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On Interpolative Type Multiple Fixed Points, Their Geometry And Applications On S-Metric Spaces*

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Abstract

The survival of a unique fixed point plays a central role in metric fixed point theory and has numerous applications in day-to-day life. However, if a self map has multiple fixed points, then looking at the geometry of the collection of fixed points is extremely appealing and natural. As a result, it is interesting to study the fixed figure problems utilizing interpolative techniques via S-metric spaces. In the present work, we examine novel hypotheses to explore the geometry of the collection of fixed points by establishing the existence of multiple fixed points via interpolative technique in S-metric spaces. Further, we exclude the possibility of an identity map in fixed circle (disc) conclusions. We verify the established conclusions by non-trivial illustrative examples. We conclude the work by discussing the parametric rectified linear unit activation function which is beneficial in the study of neural networks and solving integral equations which is beneficial in numerous mathematical models.

1 Introduction and Motivation

In 2018, Karapınar [20] embraced the interpolative technique to determine a fixed point by bringing in the generalized Kannan-type contraction. S-metric spaces [38] have been initiated as an extension of metric spaces which need not be a metric space. Numerous counterexamples are available in the literature to support this fact. The reason behind this is the fact that the triangle inequality may not be verified by elements of the underlying set. So the claim that each metric gives rise to an S-metric, is not true. A present-day perspective to the study of metric fixed point theory is to investigate the geometry of the set of fixed points with numerous applications. The investigation of the fixed figure, which is contained in the collection of fixed points is equivalent to the investigation of the fixed point. In the case of metric giving rise to an S-metric, a radius of a circle $C_{u_0,\frac{r}{2}}$ in metric space is half of the radius of the circle $C^S_{u_0,r}$ on the S-metric space.

The collection of multiple fixed points may embrace some geometric figure. For instance, a circle, a disc, an ellipse, an elliptic disc, or a hyperbola, we investigate its geometry via interpolative technique in S-metric space (for more detail in the metric case, see [14]-[15], [35]). We look at new hypotheses which are essential for the collection of multiple fixed points to incorporate a disc or a circle. For this, we make some necessary amendments to well-known fixed-point techniques. Further, we establish a characterization to exclude the possibility of an identity map in fixed circle (disc) conclusions. Established conclusions are verified by appropriate non-trivial examples. It is interesting to observe that a self-map fixing a disc also fixes a circle. We also provide some new illustrative examples of an S-metric which does not arise from a metric to contradict the fact that S-metric describes a metric. Towards the end, we discuss the parametric rectified linear unit activation function in the environment of fixed circle and fixed disc since it is interesting to provide the real-life application of fixed circle as well as fixed disc conclusions and activation functions perform an essential role in the study of neural networks. Also, we solve an integral equation via interpolative conclusions which is applicable in science and engineering besides other areas. It is well-known that numerous

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day-to-day problems can be solved by utilizing techniques of metric fixed point theory (see, Tomar and Joshi [41]) for interesting conclusions. Established fixed point and fixed circle (disc) conclusions encourage more applications and explorations in S-metric spaces.

2 Preliminaries

First, we start by recalling the interpretation of S-metric spaces and then discuss essential concepts and conclusions which would be applicable in the next section.

Definition 1 ([38]) *An S-metric on a nonempty set $\mathcal{U}$ is a function $\mathcal{S} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to [0, \infty)$ satisfying the subsequent postulates:*

$(S1)$ $\mathcal{S}(\mathfrak{u}, v, w) = 0$ *if and only if* $\mathfrak{u} = v = w$,

$(S2)$ $\mathcal{S}(\mathfrak{u}, v, w) \leq \mathcal{S}(\mathfrak{u}, \mathfrak{u}, a) + \mathcal{S}(v, v, a) + \mathcal{S}(w, w, a)$, $\mathfrak{u}, v, w, a \in \mathcal{U}$.

The pair $(\mathcal{U}, \mathcal{S})$ is known as an S-metric space.

Lemma 1 ([38]) *In an S-metric space $(\mathcal{U}, \mathcal{S})$, $\mathcal{S}(\mathfrak{u}, \mathfrak{u}, v) = \mathcal{S}(v, v, \mathfrak{u})$ for $\mathfrak{u}, v \in \mathcal{U}$.*

A metric and an S-metric have been compared in many works (see [9], [12], and [33] for more details). Hieu et al. [12], provided a subsequent association between a metric and an S-metric space:

$$\mathcal{S}_d(\mathfrak{u}, v, w) = d(\mathfrak{u}, w) + d(v, w) \text{ for } u, v, w \in \mathcal{U}.$$

Here $\mathcal{S}_d : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to [0, \infty)$ is the S-metric that arises from the standard metric d [33]. However, this is not always true. There may exist an S-metric which does not arise from a metric. The subsequent example supports this fact.

Example 1 ([33]) *Suppose the function $\mathcal{S} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to [0, \infty)$ is described on a set of real numbers ($\mathcal{U} = \mathbb{R}$) as:*

$$\mathcal{S}(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = |\mathfrak{u} - \mathfrak{w}| + |\mathfrak{u} + \mathfrak{w} - 2\mathfrak{v}| \quad \textit{for } \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in \mathcal{U}.$$

Then $\mathcal{S}$ is an S-metric on $\mathcal{U}$ which does not arise from any standard metric d.

Gupta [9], has shown that each S-metric describes a metric, that is,

$$d_S(\mathfrak{u}, \mathfrak{v}) = \mathcal{S}(\mathfrak{u}, \mathfrak{u}, \mathfrak{v}) + \mathcal{S}(\mathfrak{v}, \mathfrak{v}, \mathfrak{u}) \text{ for } \mathfrak{u}, \mathfrak{v} \in \mathcal{U}.$$

However, the function d_S does not essentially describe a metric since all the elements of $\mathcal{U}$ do not verify the triangle inequality everywhere (see, [33] for more details) as observed in the subsequent example:

Example 2 ([33]) *Suppose the function $\mathcal{S} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to [0, \infty)$ is described on $\mathcal{U} = \{1, 2, 3\}$ as:*

$$\mathcal{S}(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \begin{cases} 1 & \textit{if } \mathfrak{u} \neq \mathfrak{v} \neq \mathfrak{w}, \\ 0 & \textit{if } \mathfrak{u} = \mathfrak{v} = \mathfrak{w}, \end{cases} \quad \textit{for } \mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in \mathcal{U},$$

$$\mathcal{S}(1, 1, 2) = \mathcal{S}(2, 2, 1) = 5, \;\; \mathcal{S}(2, 2, 3) = \mathcal{S}(3, 3, 2) = \mathcal{S}(1, 1, 3) = \mathcal{S}(3, 3, 1) = 2.$$

Then $\mathcal{S}$ is an S-metric on $\mathcal{U}$ which neither arises from any standard metric nor gives rise to any standard metric d_S.

The following theorem establishes the association of an S-metric with a b-metric [3].

Theorem 1 ([39]) *Let $(\mathcal{U},\mathcal{S})$ be an S -metric space. Let $d^S:\mathcal{U}\times\mathcal{U}\to\mathbb{R}^+$ be the function defined by*

$$d^S(\mathfrak{u},\mathfrak{v})=\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{v}) \quad \text{for } \mathfrak{u},\mathfrak{v}\in\mathcal{U}.$$

Then subsequent conclusions hold:

1. d^S *is a b-metric on* $\mathcal{U}$,
2. $\mathfrak{u}_n\to\mathfrak{u}$ *in* $(\mathcal{U},\mathcal{S})$ *if and only if* $\mathfrak{u}_n\to\mathfrak{u}$ *in* $(\mathcal{U},d^S)$,
3. $\{\mathfrak{u}_n\}$ *is a Cauchy sequence in* $(\mathcal{U},\mathcal{S})$ *if and only if* $\{\mathfrak{u}_n\}$ *is a Cauchy sequence in* $(\mathcal{U},d^S)$.

The b-metric d^S arises from S-metric $\mathcal{S}$.

