and some generalized metric spaces [15]–[17] especially $\mathbb{S}$-metric spaces [18]–[23]. These solutions can be extended with some recent popular works on G_b-metric spaces [24]–[29]. Thus, the classical fixed-point theory studies gain depth in a different direction.

By the above motivation, we study on the existence and uniqueness theorems of a fixed-circle on $\mathbb{G}$-metric spaces. The paper consists of 5 sections. Section 2 presents some basic preliminaries and historical notes on $\mathbb{G}$-metric spaces. Also, a notion of the fixed-circle on $\mathbb{G}$-metric spaces with some examples is given. Some new existence theorems of fixed-circles and a contractive condition to exclude the identity map are established in Section 3. In Section 4, some uniqueness theorems are proven when the number of fixed-circles is non-unique. The final Section 5 presents an application to Rectified linear unit (ReLU) activation functions satisfying our main theorem.

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2 Preliminaries

In this section, we first recall the concept of the $\mathbb{S}$-metric space. Sedghi et al. [6] put forward that an $\mathbb{S}$-metric is a generalization of a $\mathbb{G}$-metric. Moreover, these two generalized metric spaces are different from each other. Therefore, this motivation makes our paper worthwhile.

Definition 1 ([6]) *Let χ be a nonempty set. A function $\mathbb{S}:\chi^3\to[0,\infty)$ is said to be an $\mathbb{S}$-metric on χ if it satisfies the following properties, for each $\alpha,\beta,\gamma,\eta\in\chi$,*

$(S1)$ $\mathbb{S}(\alpha,\beta,\gamma)\geq 0$;

$(S2)$ $\mathbb{S}(\alpha,\beta,\gamma)=0$ *if and only if* $\alpha=\beta=\gamma$;

$(S3)$ $\mathbb{S}(\alpha,\beta,\gamma)\leq\mathbb{S}(\alpha,\alpha,\eta)+\mathbb{S}(\beta,\beta,\eta)+\mathbb{S}(\gamma,\gamma,\eta)$.

A nonempty set χ together with an $\mathbb{S}$-metric is called an $\mathbb{S}$-metric space.

Definition 2 ([5]) *A nonempty set χ together with a function $\mathbb{G}:\chi^3\to[0,\infty)$ is called a $\mathbb{G}$-metric space, denoted by $(\chi,\mathbb{G})$ if $\mathbb{G}$ satisfies*

$(G1)$ $\mathbb{G}(\alpha,\beta,\gamma)=0$ *if and only if* $\alpha=\beta=\gamma$;

$(G2)$ $0<\mathbb{G}(\alpha,\alpha,\beta)$ *for all* $\alpha,\beta\in\chi$ *with* $\alpha\neq\beta$;

$(G3)$ $\mathbb{G}(\alpha,\alpha,\beta)\leq\mathbb{G}(\alpha,\beta,\gamma)$ *for all* $\alpha,\beta,\gamma\in\chi$ *with* $\beta\neq\gamma$;

$(G4)$ $\mathbb{G}(\alpha,\beta,\gamma)=\mathbb{G}(\alpha,\gamma,\beta)=\mathbb{G}(\beta,\gamma,\alpha)=...$, *(symmetry in all three variables);*

$(G5)$ $\mathbb{G}(\alpha,\beta,\gamma)\leq\mathbb{G}(\alpha,\eta,\eta)+\mathbb{G}(\eta,\beta,\gamma)$ *for all* $\alpha,\beta,\gamma,\eta\in\chi$, *(rectangle inequality).*

Then, the nonnegative real function $\mathbb{G}$ is called a $\mathbb{G}$-metric on χ. The set χ together with such a generalized metric $\mathbb{G}$ is called a generalized metric space, or $\mathbb{G}$-metric space and denoted by $(\chi,\mathbb{G})$.

Obviously, these properties are fulfilled when $\mathbb{G}(\alpha,\beta,\gamma)$ is the perimeter of the triangle with vertices at α,β and γ in $\mathbb{R}^2$, further taking η in the interior of the triangle shows that $(G5)$ is the best possible.

In [31], it is shown that there exists a $\mathbb{G}$-metric that is not an $\mathbb{S}$-metric. Indeed, let $\chi=\{\alpha,\beta\}$, let

$$\mathbb{G}(\alpha,\alpha,\alpha)=\mathbb{G}(\beta,\beta,\beta)=0,\quad \mathbb{G}(\alpha,\alpha,\beta)=1,\quad \mathbb{G}(\alpha,\beta,\beta)=2$$

and extend $\mathbb{G}$ to all of $\chi\times\chi\times\chi$ by symmetry in the variables. Then, it is easy to proved that $\mathbb{G}$ is $\mathbb{G}$-metric, but $\mathbb{G}(\alpha,\beta,\beta)\neq\mathbb{G}(\alpha,\alpha,\beta)$ [5]. Also, we have

$$2=\mathbb{G}(\alpha,\beta,\beta)>1=\mathbb{G}(\alpha,\alpha,\beta)+\mathbb{G}(\beta,\beta,\beta)+\mathbb{G}(\beta,\beta,\beta),$$

which proves that $\mathbb{G}$ is not an $\mathbb{S}$-metric on χ.