In view of the above discussion, it is significant to explore novel multiple fixed-point as well as fixed circle (disc) conclusions on S-metric spaces. Özgür and Taş [34] and Sedghi et al. [38], introduced a circle and a disc respectively on an S-metric space, which is described as follows:

$$C^S_{\mathfrak{u}_0,r}=\{\mathfrak{u}\in\mathcal{U}:\mathcal{S}\left(\mathfrak{u},\mathfrak{u},\mathfrak{u}_0\right)=r,\ \mathfrak{u}_0\in\mathcal{U},\ r\in[0,\infty)\}$$

and

$$D^S_{\mathfrak{u}_0,r}=\{\mathfrak{u}\in\mathcal{U}:\mathcal{S}\left(\mathfrak{u},\mathfrak{u},\mathfrak{u}_0\right)\le r,\ \ \mathfrak{u}_0\in\mathcal{U},\ r\in[0,\infty)\}.$$

Definition 2 ([30, 34]) *Let $D^S_{\mathfrak{u}_0,r}$ be a disc (resp. circle $C^S_{\mathfrak{u}_0,r}$) on an S-metric space $(\mathcal{U},\mathcal{S})$ and $T:\mathcal{U}\to\mathcal{U}$ be a self-mapping. If $T\mathfrak{u}=\mathfrak{u}$, $\mathfrak{u}\in D^S_{\mathfrak{u}_0,r}$ (resp. $\mathfrak{u}\in C^S_{\mathfrak{u}_0,r}$), then the disc $D^S_{\mathfrak{u}_0,r}$ (resp. the circle $C^S_{\mathfrak{u}_0,r}$) is the fixed disc (resp. circle) of T.*

Remark 1 *To work on the geometry of a set of fixed points in M^b_v-metric space we refer to Joshi et al. [13], in partial metric space, we refer to Tomar et al. [16] and [42], in b-metric space, we refer to Joshi et al. [17], the geometry of a set of near fixed points in metric interval space, we refer to Tomar and Joshi [43], in b-interval metric space, we refer to Joshi and Tomar [18].*

3 Main Results

First, we establish at least one fixed point of a discontinuous interpolative contraction and then explore some new postulates to look into the geometry of the set of fixed points of a self-mapping T, that is

$$Fix(T)=\{\mathfrak{u}\in\mathcal{U}:T\mathfrak{u}=\mathfrak{u}\},$$

as novel answers to the fixed-circle problem in [32] on an S-metric space.

Theorem 2 *Let $T:\mathcal{U}\to\mathcal{U}$ be a self-mapping of a complete S-metric space $(\mathcal{U},\mathcal{S})$ and*

$$\mathcal{S}(T\mathfrak{u},T\mathfrak{u},T\mathfrak{v})\le\lambda\mathcal{N}(\mathfrak{u},\mathfrak{v}),\quad \lambda\in[0,1), \tag{1}$$

where

$$\mathcal{N}(\mathfrak{u},\mathfrak{v})=[\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})]^\alpha\,[\mathcal{S}(\mathfrak{v},\mathfrak{v},T\mathfrak{v})]^\beta\left[\frac{\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})+\mathcal{S}(\mathfrak{v},\mathfrak{v},T\mathfrak{v})}{2}\right]^\gamma\left[\frac{\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{v})+\mathcal{S}(\mathfrak{v},\mathfrak{v},T\mathfrak{u})}{2}\right]^\delta,$$

for $\mathfrak{u},\mathfrak{v}\in\mathcal{U}$, $\alpha+\beta+\gamma+\delta<1$ and $\alpha,\beta,\gamma,\delta\in(0,1)$. Then T has a fixed point in $\mathcal{U}$.

Proof. Define a sequence $\{\mathfrak{u}_n\}$ as $\mathfrak{u}_{n+1} = T\mathfrak{u}_n,\ n \in \mathbb{N}_0$ with initial point $\mathfrak{u}_0 \in \mathcal{U}$. If $\mathfrak{u}_n = \mathfrak{u}_{n+1} = T\mathfrak{u}_n$, then $\mathfrak{u}_n$ is fixed point of T. Suppose $\mathfrak{u}_n \neq \mathfrak{u}_{n+1}$ for all n. Now

$$\mathcal{S}(T\mathfrak{u}_n, T\mathfrak{u}_n, T\mathfrak{u}_{n+1}) \leq \lambda\mathcal{N}(\mathfrak{u}_n, \mathfrak{u}_{n+1}), \tag{2}$$

where

$$\begin{aligned}
\mathcal{N}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) &= [\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1})]^{\alpha} [\mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_n)]^{\beta} \left[\frac{\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, T\mathfrak{u}_n) + \mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, T\mathfrak{u}_{n+1})}{2}\right]^{\gamma} \\
&\quad \left[\frac{\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, T\mathfrak{u}_{n+1}) + \mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, T\mathfrak{u}_n)}{2}\right]^{\delta} \\
&= [\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1})]^{\alpha} [\mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_n)]^{\beta} \left[\frac{\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n,\ \mathfrak{u}_{n+1}) + \mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2})}{2}\right]^{\gamma} \\
&\quad \left[\frac{\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+2}) + \mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+1})}{2}\right]^{\delta} \\
&\leq [\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1})]^{\alpha} [\mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_n)]^{\beta} \left[\frac{\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n,\ \mathfrak{u}_{n+1}) + \mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2})}{2}\right]^{\gamma} \\
&\quad \left[\frac{\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1}) + \mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2})}{2}\right]^{\delta}.
\end{aligned}$$

Case (i) If $\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1}) \leq \mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2})$, then

$$\begin{aligned}
\mathcal{N}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) &\leq [\mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2})]^{\alpha}[\mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2})]^{\beta}[\mathcal{S}(\ \mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2})]^{\gamma} \\
&\quad [\mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2})]^{\delta} \\
&= [\mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2})]^{(\alpha+\beta+\gamma+\delta)} \\
&< \mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}).
\end{aligned}$$

From inequality (2),

$$\mathcal{S}(T\mathfrak{u}_n, T\mathfrak{u}_n, T\mathfrak{u}_{n+1}) = \mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) \leq \lambda\mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) < \mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}),$$

which is a contradiction.

Case (ii) If $\mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) \leq \mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1})$, then

$$\begin{aligned}
\mathcal{N}(\mathfrak{u}_n, \mathfrak{u}_{n+1}) &\leq [\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1})]^{\alpha}[\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1})]^{\beta}[\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1})]^{\gamma} \\
&\quad [\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1})]^{\delta} \\
&= [\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1})]^{(\alpha+\beta+\gamma+\delta)} \\
&< \mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1}).
\end{aligned}$$

From inequality (2),

$$\mathcal{S}(T\mathfrak{u}_n, T\mathfrak{u}_n, T\mathfrak{u}_n) = \mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) \leq \lambda\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}_{n+1}).$$

By repeating this argument

$$\mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1}, \mathfrak{u}_{n+2}) \leq \lambda^{n+1}\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, \mathfrak{u}_1) \to 0, \quad \text{as } n \to \infty,$$

that is,

$$\lim_{n\to\infty} \mathcal{S}(\mathfrak{u}_{n+1}, \mathfrak{u}_{n+1},\ \mathfrak{u}_{n+2}) = 0.$$

Furthermore, for $n > m$

$$\begin{aligned}
\mathcal{S}(\mathfrak{u}_m, \mathfrak{u}_m, \mathfrak{u}_n) &\leq \mathcal{S}(\mathfrak{u}_m, \mathfrak{u}_m, \mathfrak{u}_{m+1}) + \mathcal{S}(\mathfrak{u}_{m+1}, \mathfrak{u}_{m+1}, \mathfrak{u}_{m+2}) + \cdots + \mathcal{S}(\mathfrak{u}_{n-1}, \mathfrak{u}_{n-1}, \mathfrak{u}_n) \\
&\leq \lambda^m \mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, \mathfrak{u}_1) + \lambda^{m+1}\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, \mathfrak{u}_1) + \cdots + \lambda^{n-1}\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, \mathfrak{u}_1) \\
&= (\lambda^m + \lambda^{m+1} + \cdots + \lambda^{n-1})\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, \mathfrak{u}_1) \\
&= \frac{\lambda^m(1-\lambda^{n-m})}{1-\lambda}\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_1, \mathfrak{u}_1) \to 0, \quad \text{as } m, n \to \infty,
\end{aligned}$$

that is, $\{\mathfrak{u}_n\}$ is a Cauchy sequence. Now, utilizing the definition of completeness of $(\mathcal{U}, \mathcal{S})$, we have $\mathfrak{u}^* \in \mathcal{U}$ so that $\{\mathfrak{u}_n\}$ converges to $\mathfrak{u}^*$.

$$\begin{aligned}
\mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, T\mathfrak{u}^*) &= \mathcal{S}(T\mathfrak{u}_{n-1}, T\mathfrak{u}_{n-1}, T\mathfrak{u}^*) \\
&\leq \lambda[\mathcal{S}(\mathfrak{u}_{n-1}, \mathfrak{u}_{n-1}, T\mathfrak{u}_{n-1})]^\alpha [\mathcal{S}(\mathfrak{u}^*, \mathfrak{u}^*, T\mathfrak{u}^*)]^\beta \left[\frac{\mathcal{S}(\mathfrak{u}_{n-1}, \mathfrak{u}_{n-1}, T\mathfrak{u}_{n-1}) + \mathcal{S}(\mathfrak{u}^*,\ \mathfrak{u}^*, T\mathfrak{u}^*)}{2}\right]^\gamma \\
&\quad \left[\frac{\mathcal{S}(\mathfrak{u}_{n-1}, \mathfrak{u}_{n-1}, T\mathfrak{u}^*) + \mathcal{S}(\mathfrak{u}^*, \mathfrak{u}^*, T\mathfrak{u}_{n-1})}{2}\right]^\delta \\
&= \lambda[\mathcal{S}(\mathfrak{u}_{n-1}, \mathfrak{u}_{n-1}, \mathfrak{u}_n)]^\alpha [\mathcal{S}(\mathfrak{u}^*, \mathfrak{u}^*, T\mathfrak{u}^*)]^\beta \left[\frac{\mathcal{S}(\mathfrak{u}_{n-1}, \mathfrak{u}_{n-1}, \mathfrak{u}_n) + \mathcal{S}(\mathfrak{u}^*, \mathfrak{u}^*, T\mathfrak{u}^*)}{2}\right]^\gamma \\
&\quad \left[\frac{\mathcal{S}(\mathfrak{u}_{n-1}, \mathfrak{u}_{n-1}, T\mathfrak{u}^*) + \mathcal{S}(\mathfrak{u}^*, \mathfrak{u}^*, \mathfrak{u}_n)}{2}\right]^\delta \\
&\to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

that is, $\lim_{n\to\infty} \mathcal{S}(\mathfrak{u}_n, \mathfrak{u}_n, T\mathfrak{u}^*) = 0$. So $\{\mathfrak{u}_n\}$ converges to $T\mathfrak{u}^*$. Utilizing the definition of limit $T\mathfrak{u}^* = \mathfrak{u}^*$, that is, $\mathfrak{u}^*$ is a fixed point of T.

■

Next, we give the subsequent examples to justify Theorem 2 and to indicate the significant fact that the fixed point of a discontinuous mapping satisfying a generalized interpolative contraction may not essentially be unique. As a result, establishing the uniqueness of fixed points for such contractions will be an interesting topic for subsequent works.

Example 3 *Let $\mathcal{U} = [0, \infty)$ be equipped with the S-metric $\mathcal{S}$ described as in Example 1. Define the self-mapping $T : \mathcal{U} \to \mathcal{U}$ as*

$$T\mathfrak{u} = \begin{cases} 2, & \mathfrak{u} \in [0, 3), \\ e^{-\mathfrak{u}}, & \mathfrak{u} \in [3, \infty), \end{cases} \quad \textit{for } \mathfrak{u} \in \mathcal{U}.$$

Then, T validates the hypotheses of Theorem 2 for $\lambda = \frac{2}{3}$, $\alpha = \beta = \frac{1}{6}$ and $\gamma = \delta = \frac{1}{5}$. Consequently, 2 is a fixed point of T.

Example 4 *Let $\mathcal{U} = \mathbb{R}$ be equipped with the S -metric $\mathcal{S}$ described as in Example 1. Define the self-mapping $T : \mathcal{U} \to \mathcal{U}$ as*

$$T\mathfrak{u} = \begin{cases} \mathfrak{u}e^{-\mathfrak{u}}, & \mathfrak{u} \in (-2, \infty), \\ \frac{1}{\mathfrak{u}+1.5}, & \mathfrak{u} \in (-\infty, -2], \end{cases} \quad \textit{for } \mathfrak{u} \in \mathcal{U}.$$

Then T validates the hypotheses of Theorem 2 for $\lambda = 0.5$, $\alpha = \beta = 0.6$ and $\gamma = \delta = 0.15$. Consequently, -2 and 0 are two fixed points of T.

Remark 2 *1.*

2. Selecting the values of constants α, β, and γ in an appropriate manner we attain the definitions of interpolative Kannan contraction [20], interpolative Chatterjea contraction [36], and interpolative Reich-Rus-Ćirić type contraction [21]. Consequently, following the procedure of Theorems 2, we attain distinct conclusions which generalize Banach [4], Chatterjea [5], Errai et al. [7], Kannan [19], and Reich [37] in metric, S-metric as well as b-metric spaces.

3. *Theorem 2 is a generalization, extension, and improvement of celebrated and recent conclusions to S-metric space via discontinuous interpolative contraction. See for instance, [1]-[2], [6]–[8], [20]–[29], [36], and references therein.*

Now, inspired by the reality that the collection of multiple fixed points may contain some geometrical shapes, we frame some postulates for the survival of fixed-disc (circle) in S-metric spaces besides slightly modifying the inequality (20).

Theorem 3 *Let $T:\mathcal{U}\to\mathcal{U}$ be a self-mapping of an S-metric space $(\mathcal{U},\mathcal{S})$ and r defined as*

$$r=\inf\left\{\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u}):\mathfrak{u}\notin Fix(T)\right\}. \tag{3}$$

If there exists $\mathfrak{u}_0\in\mathcal{U}$ so that

$$1\leq\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})<\mathcal{N}_1(\mathfrak{u}), \tag{4}$$

where,

$$\mathcal{N}_1(\mathfrak{u})=\left[\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{u}_0)\right]^{\alpha}\left[\mathcal{S}(\mathfrak{u},\ \mathfrak{u},T\mathfrak{u})\right]^{\beta}\left[\frac{\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})+\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{u}_0)}{2}\right]^{\gamma}\left[\frac{\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u}_0)+\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u})}{2}\right]^{\delta}$$

and

$$0<\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u})\leq r, \tag{5}$$

for $\mathfrak{u}\in\mathcal{U}-Fix(T)$, $\alpha+\beta+\gamma+\delta<1$ and $\alpha,\beta,\gamma,\delta\in(0,1)$, then $D^S_{\mathfrak{u}_0,r}$ is a fixed disc of T. Also, $C^S_{\mathfrak{u}_0,r}$ is a fixed circle of T.

Proof. At first, we show $\mathfrak{u}_0\in Fix(T)$. To demonstrate this, we suppose $\mathfrak{u}_0\notin Fix(T)$, that is, $\mathfrak{u}_0\neq T\mathfrak{u}_0$. Using the inequality (22), we get

$$\begin{aligned}1 &\leq \mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u}_0)<\mathcal{N}_1(\mathfrak{u}_0)\\ &= \left[\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0)\right]^{\alpha}\left[\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u}_0)\right]^{\beta}\\ &\quad\left[\frac{\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u}_0)+\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0)}{2}\right]^{\gamma}\left[\frac{\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u}_0)+\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u}_0)}{2}\right]^{\delta}\\ &= 0,\end{aligned}$$

which is a contradiction. So $\mathfrak{u}_0\in Fix(T)$, that is,

$$\mathfrak{u}_0=T\mathfrak{u}_0. \tag{6}$$

To demonstrate that $D^S_{\mathfrak{u}_0,r}$ is a fixed disc of T, we have the subsequent cases:

Case 1. If $r=0$, then we obtain $D^S_{\mathfrak{u}_0,r}=\{\mathfrak{u}_0\}$ and by the equality (6), we say $\mathfrak{u}_0\in Fix(T)$.