Also in [31], there exists an example of an $\mathbb{S}$-metric that is not a $\mathbb{G}$-metric. For example, let $\chi=\mathbb{R}$ and let

$$\mathbb{S}(\alpha,\beta,\gamma)=|\beta+\gamma-2\alpha|+|\beta-\gamma|,$$

for all $\alpha,\beta,\gamma\in\chi$. $(\chi,\mathbb{S})$ is an $\mathbb{S}$-metric [6]. We have

$$\mathbb{S}(1,0,2)=|0+2-2|+|0-2|=2,$$

$$\mathbb{S}(2,0,1)=|0+1-4|+|0-1|=4.$$

Then we get

$$\mathbb{S}(1,0,2)\neq\mathbb{S}(2,0,1),$$

which proves that $\mathbb{S}$ is not a $\mathbb{G}$-metric.

Because of these relationships above it makes sense to study the fixed-circle problem on $\mathbb{G}$-metric.

Next, we recall here some notions, lemmas and examples we will use.

Example 1 ([32]) *If χ is a nonempty subset of $\mathbb{R}$, then the function $\mathbb{G} : \chi^3 \to [0, \infty)$ given by*

$$\mathbb{G}(\alpha, \beta, \gamma) = |\alpha - \beta| + |\alpha - \gamma| + |\beta - \gamma|$$

for all $\alpha, \beta, \gamma \in \chi$, is a $\mathbb{G}$-metric on χ.

Definition 3 ([5]) *A $\mathbb{G}$-metric space $(\chi, \mathbb{G})$ is called be symmetric if*

$$\mathbb{G}(\alpha, \beta, \beta) = \mathbb{G}(\beta, \alpha, \alpha),$$

for all $\alpha, \beta \in \chi$.

The mapping given in Example 1 is symmetric $\mathbb{G}$-metric. There also exist $\mathbb{G}$-metric spaces that are not symmetric, as we see in the following example.

Example 2 ([5]) *Let $\chi = \{0, 1, 2\}$ and the function $\mathbb{G} : \chi^3 \to [0, \infty)$ be defined by the following table.*

(α, β, γ)	$G(\alpha, \beta, \gamma)$
(0,0,0), (1,1,1), (2,2,2)	0
(0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,0,1),(1,1,0)	1
(1,2,2),(2,1,2),(2,2,1)	2
(0,0,2),(0,2,0),(2,0,0),(0,2,2),(2,0,2),(2,2,0)	3
(1,1,2),(1,2,1),(2,1,1),(0,1,2),(0,2,1),(1,0,2)	4
(1,2,0),(2,0,1),(2,1,0)	4

Then, $\mathbb{G}$ is a $\mathbb{G}$-metric on χ, but it is not symmetric because $\mathbb{G}(1, 1, 2) = 4 \neq 2 = \mathbb{G}(2, 2, 1)$.

The relationships between a metric and a $\mathbb{G}$-metric were given in the following lemmas.

Lemma 1 ([5]) *If (χ, d) is a metric space, then (χ, d) can define $\mathbb{G}$-metrics on χ by*

$$\mathbb{G}_m^d(\alpha, \beta, \gamma) = max\{d(\alpha, \beta), d(\beta, \gamma), d(\gamma, \alpha)\}$$

and

$$\mathbb{G}_s^d(\alpha, \beta, \gamma) = d(\alpha, \beta) + d(\beta, \gamma) + d(\gamma, \alpha),$$

for all $\alpha, \beta, \gamma \in \chi$.

Lemma 2 ([32]) *If $(\chi, \mathbb{G})$ is a $\mathbb{G}$-metric space, then $(\chi, \mathbb{G})$ can define metrics on χ by*

$$d_m^{\mathbb{G}} = max\{\mathbb{G}(\alpha, \beta, \beta), \mathbb{G}(\beta, \alpha, \alpha)\}$$

and

$$d_s^{\mathbb{G}} = \mathbb{G}(\alpha, \beta, \beta) + \mathbb{G}(\beta, \alpha, \alpha),$$

for all $\alpha, \beta \in \chi$.

Let us recall some examples of $\mathbb{G}$-metric.

Example 3 ([33]) *Every nonempty set χ can be provided with the discrete $\mathbb{G}$-metric, which is defined by*

$$\mathbb{G}(\alpha, \beta, \gamma) = \begin{cases} 0 & \alpha = \beta = \gamma, \\ 1 & otherwise, \end{cases}$$

for all $\alpha, \beta, \gamma \in \chi$. Then, $\mathbb{G}$ is a $\mathbb{G}$-metric on χ.

Example 4 ([33]) *Let $\chi = [0,\infty)$ and $\mathbb{G}$ be defined by*

$$\mathbb{G}(\alpha,\beta,\gamma) = \begin{cases} 0 & \alpha=\beta=\gamma, \\ \max\{\alpha,\beta,\gamma\} & \textit{otherwise}, \end{cases}$$

for all $\alpha,\beta,\gamma \in \chi$. Then, $\mathbb{G}$ is a $\mathbb{G}$-metric on χ.

Example 5 ([33]) *Let $\chi = [0,\infty)$ and the function $\mathbb{G} : \chi^3 \to [0,\infty)$ be defined by*

$$\mathbb{G}(\alpha,\beta,\gamma) = \begin{cases} 0 & \alpha=\beta=\gamma, \\ 1 & \alpha=\beta, \alpha=\gamma \text{ or } \beta=\gamma, \\ 2 & \alpha\neq\beta\neq\gamma, \end{cases}$$

for all $\alpha,\beta,\gamma \in \chi$. Then, $\mathbb{G}$ is a $\mathbb{G}$-metric on χ.

Next, we introduce the concept of a circle on a $\mathbb{G}$-metric space.

Definition 4 *Let $(\chi,\mathbb{G})$ be a $\mathbb{G}$-metric space. A circle of center $\alpha_0 \in \chi$ and radius $r \in (0,\infty)$ is defined as follows:*

$$C_{\mathbb{G}}(\alpha_0, r) = \{\alpha \in \chi : \mathbb{G}(\alpha_0,\alpha,\alpha) = r\}.$$

Example 6 *Let $\chi = [0,\infty)$ and the function $\mathbb{G} : \chi^3 \to [0,\infty)$ be defined by*

$$\mathbb{G}(\alpha,\beta,\gamma) = |\alpha-\beta| + |\beta-\gamma| + |\gamma-\alpha|,$$

for each $\alpha,\beta,\gamma \in \chi$ [32]. So, it can be easily seen that $\mathbb{G}$ is a $\mathbb{G}$-metric on χ and the pair $(\chi,\mathbb{G})$ is a $\mathbb{G}$-metric space. Clearly, from Lemma 1, this $\mathbb{G}$-metric is generated by taking the usual metric on $\mathbb{R}$. Then, we have the circle $C_{\mathbb{G}}(5,10)$ as follows

$$\begin{aligned} C_{\mathbb{G}}(5,10) &= \{\alpha \in \chi : \mathbb{G}(5,\alpha,\alpha) = 10\} \\ &= \{\alpha \in \chi : |5-\alpha| + |\alpha-\alpha| + |\alpha-5| = 10\} \\ &= \{\alpha \in \chi : |5-\alpha| = 5\} \\ &= \{\alpha \in \chi : \alpha = 0 \wedge \alpha = 10\} \\ &= \{0,10\}. \end{aligned}$$

Example 7 *Let $\chi = \mathbb{R}^2$ and d be a metric space. Let the function $\mathbb{G} : \chi^3 \to [0,\infty)$ be defined by*

$$\mathbb{G}(\alpha,\beta,\gamma) = max\{d(\alpha,\beta), d(\beta,\gamma), d(\gamma,\alpha)\}$$

for all $\alpha,\beta,\gamma \in \chi$. Then, $(\chi,\mathbb{G})$ be a $\mathbb{G}$-metric space. Let us consider the function $d : \chi\times\chi \to \mathbb{R}$ as

$$d(\alpha,\beta) = \sqrt{(\alpha_1-\beta_1)^2 + (\alpha_2-\beta_2)^2}$$

for all $\alpha = (\alpha_1,\alpha_2), \beta = (\beta_1,\beta_2) \in \chi$. Then, we get

$$\begin{aligned} C_{\mathbb{G}}((1,0),1) &= \{\alpha \in \chi : \mathbb{G}((1,0),\alpha,\alpha) = 1\} \\ &= \{\alpha \in \chi : max\{d((1,0),\alpha), d(\alpha,\alpha), d(\alpha,(1,0)) = 1\}\} \\ &= \{\alpha \in \chi : d((1,0),\alpha) = 1\} \\ &= \{\alpha \in \chi : \sqrt{(\alpha_1-1)^2 + (\alpha_2-0)^2} = 1\} \\ &= \{\alpha \in \chi : (\alpha_1-1)^2 + \alpha_2^2 = 1\}. \end{aligned}$$

Example 8 *Let $\chi = \mathbb{R}^3$ and d be a metric space. Let the function $\mathbb{G} : \chi^3 \to [0, \infty)$ be defined by*

$$\mathbb{G}(\alpha, \beta, \gamma) = \frac{1}{3}\left(d(\alpha, \beta) + d(\beta, \gamma) + d(\gamma, \alpha)\right)$$

for all $\alpha, \beta, \gamma \in \chi$. Then, $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space. Let us consider the function $d : \chi \times \chi \to \mathbb{R}$ as

$$d(\alpha, \beta) = |\alpha_1 - \beta_1| + |\alpha_2 - \beta_2| + |\alpha_3 - \beta_3|$$

for all $\alpha = (\alpha_1, \alpha_2, \alpha_3), \beta = (\beta_1, \beta_2, \beta_3) \in \chi$. Then, we obtain

$$\begin{aligned} C_{\mathbb{G}}\left((0,0,0), 2\right) &= \{\alpha \in \chi : \mathbb{G}\left((0,0,0), \alpha, \alpha\right) = 2\} \\ &= \{\alpha \in \chi : \frac{1}{3}\left(d((0,0,0), \alpha) + d(\alpha, \alpha) + d(\alpha, (0,0,0))\right) = 2\} \\ &= \{\alpha \in \chi : \frac{2}{3} d((0,0,0), \alpha) = 2\} \\ &= \{\alpha \in \chi : d((0,0,0), \alpha) = 3\} \\ &= \{\alpha \in \chi : |\alpha_1 - 0| + |\alpha_2 - 0| + |\alpha_3 - 0| = 3\} \\ &= \{\alpha \in \chi : |\alpha_1| + |\alpha_2| + |\alpha_3| = 3\}. \end{aligned}$$

Example 9 *Let $\chi = \mathbb{R}^3$ and the function $\mathbb{G} : \chi^3 \to [0, \infty)$ be defined by*

$$\mathbb{G}(\alpha, \beta, \gamma) = \sum_{i=1}^{3} \left(|\alpha_i - \beta_i| + |\beta_i - \gamma_i| + |\gamma_i - \alpha_i|\right)$$

for all $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3)$, $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \chi$. Then, $\mathbb{G}$ is a $\mathbb{G}$-metric on χ. If we choose $\alpha_0 = 0 = (0, 0, 0)$ and $r = 2$, then we get

$$C_{\mathbb{G}}(0, 2) = \{x \in \chi : \mathbb{G}\left(0, \alpha, \alpha\right) = 2\} = \{\alpha \in \chi : |\alpha_1| + |\alpha_2| + |\alpha_3| = 1\}.$$