Case 2. If $r>0$ and $\mathfrak{u}\in D^S_{\mathfrak{u}_0,r}$, so that $\mathfrak{u}\notin Fix(T)$. From the inequalities (22), (23) and the equality (6), we find

$$\begin{aligned}1 &\leq \mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})<\mathcal{N}_1(\mathfrak{u})\\ &= \left[\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{u}_0)\right]^{\alpha}\left[\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})\right]^{\beta}\\ &\quad\left[\frac{\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})+\ \mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{u}_0)}{2}\right]^{\gamma}\left[\frac{\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u}_0)+\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u})}{2}\right]^{\delta}\\ &\leq r^{\alpha}\left[\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})\right]^{\beta}\left[\frac{\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})+r}{2}\right]^{\gamma}\left[\frac{r+r}{2}\right]^{\delta}\\ &\leq \left[\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})\right]^{\alpha+\beta+\gamma+\delta},\end{aligned}$$

which is a contradiction with $\alpha+\beta+\gamma+\delta<1$. Hence, $x\in Fix(T)$.

As a result, $D^S_{\mathfrak{u}_0,r}$ is a fixed disc of T. Also, we may observe that $C^S_{\mathfrak{u}_0,r}$ is a fixed circle of T, since $C^S_{\mathfrak{u}_0,r}$ is a boundary of $D^S_{\mathfrak{u}_0,r}$. ■

Example 5 *Let $\mathcal{U} = \left\{-1, 0, \frac{1}{2}, 1, 3, 4\right\}$ be equipped with the S-metric $\mathcal{S}$ described as in Example 1. Define the self-mapping $T : \mathcal{U} \to \mathcal{U}$ as*

$$T\mathfrak{u} = \begin{pmatrix} -1 & 0 & \frac{1}{2} & 1 & 3 & 4 \\ -1 & 0 & \frac{1}{2} & 1 & 4 & 4 \end{pmatrix}, \text{ for } \mathfrak{u} \in \mathcal{U}.$$

Then, T validates the hypotheses of Theorem 3 for $\mathfrak{u}_0 = 0$, $\alpha = \gamma = \delta = \frac{1}{4}$ and $\beta = \frac{1}{8}$. As expected, for $\mathfrak{u} = 3$, we obtain

$$1 \leq \mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) = 2 \leq 3.88 \simeq \mathcal{N}_1(3)$$

and

$$r = \inf \{\mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) : \mathfrak{u} = 3\} = 2.$$

Noticeably, T fixes the disc $D^S_{0,2} = \left\{-1, 0, \frac{1}{2}, 1\right\}$ and the circle $C^S_{0,2} = \{-1, 1\}$.

Theorem 4 *Let $T : \mathcal{U} \to \mathcal{U}$ be a self-mapping of an S-metric space $(\mathcal{U}, \mathcal{S})$ and satisfy the inequality (23) and r described as in (3). If there exists $\mathfrak{u}_0 \in \mathcal{U}$ so that*

$$1 \leq \mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) < \mathcal{N}_2(\mathfrak{u}), \tag{7}$$

where,

$$\mathcal{N}_2(\mathfrak{u}) = [\mathcal{S}(\mathfrak{u}, \mathfrak{u}, \mathfrak{u}_0)]^\alpha [\mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u})]^\beta [\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, T\mathfrak{u})]^\gamma,$$

for $\mathfrak{u} \in \mathcal{U} - Fix(T)$, $\alpha + \beta + \gamma < 1$ and $\alpha, \beta, \gamma \in (0, 1)$, then $D^S_{\mathfrak{u}_0,r}$ is a fixed disc of T. Also, $C^S_{\mathfrak{u}_0,r}$ is a fixed circle of T.

Proof. Firstly, we prove $\mathfrak{u}_0 \in Fix(T)$. Suppose to the contrary that $\mathfrak{u}_0 \notin Fix(T)$. Using the inequality (24), we get

$$\begin{aligned} 1 &\leq \mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, T\mathfrak{u}_0) < \mathcal{N}_2(\mathfrak{u}_0) \\ &= [\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, \mathfrak{u}_0)]^\alpha [\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, T\mathfrak{u}_0)]^\beta [\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, T\mathfrak{u}_0)]^\gamma \\ &= 0, \end{aligned}$$

which is a contradiction. So $\mathfrak{u}_0 \in Fix(T)$. Now, we have the subsequent cases:

Case 1. If $r = 0$, then we obtain $D^S_{\mathfrak{u}_0,r} = \{\mathfrak{u}_0\}$ and $\mathfrak{u}_0 = T\mathfrak{u}_0$ since $\mathfrak{u}_0 \in Fix(T)$.

Case 2. If $r > 0$ and $\mathfrak{u} \in D^S_{\mathfrak{u}_0,r}$ so that $\mathfrak{u} \notin Fix(T)$. From the inequalities (23) and (24), we attain

$$\begin{aligned} 1 &\leq \mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) < \mathcal{N}_2(\mathfrak{u}) \\ &= [\mathcal{S}(\mathfrak{u}, \mathfrak{u}, \mathfrak{u}_0)]^\alpha [\mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u})]^\beta [\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, T\mathfrak{u})]^\gamma \\ &\leq r^\alpha [\mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u})]^\beta r^\gamma \leq [\mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u})]^{\alpha+\beta+\gamma}, \end{aligned}$$

which is a contradiction with $\alpha + \beta + \gamma < 1$. Hence, $\mathfrak{u} \in Fix(T)$.

As a result, $D^S_{\mathfrak{u}_0,r}$ is a fixed disc of T. Also, we may observe that $C^S_{\mathfrak{u}_0,r}$ is a fixed circle of T, since $C^S_{\mathfrak{u}_0,r}$ is a boundary of $D^S_{\mathfrak{u}_0,r}$. ■

Example 6 *Let $\mathcal{U} = \mathbb{R}$ be equipped with the S-metric $\mathcal{S}$ described as in Example 1. Define the self-mapping $T : \mathcal{U} \to \mathcal{U}$ as*

$$T\mathfrak{u} = \begin{cases} \mathfrak{u} & \text{if } \mathfrak{u} \leq 8, \\ \mathfrak{u} + 1 & \text{if } \mathfrak{u} > 8, \end{cases} \quad \text{for } \mathfrak{u} \in \mathcal{U}.$$

Then, T validates the hypotheses of Theorem 4 for $\mathfrak{u}_0 = 0$, $\alpha = \frac{1}{2}$ and $\beta = \gamma = \frac{1}{8}$. As expected, for $\mathfrak{u} \in (8, \infty)$, we obtain

$$1 \leq \mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) = 2 \leq 6.23 \simeq \mathcal{N}_2(\mathfrak{u})$$

and

$$r = \inf \{\mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) : \mathfrak{u} \in (8, \infty)\} = 2.$$

Noticeably, T fixes the disc $D^S_{0,2} = [-1, 1]$ and the circle $C^S_{0,2} = \{-1, 1\}$.

Theorem 5 *Let $T:\mathcal{U}\to\mathcal{U}$ be a self-mapping of an S-metric space $(\mathcal{U},\mathcal{S})$ and satisfy the inequality (23) and r as described in (3). If there exists $\mathfrak{u}_0\in\mathcal{U}$ so that*

$$1\leq\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})<\mathcal{N}_3(\mathfrak{u}),\tag{8}$$

where

$$\mathcal{N}_3(u)=\left[\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{u}_0)\right]^{\alpha}\left[\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u}_0)\right]^{\beta}\left[\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u})\right]^{\gamma},$$

for $\mathfrak{u}\in\mathcal{U}-Fix(T)$, $\alpha+\beta+\gamma<1$ and $\alpha,\beta,\gamma\in(0,1)$, then $D^S_{\mathfrak{u}_0,r}$ is a fixed disc of T. Also, $C^S_{\mathfrak{u}_0,r}$ is a fixed circle of T.

Proof. At first, we demonstrate $\mathfrak{u}_0\in Fix(T)$. Suppose to the contrary that $\mathfrak{u}_0\notin Fix(T)$. Using the inequality (25), we get

$$\begin{aligned}1&\leq\mathcal{S}(u_0,\mathfrak{u}_0,T\mathfrak{u}_0)<\mathcal{N}_3(\mathfrak{u}_0)\\&=\left[\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0)\right]^{\alpha}\left[\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u}_0)\right]^{\beta}\left[\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u}_0)\right]^{\gamma}\\&=0,\end{aligned}$$

which is a contradiction and so $\mathfrak{u}_0\in Fix(T)$, that is,

$$\mathfrak{u}_0=T\mathfrak{u}_0.\tag{9}$$

To establish $D^S_{\mathfrak{u}_0,r}$ is the fixed disc, we have the subsequent cases:

Case 1. If $r=0$, then we obtain $D^S_{\mathfrak{u}_0,r}=\{\mathfrak{u}_0\}$ and by the equality (9), we get $\mathfrak{u}_0\in Fix(T)$.

Case 2. If $r>0$ and $\mathfrak{u}\in D^S_{\mathfrak{u}_0,r}$ so that $\mathfrak{u}\notin Fix(T)$. From the inequalities (23), (25), and equality (9), we find

$$\begin{aligned}1&\leq\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})<\mathcal{N}_3(\mathfrak{u})\\&=\left[\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{u}_0)\right]^{\alpha}\left[\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u}_0)\right]^{\beta}\left[\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u})\right]^{\gamma}\\&\leq r^{\alpha}r^{\beta}r^{\gamma}=r^{\alpha+\beta+\gamma}\leq\left[\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})\right]^{\alpha+\beta+\gamma},\end{aligned}$$

which is a contradiction with $\alpha+\beta+\gamma<1$. Hence, $\mathfrak{u}\in Fix(T)$.

As a result, $D^S_{\mathfrak{u}_0,r}$ is a fixed disc of T. Also, we may observe that $C^S_{\mathfrak{u}_0,r}$ is a fixed circle of T, since $C^S_{\mathfrak{u}_0,r}$ is a boundary of $D^S_{\mathfrak{u}_0,r}$. ■

Theorem 6 *Let $T:\mathcal{U}\to\mathcal{U}$ be a self-mapping of an S-metric space $(\mathcal{U},\mathcal{S})$ and r as described in (3). If there exists $\mathfrak{u}_0\in\mathcal{U}$ so that*

$$1\leq\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})<\mathcal{N}_4(\mathfrak{u}),\tag{10}$$

where

$$\mathcal{N}_4(\mathfrak{u})=\left[\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{u}_0)\right]^{\alpha}\left[\mathcal{S}(\mathfrak{u},\ \mathfrak{u},T\mathfrak{u}_0)\right]^{\beta}\left[\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})\right]^{\gamma},$$

for $\mathfrak{u}\in\mathcal{U}-Fix(T)$, $\alpha+\beta+\gamma<1$ and $\alpha,\beta,\gamma\in(0,1)$, then $D^S_{\mathfrak{u}_0,r}$ is a fixed disc of T. Also, $C^S_{\mathfrak{u}_0,r}$ is a fixed circle of T.

Proof. Now we prove $\mathfrak{u}_0\in Fix(T)$. Suppose to the contrary that $\mathfrak{u}_0\notin Fix(T)$. Using the inequality (26), we get

$$\begin{aligned}1&\leq\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u}_0)<\mathcal{N}_4(\mathfrak{u}_0)\\&=\left[\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0)\right]^{\alpha}\left[\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u}_0)\right]^{\beta}\left[\mathcal{S}(\mathfrak{u}_0,\mathfrak{u}_0,T\mathfrak{u}_0)\right]^{\gamma}\\&=0,\end{aligned}$$

which is a contradiction. Thereby, $\mathfrak{u}_0\in Fix(T)$, that is,

$$\mathfrak{u}_0=T\mathfrak{u}_0.\tag{11}$$

Let us consider the subsequent cases:

Case 1. If $r = 0$, then we obtain $D^S_{\mathfrak{u}_0,r} = \{\mathfrak{u}_0\}$ and by the equality (11), we get $\mathfrak{u}_0 \in Fix(T)$.

Case 2. If $r > 0$ and $\mathfrak{u} \in D^S_{\mathfrak{u}_0,r}$ so that $\mathfrak{u} \notin Fix(T)$. From the inequality (26) and the equality (11), we obtain

$$\begin{aligned} 1 &\leq \mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u}) < \mathcal{N}_4(\mathfrak{u}) \\ &= [\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{u}_0)]^{\alpha}\,[\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u}_0)]^{\beta}\,[\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})]^{\gamma} \\ &\leq r^{\alpha}r^{\beta}\,[\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})]^{\gamma} \leq [\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})]^{\alpha+\beta+\gamma}\,, \end{aligned}$$

which is a contradiction with $\alpha+\beta+\gamma < 1$. So $\mathfrak{u} \in Fix(T)$.

As a result, $D^S_{\mathfrak{u}_0,r}$ is a fixed disc of T. Also, we may observe that $C^S_{\mathfrak{u}_0,r}$ is a fixed circle of T, since $C^S_{\mathfrak{u}_0,r}$ is a boundary of $D^S_{\mathfrak{u}_0,r}$. ■

Example 7 *Let us consider the S-metric $\mathcal{S}$ defined as in Example 1 and the self-mapping $T : \mathcal{U} \to \mathcal{U}$ defined as in Example 6. Then, T validates the hypotheses of Theorem 5 and Theorem 6 for $\mathfrak{u}_0 = 0$, $\alpha = \frac{1}{2}$, $\beta = \gamma = \frac{1}{8}$. As expected, for $\mathfrak{u} \in (8,\infty)$, we obtain*

$$1 \leq \mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u}) = 2 \leq 8.06 \simeq \mathcal{N}_3(\mathfrak{u}),$$

$$1 \leq \mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u}) = 2 \leq 6.14 \simeq \mathcal{N}_4(\mathfrak{u}),$$

and

$$r = \inf\{\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u}) : \mathfrak{u} \in (8,\infty)\} = 2.$$

Noticeably, T fixes the disc $D^S_{0,2} = [-1,1]$ and the circle $C^S_{0,2} = \{-1,1\}$.

Next, following Joshi et al. [16] (see also, [41]), we frame some novel postulates to establish the greatest fixed disc via S-metric.

Theorem 7 *If in Theorems 3 or 4 or 5 or 6, self mapping T satisfy*

$$\mathcal{S}(T\mathfrak{u},T\mathfrak{u},T\mathfrak{v}) \leq [\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{v})]^{\alpha}[\mathcal{S}(\mathfrak{u},\ \mathfrak{u},T\mathfrak{u})]^{\beta}[\mathcal{S}(\mathfrak{v},\mathfrak{v},T\mathfrak{v})]^{\gamma}\left[\frac{\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{v})+\mathcal{S}(\mathfrak{v},\mathfrak{v},T\mathfrak{u})}{2}\right]^{\delta}, \tag{12}$$

for $\mathfrak{u} \in D^S_{\mathfrak{u}_0,r}$, $\mathfrak{v} \in \mathcal{U} \setminus D^S_{\mathfrak{u}_0,r}$, $\alpha+\beta+\gamma < 1$ and $\alpha,\beta,\gamma \in (0,1)$, then there exists no fixed disc of a self mapping T that possesses a radius greater than r, that is, $D^S_{\mathfrak{u}_0,r}$ is the greatest fixed disc of a self mapping T.