If we choose $\alpha_0 = (1, 1, 2)$ and $r = 2$, then we get

$$\begin{aligned} C_{\mathbb{G}}((1,1,2), 2) &= \{\alpha \in \chi : \mathbb{G}\left((1,1,2), \alpha, \alpha\right) = 2\} \\ &= \{\alpha \in \chi : |\alpha_1 - 1| + |\alpha_2 - 1| + |\alpha_3 - 2| = 1\}. \end{aligned}$$

3 Existence Theorems for Fixed-Circles of Self-Mappings

Firstly, we introduce the concept of a fixed-circle on a $\mathbb{G}$-metric space. After that, we present some existence theorems for fixed-circles of self-mappings and obtain some examples of mappings which have or not fixed-circles.

Definition 5 *Let $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space and $C_{\mathbb{G}}(\alpha_0, r)$ be a circle. For a self-mapping $T : \chi \to \chi$, if $T\alpha = \alpha$ for all $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$ then, the circle $C_{\mathbb{G}}(\alpha_0, r)$ is said to be a fixed-circle of T.*

Theorem 1 *Let $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space and $C_{\mathbb{G}}(\alpha_0, r)$ be any circle on χ. Let us define the mapping $\phi : \chi \to [0, \infty)$ as*

$$\phi(\alpha) = \mathbb{G}(\alpha, \alpha, \alpha_0), \tag{1}$$

for all $\alpha \in \chi$. If there exists a self-mapping $T : \chi \to \chi$ satisfying

$$\mathbb{G}(\alpha, \alpha, T\alpha) \leq \phi(\alpha) - \phi(T\alpha) \tag{2}$$

and

$$\mathbb{G}(T\alpha, T\alpha, \alpha_0) \geq r, \tag{3}$$

for all $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$, then the circle $C_{\mathbb{G}}(\alpha_0, r)$ is a fixed-circle of T.

Proof. Let us consider the mapping ϕ defined in (1). Let $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$ be any arbitrary point. We show that $T\alpha = \alpha$ whenever $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$. Using the inequalities (2), (3) and the notion of the mapping ϕ, we obtain

$$\begin{aligned} \mathbb{G}(\alpha, \alpha, T\alpha) &\leq \phi(\alpha) - \phi(T\alpha) \\ &\leq \mathbb{G}(\alpha, \alpha, \alpha_0) - \mathbb{G}(T\alpha, T\alpha, \alpha_0) \\ &\leq r - \mathbb{G}(T\alpha, T\alpha, \alpha_0) \leq r - r = 0 \end{aligned} \tag{4}$$

and so $T\alpha = \alpha$. Consequently, $C_{\mathbb{G}}(\alpha_0, r)$ is a fixed-circle of T. ■

Remark 1 *Notice that the inequality (2) guarantees that $T\alpha$ is not in the interior of the circle $C_{\mathbb{G}}(\alpha_0, r)$ for $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$. Then, there exist two cases.*

Case 1. *If $T\alpha$ is exterior of the circle $C_{\mathbb{G}}(\alpha_0, r)$, then we have*

$$\mathbb{G}(T\alpha, T\alpha, \alpha_0) > r.$$

This is a contradiction with the inequality (4) since the distance function is a nonnegative function.

Case 2. *Then, the point $T\alpha$ should be lies on of the circle $C_{\mathbb{G}}(\alpha_0, r)$ and so $\mathbb{G}(T\alpha, T\alpha, \alpha_0) = r$. In this case, using (4) we obtain*

$$\mathbb{G}(\alpha, \alpha, T\alpha) \leq r - \mathbb{G}(T\alpha, T\alpha, \alpha_0) = r - r = 0.$$

That is $T\alpha = \alpha$ and the self-mapping T fixes the circle $C_{\mathbb{G}}(\alpha_0, r)$.

Theorem 2 *Let $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space, $C_{\mathbb{G}}(\alpha_0, r)$ be any circle on χ and the mapping ϕ be defined as in (1). If there exists a self-mapping $T : \chi \to \chi$ satisfying*

$$\mathbb{G}(\alpha, \alpha, T\alpha) \leq \phi(\alpha) + \phi(T\alpha) - 2r \tag{5}$$

and

$$\mathbb{G}(\alpha, T\alpha, T\alpha) + \mathbb{G}(T\alpha, T\alpha, \alpha_0) \leq r, \tag{6}$$

for all $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$, then the circle $C_{\mathbb{G}}(\alpha_0, r)$ is a fixed-circle of T.

Proof. Let $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$. Then using the inequalities (5), (6) and the properties $(G4)$, $(G5)$ of a $\mathbb{G}$-metric space, we have

$$\begin{aligned} \mathbb{G}(\alpha, \alpha, T\alpha) &\leq \phi(\alpha) + \phi(T\alpha) - 2r \\ &= \mathbb{G}(\alpha, \alpha, \alpha_0) + \mathbb{G}(T\alpha, T\alpha, \alpha_0) - 2r \\ &\leq \mathbb{G}(\alpha, T\alpha, T\alpha) + \mathbb{G}(T\alpha, \alpha, \alpha_0) + \mathbb{G}(T\alpha, T\alpha, \alpha_0) - 2r \\ &= \mathbb{G}(\alpha, T\alpha, T\alpha) + \mathbb{G}(\alpha, T\alpha, \alpha_0) + \mathbb{G}(T\alpha, T\alpha, \alpha_0) - 2r \\ &\leq \mathbb{G}(\alpha, T\alpha, T\alpha) + \mathbb{G}(\alpha, T\alpha, T\alpha) + \mathbb{G}(T\alpha, T\alpha, \alpha_0) + \mathbb{G}(T\alpha, T\alpha, \alpha_0) - 2r \\ &= 2\mathbb{G}(\alpha, T\alpha, T\alpha) + 2\mathbb{G}(T\alpha, T\alpha, \alpha_0) - 2r \\ &\leq 2r - 2r = 0 \end{aligned}$$

and so

$$\mathbb{G}(\alpha, \alpha, T\alpha) = 0$$

which implies $T\alpha = \alpha$. As a result, $C_{\mathbb{G}}(\alpha_0, r)$ is a fixed-circle of T. ■

Remark 2 *Notice that the inequality (5) guarantees that $T\alpha$ is not in the interior of the circle $C_{\mathbb{G}}(\alpha_0, r)$ for $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$. Also, the inequality (6) guarantees that $T\alpha$ is not exterior of the circle $C_{\mathbb{G}}(\alpha_0, r)$ for $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$. In this case, $T\alpha \in C_{\mathbb{G}}(\alpha_0, r)$ for each $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$.*

Remark 3 *(1) Theorems on the existence of fixed-circles of a self-mappings on metric spaces are given in [10]. (2) Theorems on the existence of fixed-circles of a self-mappings on $\mathbb{S}$-metric spaces are given in [20, 22].*

Next, we give some examples of a self-mapping with a fixed-circle.

Example 10 *Let $\chi = [0,\infty)$ and the function $\mathbb{G} : \chi^3 \to [0,\infty)$ be defined as in Example 1. Then, $(\chi, \mathbb{G})$ is a $\mathbb{G}$-metric space. Let us consider the circle $C_{\mathbb{G}}(0,2)$ and define the self-mapping $T : \chi \to \chi$ as*

$$T\alpha = 3\alpha^2 + \alpha - 3,$$

for all $\alpha \in \chi$. Then, the self-mapping T satisfies the conditions of the Theorem 1 and Theorem 2. Hence,

$$\begin{aligned} C_{\mathbb{G}}(0,2) &= \{\alpha \in \chi : \mathbb{G}(0,\alpha,\alpha) = 2\} \\ &= \{\alpha \in \chi : |0-\alpha| + |\alpha-\alpha| + |\alpha-0| = 2\} \\ &= \{\alpha \in \chi : 2|\alpha| = 2\} \\ &= \{-1, 1\}. \end{aligned}$$

Thus, $C_{\mathbb{G}}(0,2)$ is a fixed-circle of T.

Example 11 *Let $\chi = \mathbb{R}$ and the function $\mathbb{G} : \chi^3 \to [0,\infty)$ be defined as in Example 1. Then, $(\chi, \mathbb{G})$ is a $\mathbb{G}$-metric space. Let us consider the circle $C_{\mathbb{G}}(0,4)$ and define the self-mapping $T : \chi \to \chi$ as*

$$T\alpha = \begin{cases} \alpha & \alpha \in \{-2, 2\}, \\ 18 & \textit{otherwise}, \end{cases}$$

for all $\alpha \in \chi$. Then the self-mapping T satisfies the conditions of the Theorem 1 and Theorem 2. Thus, $C_{\mathbb{G}}(0,4) = \{-2,2\}$ is a fixed-circle of T. Notice that $C_{\mathbb{G}}(10,16) = \{2,18\}$ is another fixed-circle of T and so the fixed-circle is not unique for a giving self-mapping. On the other hand, the circle $C_{\mathbb{G}}(0,2) = \{-1,1\}$ is not a fixed-circle of T.

Example 12 *Let $\chi = [0,\infty)$ and the function $\mathbb{G} : \chi^3 \to [0,\infty)$ be defined as in Example 1. Let us consider the circles $C_{\mathbb{G}}(0,2) = \{-1,1\}$ and $C_{\mathbb{G}}(4,10) = \{-1,9\}$ and define the self-mapping $T : \chi \to \chi$ as*

$$T\alpha = \begin{cases} \frac{1}{\alpha} & \alpha \in C_{\mathbb{G}}(0,2), \\ 9 & \textit{otherwise}, \end{cases}$$

for all $\alpha \in \chi$. Then, the self-mapping T satisfies the conditions of the Theorem 1 and Theorem 2. Notice that the fixed-circle is not unique.

In the following example, we give an example of a self-mapping that satisfies the inequalities (2) and (6) but does not satisfy the inequalities (3) and (5).

Example 13 *Let $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space, $C_{\mathbb{G}}(\alpha_0, r)$ be a circle on χ and the self-mapping $T : \chi \to \chi$ be defined as*

$$T\alpha = \alpha_0,$$

for all $\alpha \in \chi$. Then, the self-mapping T does not satisfy the inequality (3) of Theorem 1 and the inequality (5) of Theorem 2. Thus, T does not fix a circle $C_{\mathbb{G}}(\alpha_0, r)$.

Now we present a contractive condition exclude the identity map $I_\chi : \chi \to \chi$ defined by

$$I_\chi(\alpha) = \alpha,$$

for all $\alpha \in \chi$ in the above existence theorems.