Proof. Assume that there exist two fixed discs $D^S_{\mathfrak{u}_0,r}$ and $D^S_{\mathfrak{u}'_0,r'}$; $r < r'$ of T, that is, T validates all postulates of Theorems 3 or 4 or 5 or 6 for both the discs $D^S_{\mathfrak{u}_0,r}$ and $D^S_{\mathfrak{u}'_0,r'}$. Let $\mathfrak{u} \in D^S_{\mathfrak{u}_0,r}$ and $\mathfrak{v} \in D^S_{\mathfrak{u}'_0,r'}$, that is, $T\mathfrak{u} = \mathfrak{u}$ and $T\mathfrak{v} = \mathfrak{v}$. Then using inequality (12),

$$\mathcal{S}(T\mathfrak{u},T\mathfrak{u},T\mathfrak{v}) \leq [\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{v})]^{\alpha}[\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u})]^{\beta}[\mathcal{S}(\mathfrak{v},\mathfrak{v},T\mathfrak{v})]^{\gamma}\left[\frac{\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{v})+\mathcal{S}(\mathfrak{v},\mathfrak{v},T\mathfrak{u})}{2}\right]^{\delta},$$

$$\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{v}) \leq [\mathcal{S}(\ \mathfrak{u},\mathfrak{u},\mathfrak{v})]^{\alpha}[\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{u})]^{\beta}[\mathcal{S}(\mathfrak{v},\mathfrak{v},\mathfrak{v})]^{\gamma}\left[\frac{\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{v})+\mathcal{S}(\mathfrak{v},\mathfrak{v},\mathfrak{u})}{2}\right]^{\delta},$$

$$\mathcal{S}(\mathfrak{u},\mathfrak{u},\mathfrak{v}) \leq 0,$$

which is a contradiction. Hence, $D^S_{\mathfrak{u}_0,r}$ is the greatest fixed disc of T having maximum radius r. ■

Following Mlaiki et al. [31], we define a common fixed circle in S-metric space.

Definition 3 *Let $C^S_{\mathfrak{u}_0,r}$ be a circle on an S-metric space $(\mathcal{U},\mathcal{S})$ and $A,B : \mathcal{U} \to \mathcal{U}$ are two self-mappings. If $A\mathfrak{u} = B\mathfrak{u} = \mathfrak{u}$, $\mathfrak{u} \in C^S_{\mathfrak{u}_0,r}$, then the circle $C^S_{\mathfrak{u}_0,r}$ is the common fixed circle of a pair of self mappings A and B.*

Theorem 8 *Let $A, B : \mathcal{U} \to \mathcal{U}$ be self-mappings of an S-metric space $(\mathcal{U}, \mathcal{S})$ and $A\mathfrak{u}_0 = B\mathfrak{u}_0 = \mathfrak{u}_0$ and let r defined as*

$$r = \min\{r_1, r_2, r_3\}, \tag{13}$$

where

$$\begin{aligned} r_1 &= \inf\{\mathcal{S}(\mathfrak{u}, \mathfrak{u}, A\mathfrak{u}) : \mathfrak{u} \neq A\mathfrak{u}\}, \\ r_2 &= \inf\{\mathcal{S}(\mathfrak{u}, \mathfrak{u}, B\mathfrak{u}) : \mathfrak{u} \neq B\mathfrak{u}\}, \\ r_3 &= \inf\{\mathcal{S}(A\mathfrak{u}, A\mathfrak{u}, B\mathfrak{u}) : A\mathfrak{u} \neq B\mathfrak{u}\}. \end{aligned}$$

If there exists $\mathfrak{u}_0 \in \mathcal{U}$ so that

$$1 \leq \mathcal{S}(A\mathfrak{u}, A\mathfrak{u}, B\mathfrak{u}) < \mathcal{N}_5(\mathfrak{u}), \tag{14}$$

where

$$\begin{aligned} \mathcal{N}_5(\mathfrak{u}) &= [\mathcal{S}(A\mathfrak{u}, A\mathfrak{u}, B\mathfrak{u}_0)]^{\alpha} [\mathcal{S}(B\mathfrak{u}, B\mathfrak{u}, A\mathfrak{u}_0)]^{\beta} \left[\frac{\mathcal{S}(A\mathfrak{u}, A\mathfrak{u}, B\mathfrak{u}_0) + \mathcal{S}(B\mathfrak{u}, B\mathfrak{u}, A\mathfrak{u}_0)}{2}\right]^{\gamma} \\ &\quad \times \left[\frac{\mathcal{S}(\mathfrak{u}, \mathfrak{u}, A\mathfrak{u}_0) + \mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}_0)}{2}\right]^{\delta} \end{aligned}$$

and

$$0 < \mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, A\mathfrak{u}) \leq r \ \text{ and } \ 0 < \mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, B\mathfrak{u}) \leq r, \tag{15}$$

for $\alpha + \beta + \gamma + \delta < 1$, $\alpha, \beta, \gamma, \delta \in (0, 1)$, and mapping A (or B) satisfies the postulates of Theorems 3 or 4 or 5 or 6, then $D^S_{\mathfrak{u}_0,r}$ is a common fixed disc of self-mappings A and B. Also, $C^S_{\mathfrak{u}_0,r}$ is a common fixed circle of pair of self mappings A and B.

Proof. To show that $D^S_{\mathfrak{u}_0,r}$ is a common fixed disc of A and B, we have the subsequent cases:

Case 1. If $r = 0$, then $D^S_{\mathfrak{u}_0,r} = \{\mathfrak{u}_0\}$ and $A\mathfrak{u}_0 = B\mathfrak{u}_0 = \mathfrak{u}_0$.

Case 2. If $r > 0$ and $\mathfrak{u} \in D^S_{\mathfrak{u}_0,r}$ be any point with $A\mathfrak{u} \neq B\mathfrak{u}$, that is, $S(A\mathfrak{u}, A\mathfrak{u}, B\mathfrak{u}) > r$. From the inequality (14), we find

$$\begin{aligned} 1 &\leq \mathcal{S}(A\mathfrak{u}, A\mathfrak{u}, B\mathfrak{u}) < \mathcal{N}_5(\mathfrak{u}) \\ &= [\mathcal{S}(A\mathfrak{u}, A\mathfrak{u}, B\mathfrak{u}_0)]^{\alpha} [\mathcal{S}(B\mathfrak{u}, B\mathfrak{u}, A\mathfrak{u}_0)]^{\beta} \\ &\quad \left[\frac{\mathcal{S}(A\mathfrak{u}, A\mathfrak{u}, B\mathfrak{u}_0) + \mathcal{S}(B\mathfrak{u}, B\mathfrak{u}, A\mathfrak{u}_0)}{2}\right]^{\gamma} \left[\frac{\mathcal{S}(\mathfrak{u}, \mathfrak{u}, A\mathfrak{u}_0) + \mathcal{S}(\mathfrak{u}, \mathfrak{u}, B\mathfrak{u}_0)}{2}\right]^{\delta} \\ &= [\mathcal{S}(A\mathfrak{u}, A\mathfrak{u}, \mathfrak{u}_0)]^{\alpha} [\mathcal{S}(B\mathfrak{u}, B\mathfrak{u}, \mathfrak{u}_0)]^{\beta} \\ &\quad \left[\frac{\mathcal{S}(A\mathfrak{u}, A\mathfrak{u}, \mathfrak{u}_0) + \mathcal{S}(B\mathfrak{u}, B\mathfrak{u}, \mathfrak{u}_0)}{2}\right]^{\gamma} \left[\frac{\mathcal{S}(\mathfrak{u}, \mathfrak{u}, \mathfrak{u}_0) + \mathcal{S}(\mathfrak{u}, \mathfrak{u}, \mathfrak{u}_0)}{2}\right]^{\delta} \\ &\leq r^{\alpha} r^{\beta} r^{\gamma} r^{\delta} \\ &= r^{\alpha+\beta+\gamma+\delta}, \end{aligned}$$

which is a contradiction with $\alpha + \beta + \gamma + \delta < 1$. Hence,

$$A\mathfrak{u} = B\mathfrak{u}. \tag{16}$$

Since, A (or B) satisfies the postulates of Theorems 3 or 4 or 5 or 6, we get

$$A\mathfrak{u} = \mathfrak{u} \ \ (\text{or } B\mathfrak{u} = \mathfrak{u}). \tag{17}$$

By the equalities (16) and (17), we obtain $A\mathfrak{u} = \mathfrak{u} = B\mathfrak{u}$, $\mathfrak{u} \in D^S_{\mathfrak{u}_0,r}$.