Theorem 3 *Let $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space, $C_{\mathbb{G}}(\alpha_0, r)$ be a circle on χ, $T : \chi \to \chi$ be a self-mapping and the mapping ϕ be defined as in (1). T satisfies the condition*

$$(I_{\mathbb{G}})\ \mathbb{G}(\alpha, \alpha, T\alpha) \leq h\left[\phi(\alpha) - \phi(T\alpha)\right],$$

for all $\alpha \in \chi$ and some $h \in \left[0, \frac{1}{4}\right)$ if and only if $T = I_{\chi}$.

Proof. Let T satisfies the condition $(I_{\mathbb{G}})$ and $\alpha \neq T\alpha$. Then, using the conditions $(G4)$ and $(G5)$, we have

$$\begin{aligned}
\mathbb{G}(\alpha, \alpha, T\alpha) &\leq h\left[\phi(\alpha) - \phi(T\alpha)\right] \\
&= h\left[\mathbb{G}(\alpha, \alpha, \alpha_0) - \mathbb{G}(T\alpha, T\alpha, \alpha_0)\right] \\
&\leq h\left[\mathbb{G}(\alpha, T\alpha, T\alpha) + \mathbb{G}(T\alpha, \alpha, \alpha_0) - \mathbb{G}(T\alpha, T\alpha, \alpha_0)\right] \\
&= h\left[\mathbb{G}(\alpha, T\alpha, T\alpha) + \mathbb{G}(\alpha, T\alpha, \alpha_0) - \mathbb{G}(T\alpha, T\alpha, \alpha_0)\right] \\
&= h\left[\mathbb{G}(\alpha, T\alpha, T\alpha) + \mathbb{G}(\alpha, T\alpha, T\alpha) + \mathbb{G}(T\alpha, T\alpha, \alpha_0) - \mathbb{G}(T\alpha, T\alpha, \alpha_0)\right] \\
&\leq 2h\mathbb{G}(\alpha, T\alpha, T\alpha) = 2h\mathbb{G}(T\alpha, \alpha, T\alpha) \\
&\leq 2h\left[\mathbb{G}(T\alpha, \alpha, \alpha) + \mathbb{G}(\alpha, \alpha, T\alpha)\right] \\
&= 4h\mathbb{G}(T\alpha, \alpha, \alpha) = 4h\mathbb{G}(\alpha, \alpha, T\alpha),
\end{aligned}$$

which is a contradiction. So it should be $\alpha = T\alpha$ for each $\alpha \in \chi$ and $T = I_{\chi}$. The converse statement is clear. ■

Remark 4 *If a self-mapping $T : \chi \to \chi$ does not satisfy the condition $(I_{\mathbb{G}})$ then we exclude the identity map.*

4 Uniqueness Theorems for Fixed-Circles of Self-Mappings

In this section, firstly, we give two propositions expressing the existence of mappings having more than one fixed-circle in the $\mathbb{G}$-metric space. Then, we investigate the uniqueness of fixed-circles obtained in the existence theorems.

Proposition 1 *Let $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space. For any given circles $C_{\mathbb{G}}(\alpha_0, r)$ and $C_{\mathbb{G}}(\alpha_1, \rho)$, there exists at least one self-mapping T of χ such that T fixes the circles $C_{\mathbb{G}}(\alpha_0, r)$ and $C_{\mathbb{G}}(\alpha_1, \rho)$.*

Proof. Let $C_{\mathbb{G}}(\alpha_0, r)$ and $C_{\mathbb{G}}(\alpha_1, \rho)$ be any circles on χ. Also, let ξ be a constant point in χ satisfying $\mathbb{G}(\xi, \xi, \alpha_0) \neq r$ and $\mathbb{G}(\xi, \xi, \alpha_1) \neq \rho$. Let us define the self-mapping $T : \chi \to \chi$ as

$$T\alpha = \begin{cases} \alpha & \alpha \in C_{\mathbb{G}}(\alpha_0, r) \cup C_{\mathbb{G}}(\alpha_1, \rho), \\ \xi & \text{otherwise}, \end{cases}$$

for all $\alpha \in \chi$. Let us define the mappings $\phi_1, \phi_2 : \chi \to [0, \infty)$ as

$$\phi_1(\alpha) = \mathbb{G}(\alpha, \alpha, \alpha_0)$$

and

$$\phi_2(\alpha) = \mathbb{G}(\alpha, \alpha, \alpha_1),$$

for all $\alpha \in \chi$. Then, the self-mapping T satisfies the conditions of the Theorem 1 and Theorem 2. Thus, $C_{\mathbb{G}}(\alpha_0, r)$ and $C_{\mathbb{G}}(\alpha_1, \rho)$ are the fixed-circles of T. ■

Notice that the circles $C_{\mathbb{G}}(\alpha_0, r)$ and $C_{\mathbb{G}}(\alpha_1, \rho)$ do not have to be disjoint as seen in (12).

In the following example, the self-mapping T has two fixed-circle.

Example 14 *Let $\chi = [0, \infty)$ and $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space with the $\mathbb{G}$-metric defined in as Example 1. Let us consider the circles $C_{\mathbb{G}}(2,4)$, $C_{\mathbb{G}}(3,6)$ and define the self-mapping $T : \chi \to \chi$ as*

$$T\alpha = \begin{cases} \alpha & \alpha \in \{0,4,6\}, \\ \xi & \text{otherwise,} \end{cases}$$

for all $\alpha \in \chi$ where $\xi \in \chi$. Then, the conditions of the Theorem 1 and Theorem 2 are satisfied by T for the circles $C_{\mathbb{G}}(2,4)$ and $C_{\mathbb{G}}(3,6)$, respectively. Consequently, $C_{\mathbb{G}}(2,4)$ and $C_{\mathbb{G}}(3,6)$ are the fixed-circles of T.

Corollary 1 *Let $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space. For any given circles*

$$C_{\mathbb{G}}(\alpha_1, r_1), ..., C_{\mathbb{G}}(\alpha_n, r_n),$$

there exists at least one self-mapping T of χ such that T fixes the circles

$$C_{\mathbb{G}}(\alpha_1, r_1), ..., C_{\mathbb{G}}(\alpha_n, r_n).$$

Theorem 4 *Let $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space and $C_{\mathbb{G}}(\alpha_0, r)$ be any circle on χ. Let $T : \chi \to \chi$ be a self-mapping satisfying the inequalities (2) and (3) of Theorem 1. If the contractive condition*

$$\mathbb{G}(T\alpha, T\alpha, T\beta) \leq k\mathbb{G}(\alpha, \alpha, \beta) \tag{7}$$

is satisfied for all $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$, $\beta \in \chi \setminus C_{\mathbb{G}}(\alpha_0, r)$ and some $k \in [0,1)$ by T then, $C_{\mathbb{G}}(\alpha_0, r)$ is the unique fixed-circle of T.

Proof. Assume that there exist two different fixed-circles $C_{\mathbb{G}}(\alpha_0, r)$ and $C_{\mathbb{G}}(\alpha_1, \rho)$ of the self-mapping T. Let $u \in C_{\mathbb{G}}(\alpha_0, r)$ and $v \in C_{\mathbb{G}}(\alpha_1, \rho)$ be arbitrary points such that $u \neq v$. We show that $\mathbb{G}(u, u, v) = 0$ and hence $u = v$. Using the contractive condition of T, we obtain

$$\mathbb{G}(u, u, v) = \mathbb{G}(Tu, Tu, Tv) \leq k\mathbb{G}(u, u, v)$$

which is a contradiction since $k \in [0,1)$. As a result, $C_{\mathbb{G}}(\alpha_0, r)$ is the unique fixed-circle of T. ∎

Notice that the self-mapping T given in the proof of Proposition 1 does not satisfy the contractive condition (7).

Now, we give a uniqueness condition for the fixed-circles in Theorem 2.

Theorem 5 *Let $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space and $C_{\mathbb{G}}(\alpha_0, r)$ be any circle on χ. Let $\chi \to \chi$ be a self-mapping satisfying the inequalities (5) and (6) of Theorem 2. If the contractive condition defined in (7) is satisfied for all $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$, $\beta \in \chi \setminus C_{\mathbb{G}}\alpha_0, r)$ and some $k \in [0,1)$ by T, then $C_{\mathbb{G}}(\alpha_0, r)$ is the unique fixed-circle of T.*

Proof. It can be easily proven similar to the proof of Theorem 4. ∎

It is important to investigate the uniqueness of the fixed-circles. Firstly, we determine the uniqueness conditions for the fixed-circles in Theorem 1 using the Banach type contractive condition [9].

Using the Rhoades type contractive condition [36], we obtain the following theorem.

Theorem 6 *Let $(\chi, \mathbb{G})$ be a symmetric $\mathbb{G}$-metric space and $C_{\mathbb{G}}(\alpha_0, r)$ be any circle on χ. Let $T : \chi \to \chi$ be a self-mapping satisfying the conditions in Theorem 1 and Theorem 2. If the contractive condition*

$$\mathbb{G}(T\alpha, T\alpha, T\beta) < \max\{\mathbb{G}(\alpha, \alpha, \beta), \mathbb{G}(\alpha, \alpha, T\alpha), \mathbb{G}(\beta, \beta, T\beta), \mathbb{G}(\alpha, \alpha, T\beta), \mathbb{G}(\beta, \beta, T\alpha)\} \tag{8}$$

is satisfied for all $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$ and $\beta \in \chi \setminus C_{\mathbb{G}}(\alpha_0, r)$, then the fixed-circle of T is unique.

Proof. Assume that there exist two fixed-circles $C_{\mathbb{G}}(\alpha_0, r)$ and $C_{\mathbb{G}}(\alpha_1, \rho)$ of the self-mapping T, that is, T satisfies the conditions in Theorem 1 and Theorem 2 for each circles $C_{\mathbb{G}}(\alpha_0, r)$ and $C_{\mathbb{G}}(\alpha_1, \rho)$. Let $u \in C_{\mathbb{G}}(\alpha_0, r)$, $v \in C_{\mathbb{G}}(\alpha_1, \rho)$ and $u \neq v$ be arbitrary points. We show that $\mathbb{G}(u, u, v) = 0$ and hence $u = v$. Using the inequality (8) and the symmetric property of $\mathbb{G}$-metric, we obtain

$$\begin{aligned} \mathbb{G}(u, u, v) &= \mathbb{G}(Tu, Tu, Tv) \\ &< \max\{\mathbb{G}(u, u, v), \mathbb{G}(u, u, Tu), \mathbb{G}(v, v, Tv), \mathbb{G}(u, u, Tv), \mathbb{G}(v, v, Tu)\} \\ &= \mathbb{G}(u, u, v) \end{aligned}$$

which is a contradiction. As a result, it should be $u = v$ for all $u \in C_{\mathbb{G}}(\alpha_0, r)$, $v \in C_{\mathbb{G}}(\alpha_1, \rho)$ and thus $C_{\mathbb{G}}(\alpha_0, r)$ is the unique fixed-circle of T. ■

By the Kannan type contractive condition [34], we prove the following theorem.