As a result, $D^S_{\mathfrak{u}_0,r}$ is a common fixed disc of self mappings A and B. Also, we may observe that $C^S_{\mathfrak{u}_0,r}$ is a common fixed circle of pair of self mappings A and B, since $C^S_{\mathfrak{u}_0,r}$ is a boundary of $D^S_{\mathfrak{u}_0,r}$. ■

Example 8 *Let $\mathcal{S}$ be described as in Example 1. Define the self-mappings $A, B : \mathcal{U} \to \mathcal{U}$ as*

$$A\mathfrak{u} = \begin{cases} \mathfrak{u} & \text{if } \mathfrak{u} \leq 8, \\ \mathfrak{u}+1 & \text{if } \mathfrak{u} > 8, \end{cases}$$

and

$$B\mathfrak{u} = \begin{cases} \mathfrak{u} & \text{if } \mathfrak{u} \leq 8, \\ \mathfrak{u}-1 & \text{if } \mathfrak{u} > 8, \end{cases}$$

for $\mathfrak{u} \in \mathcal{U}$. Then, A and B validates the hypotheses of Theorem 4 for $\mathfrak{u}_0 = 0$, $\alpha = \frac{1}{2}$ and $\beta = \gamma = \frac{1}{8}$. Indeed, for $\mathfrak{u} \in (8, \infty)$, we obtain

$$1 \leq \mathcal{S}(A\mathfrak{u}, A\mathfrak{u}, B\mathfrak{u}) = 4 \leq \mathcal{N}_5(\mathfrak{u}),$$

and

$$r = \min\left\{\mathcal{S}(\mathfrak{u}, \mathfrak{u}, A\mathfrak{u}),\ \mathcal{S}(\mathfrak{u}, \mathfrak{u}, B\mathfrak{u}),\ \mathcal{S}(A\mathfrak{u}, A\mathfrak{u}, B\mathfrak{u}) : \mathfrak{u} \in (8, \infty)\right\} = \min\{2, 2, 4\} = 2.$$

Thus, A, B fix the disc $D^{S}_{0,2} = [-1, 1]$ and the circle $C^{S}_{0,2} = \{-1, 1\}$.

Since, the identity map $I_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$, defined as $I_{\mathcal{U}}(\mathfrak{u}) = \mathfrak{u}$, $\mathfrak{u} \in \mathcal{U}$, fixes every disc (resp. circle). Hence, we explore a new contraction that excludes the identity map $I_{\mathcal{U}}$.

Theorem 9 *Let $T : \mathcal{U} \to \mathcal{U}$ be a self-mapping on an S-metric space $(\mathcal{U}, \mathcal{S})$ and $\mathfrak{u}_0 \in \mathcal{U}$ satisfying*

$$\mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) < \mathcal{N}_6(\mathfrak{u}), \tag{18}$$

where

$$\mathcal{N}_6(\mathfrak{u}) = [\mathcal{S}(\mathfrak{u}, \mathfrak{u}, \mathfrak{u}_0)]^{\alpha} [\mathcal{S}(\mathfrak{u},\ \mathfrak{u}, T\mathfrak{u})]^{\beta} [\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, T\mathfrak{u}_0)]^{\gamma},$$

$\mathfrak{u} \in \mathcal{U}$ and $\alpha + \beta + \gamma < 1$, $\alpha, \beta, \gamma \in (0, 1)$ if and only if $T = I_{\mathcal{U}}$.

Proof. At first, we demonstrate $\mathfrak{u}_0 \in Fix(T)$. For this, assume that $\mathfrak{u}_0 \notin Fix(T)$. Using the inequality (18), we get

$$\begin{aligned} \mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, T\mathfrak{u}_0) &< \mathcal{N}_6(\mathfrak{u}_0) \\ &= [\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, \mathfrak{u}_0)]^{\alpha} [\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, T\mathfrak{u}_0)]^{\beta} [\mathcal{S}(\mathfrak{u}_0, \mathfrak{u}_0, T\mathfrak{u}_0)]^{\gamma} \\ &= 0, \end{aligned}$$

a contradiction with $\mathfrak{u}_0 \notin Fix(T)$. So

$$\mathfrak{u}_0 = T\mathfrak{u}_0. \tag{19}$$

Let $\mathfrak{u} \in \mathcal{U}$ with $\mathfrak{u} \notin Fix(T)$. Using the inequality (18) and the equality (19), we find

$$\mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) < 0,$$

which is a contradiction. Hence, $\mathfrak{u} \in Fix(T)$. As a result, we get $T = I_{\mathcal{U}}$. The reverse statement can be easily seen using similar approaches. ■

Remark 3

1. If the S-metric arises from a metric d, then the established conclusions can be considered on a metric space.
2. Since each b-metric arise from an S-metric, the established conclusions can be considered on a b-metric space.

Remark 4 *As seen in Examples 1 and 2, it is not always possible to generate S-metric from a metric or a b-metric. Hence our conclusions proved utilizing interpolative technique are more general than the existing conclusions proved in metric, S-metric, and b-metric spaces (see, [14], [15], [17], [32], and references therein).*

Remark 5 *Examples 5, 6, 7, and 8 demonstrate that a circle (disc) in an S-metric space may not be similar to a circle (disc) in an Euclidean space. The fixed circle and fixed disc conclusions are comparable to fixed-point conclusions if the set of fixed points is a singleton set. Also, $TC^S_{\mathfrak{u}_0,r} = C^S_{\mathfrak{u}_0,r}$ $(TD^S_{\mathfrak{u}_0,r} = D^S_{\mathfrak{u}_0,r})$ does not suggest that $C^S_{\mathfrak{u}_0,r}$ $(D^S_{\mathfrak{u}_0,r})$ is a fixed circle (disc) of T. It is clear from Examples 5, 6, 7, and 8 that if a set of fixed points of a self-mapping includes a disc, then it also includes a circle. However, the reverse may not hold true. The fixed disc is not unique, that is, all the discs in the interior of a fixed disc in an S-metric space are also fixed discs, (see, Examples 5, 6, 7, and 8). A disc having a maximum radius is called the greatest disc [16].*

4 Some Multiple Fixed Point Results on b-Metric Spaces

In this section, inspired by the used technique in [40] with Theorem 1, we give the following theorems:

Theorem 10 *Let $T:\mathcal{U}\to\mathcal{U}$ be a self-mapping of a complete b-metric space $(\mathcal{U}, d^S)$ and*

$$d^S(T\mathfrak{u}, T\mathfrak{v}) \leq \lambda \mathcal{N}_{d^S}(\mathfrak{u}, \mathfrak{v}),\ \lambda \in [0,1), \tag{20}$$

where,

$$\mathcal{N}_{d^S}(\mathfrak{u},\mathfrak{v}) = \left[d^S(\mathfrak{u},T\mathfrak{u})\right]^{\alpha}\left[d^S(\mathfrak{v},T\mathfrak{v})\right]^{\beta}\left[\frac{d^S(\mathfrak{u},T\mathfrak{u})+d^S(\mathfrak{v},T\mathfrak{v})}{2}\right]^{\gamma}\left[\frac{d^S(\mathfrak{u},T\mathfrak{v})+d^S(\mathfrak{v},T\mathfrak{u})}{2}\right]^{\delta},$$

for $\mathfrak{u},\mathfrak{v}\in\mathcal{U}$, $\alpha+\beta+\gamma+\delta<1$ and $\alpha,\beta,\gamma,\delta\in(0,1)$. Then T has a fixed point in $\mathcal{U}$.