Theorem 7 *Let $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space and $C_{\mathbb{G}}(\alpha_0, r)$ be any circle on χ. Let $T : \chi \to \chi$ be a self-mapping satisfying the conditions in Theorem 1 and Theorem 2. If the following condition*

$$\mathbb{G}(T\alpha, T\alpha, T\beta) \leq \lambda \left(\mathbb{G}(\alpha, \alpha, T\alpha) + \mathbb{G}(\beta, \beta, T\beta)\right) \tag{9}$$

is satisfied for all $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$, $\beta \in \chi \setminus C_{\mathbb{G}}(\alpha_0, r)$ and $\lambda \in [0, \frac{1}{2})$, then $C_{\mathbb{G}}(\alpha_0, r)$ is a unique circle of T.

Proof. Assume that there exist two fixed-circles $C_{\mathbb{G}}(\alpha_0, r)$ and $C_{\mathbb{G}}(\alpha_1, \rho)$ of the self-mapping T, that is, T satisfies the conditions in Theorem 1 and Theorem 2 for each circles $C_{\mathbb{G}}(\alpha_0, r)$ and $C_{\mathbb{G}}(\alpha_1, \rho)$. Let $u \in C_{\mathbb{G}}(\alpha_0, r)$, $v \in C_{\mathbb{G}}(\alpha_1, \rho)$ and $u \neq v$ be arbitrary points. Using the inequality (9), we have

$$\begin{aligned} \mathbb{G}(u, u, v) &= \mathbb{G}(Tu, Tu, Tv) \\ &\leq \lambda \left(\mathbb{G}(u, u, Tu) + \mathbb{G}(v, v, Tv)\right) \\ &= \lambda \left(\mathbb{G}(u, u, u) + \mathbb{G}(v, v, v)\right) \\ &= 0. \end{aligned}$$

So, we obtain $\mathbb{G}(u, u, v) = 0$, that is $u = v$. But, this is a contradiction. Consequently, $C_{\mathbb{G}}(\alpha_0, r)$ is a unique fixed-circle of T. ■

From the Reich type contractive condition [35], we obtain another uniqueness theorem as follows.

Theorem 8 *Let $(\chi, \mathbb{G})$ be a $\mathbb{G}$-metric space and $C_{\mathbb{G}}(\alpha_0, r)$ be any circle on χ. Let $T : \chi \to \chi$ be a self-mapping satisfying the conditions in Theorem 1 and Theorem 2. If the following condition*

$$\mathbb{G}(T\alpha, T\alpha, T\beta) \leq x\mathbb{G}(\alpha, \alpha, T\alpha) + y\mathbb{G}(\beta, \beta, T\beta) + z\mathbb{G}(\alpha, \alpha, \beta) \tag{10}$$

is satisfied for all $\alpha \in C_{\mathbb{G}}(\alpha_0, r)$, $\beta \in \chi \setminus C_{\mathbb{G}}(\alpha_0, r)$ such that $x + y + z < 1$, then $C_{\mathbb{G}}(\alpha_0, r)$ is a unique circle of T.

Proof. Assume that there exist two fixed-circles $C_{\mathbb{G}}(\alpha_0, r)$ and $C_{\mathbb{G}}(\alpha_1, \rho)$ of the self-mapping T, that is, T satisfies the conditions in Theorem 1 and Theorem 2 for each circles $C_{\mathbb{G}}(\alpha_0, r)$ and $C_{\mathbb{G}}(\alpha_1, \rho)$. Let $u \in C_{\mathbb{G}}(\alpha_0, r)$, $v \in C_{\mathbb{G}}(\alpha_1, \rho)$ and $u \neq v$ be arbitrary points. Using the inequality (10), we have

$$\begin{aligned} \mathbb{G}(u, u, v) &= \mathbb{G}(Tu, Tu, Tv) \\ &\leq x\mathbb{G}(u, u, Tu) + y\mathbb{G}(v, v, Tv) + z\mathbb{G}(u, u, v) \\ &= z\mathbb{G}(u, u, v). \end{aligned}$$

Since $x + y + z < 1$, clearly $z < 1$. Consequently, it is a contradiction. So, $C_{\mathbb{G}}(\alpha_0, r)$ is a unique fixed-circle of T. ■

5 An Application to Activation Functions

Activation functions are important in neural networks and applicable areas. There are a lot of activation functions in the literature. In the context of artificial neural networks, one of them is Rectified Linear Unit (ReLU) activation function (see [37] and the references therein). This activation function is defined by

$$ReLU(x) = x^+ = \max\{0, x\} = \begin{cases} 0 & x \leq 0, \\ x & x > 0, \end{cases}$$

where x is the input to a neuron. This function can be considered as the positive part of its argument.

Let us consider this function on $\chi = \mathbb{R}$ with the $\mathbb{G}$-metric defined as in Example 1. The function $ReLU$ satisfies the conditions of Theorem 1 and Theorem 2 for the circle $C_{\mathbb{G}}(2,1) = \left\{\frac{3}{2}, \frac{5}{2}\right\}$. Consequently, the circle $C_{\mathbb{G}}(2,1)$ is a fixed-circle of $ReLU$. On the other hand, the function $ReLU$ satisfies the conditions of Theorem 1 and Theorem 2 for the circle $C_{\mathbb{G}}(3,2) = \{2,4\}$. Hence, the circle $C_{\mathbb{G}}(3,2)$ is another fixed circle of $ReLU$. Consequently, the fixed-circle of this activation function is not unique. This situation is important for increasing the number of fixed-points used in neural networks.

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