Theorem 11 *Let $T:\mathcal{U}\to\mathcal{U}$ be a self-mapping of a b-metric space $(\mathcal{U}, d^S)$ and r defined as*

$$\mu = \inf\left\{d^S(\mathfrak{u},T\mathfrak{u}) : \mathfrak{u}\notin Fix(T)\right\}. \tag{21}$$

If there exists $\mathfrak{u}_0\in\mathcal{U}$ so that

$$1 \leq d^S(\mathfrak{u},T\mathfrak{u}) < \mathcal{N}_{d^S1}(\mathfrak{u}), \tag{22}$$

where,

$$\mathcal{N}_{d^S1}(\mathfrak{u}) = \left[d^S(\mathfrak{u},\mathfrak{u}_0)\right]^{\alpha}\left[d^S(\mathfrak{u},T\mathfrak{u})\right]^{\beta}\left[\frac{d^S(\mathfrak{u},T\mathfrak{u})+d^S(\mathfrak{u},\mathfrak{u}_0)}{2}\right]^{\gamma}\left[\frac{d^S(\mathfrak{u},T\mathfrak{u}_0)+d^S(\mathfrak{u}_0,T\mathfrak{u})}{2}\right]^{\delta}$$

and

$$0 < d^S(\mathfrak{u}_0,T\mathfrak{u}) \leq \mu, \tag{23}$$

for $\mathfrak{u}\in\mathcal{U}-Fix(T)$, $\alpha+\beta+\gamma+\delta<1$ and $\alpha,\beta,\gamma,\delta\in(0,1)$, then $D^{d^S}_{\mathfrak{u}_0,\mu}=\{\mathfrak{u}\in\mathcal{U}: d^S(\mathfrak{u},\mathfrak{u}_0)\leq\mu\}$ is a fixed disc of T. Also, $C^{d^S}_{\mathfrak{u}_0,\mu}=\{\mathfrak{u}\in\mathcal{U}: d^S(\mathfrak{u},\mathfrak{u}_0)=\mu\}$ is a fixed circle of T.

Theorem 12 *Let $T:\mathcal{U}\to\mathcal{U}$ be a self-mapping of a b-metric space $(\mathcal{U}, d^S)$ and satisfy the inequality (23) and μ described as in (21). If there exists $\mathfrak{u}_0\in\mathcal{U}$ so that*

$$1 \leq d^S(\mathfrak{u},T\mathfrak{u}) < \mathcal{N}_{d^S2}(\mathfrak{u}), \tag{24}$$

where,

$$\mathcal{N}_{d^S2}(\mathfrak{u}) = \left[d^S(\mathfrak{u},\mathfrak{u}_0)\right]^{\alpha}\left[d^S(\mathfrak{u},T\mathfrak{u})\right]^{\beta}\left[d^S(\mathfrak{u}_0,T\mathfrak{u})\right]^{\gamma},$$

for $\mathfrak{u}\in\mathcal{U}-Fix(T)$, $\alpha+\beta+\gamma<1$ and $\alpha,\beta,\gamma\in(0,1)$. Then $D^{d^S}_{\mathfrak{u}_0,\mu}$ is a fixed disc of T. Also, $C^{d^S}_{\mathfrak{u}_0,\mu}$ is a fixed circle of T.

Theorem 13 *Let $T : \mathcal{U} \to \mathcal{U}$ be a self-mapping of a b-metric space $(\mathcal{U}, d^S)$ and satisfy the inequality (23) and μ as described in (21). If there exists $\mathfrak{u}_0 \in \mathcal{U}$ so that*

$$1 \leq d^S(\mathfrak{u}, T\mathfrak{u}) < \mathcal{N}_{d^S 3}(\mathfrak{u}), \tag{25}$$

where,

$$\mathcal{N}_{d^S 3}(u) = \left[d^S(\mathfrak{u}, \mathfrak{u}_0)\right]^{\alpha} \left[d^S(\mathfrak{u}, T\mathfrak{u}_0)\right]^{\beta} \left[d^S(\mathfrak{u}_0, T\mathfrak{u})\right]^{\gamma},$$

for $\mathfrak{u} \in \mathcal{U} - Fix(T)$, $\alpha + \beta + \gamma < 1$ and $\alpha, \beta, \gamma \in (0,1)$, then $D^{d^S}_{\mathfrak{u}_0,\mu}$ is a fixed disc of T. Also, $C^{d^S}_{\mathfrak{u}_0,\mu}$ is a fixed circle of T.

Theorem 14 *Let $T : \mathcal{U} \to \mathcal{U}$ be a self-mapping of a b-metric space $(\mathcal{U}, d^S)$ and μ as described in (21). If there exists $\mathfrak{u}_0 \in \mathcal{U}$ so that*

$$1 \leq d^S(\mathfrak{u}, T\mathfrak{u}) < \mathcal{N}_{d^S 4}(\mathfrak{u}), \tag{26}$$

where,

$$\mathcal{N}_{d^S 4}(\mathfrak{u}) = \left[d^S(\mathfrak{u}, \mathfrak{u}_0)\right]^{\alpha} \left[d^S(\ \mathfrak{u}, T\mathfrak{u}_0)\right]^{\beta} \left[d^S(\mathfrak{u}, T\mathfrak{u})\right]^{\gamma},$$

for $\mathfrak{u} \in \mathcal{U} - Fix(T)$, $\alpha + \beta + \gamma < 1$ and $\alpha, \beta, \gamma \in (0,1)$, then $D^{d^S}_{\mathfrak{u}_0,\mu}$ is a fixed disc of T. Also, $C^{d^S}_{\mathfrak{u}_0,\mu}$ is a fixed circle of T.

The proofs of Theorems 10, 11, 12, 13, and 14 are clear from the proved theorems in the previous section.

5 An Application to PReLU

Activation functions are very important in neural networks. There are many examples of activation functions. Some of them are partitioned. One of these partitioned activation functions is a "Parametric Rectified Linear Unit (PReLU)" (see [11] for more details) be described as follows:

$$PReLU(\mathfrak{u}) = \begin{cases} \lambda\mathfrak{u} & \text{if } \mathfrak{u} < 0, \\ \mathfrak{u} & \text{if } \mathfrak{u} \geq 0, \end{cases} \quad \text{for } \mathfrak{u} \in \mathcal{U}.$$

Now, let $U = [0, \infty) \cup \{-2\}$ and $\lambda = \frac{1}{2}$. Then we have

$$PReLU(\mathfrak{u}) = \begin{cases} \frac{\mathfrak{u}}{2} & \text{if } \mathfrak{u} < 0, \\ \mathfrak{u} & \text{if } \mathfrak{u} \geq 0, \end{cases} = \begin{cases} -1 & \text{if } \mathfrak{u} = -2, \\ \mathfrak{u} & \text{if } \mathfrak{u} \in [0, \infty). \end{cases}$$

Let S-metric $\mathcal{S}$ be described as in Example 1. The function $PReLU$ verifies the hypotheses of Theorem 3 with $\mathfrak{u}_0 = 0$, $\alpha = \frac{1}{4}$ and $\beta = \gamma = \delta = \frac{1}{4}$. As a matter of fact, for $\mathfrak{u} = -2$, we have

$$1 \leq \mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) = 2 \leq 2.01 \simeq \mathcal{N}_1(-2).$$

Also, the function $PReLU$ validates the hypotheses of Theorem 4 with $\mathfrak{u}_0 = 0$, $\alpha = \frac{1}{2}$ and $\beta = \delta = \frac{1}{8}$. As expected, for $\mathfrak{u} = -2$, we have

$$1 \leq \mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) = 2 \leq 2.37 \simeq \mathcal{N}_2(-2).$$

If, the function $PReLU$ validates the hypotheses of Theorem 5 and Theorem 6 with $\mathfrak{u}_0 = 0$, $\alpha = \beta = \delta = \frac{1}{4}$. As a matter of fact, for $\mathfrak{u} = -2$, we obtain

$$1 \leq \mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) = 2 \leq 2.34 \simeq \mathcal{N}_3(-2),$$

$$1 \leq \mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) = 2 \leq 2.34 \simeq \mathcal{N}_4(-2),$$

and

$$r = \inf \{\mathcal{S}(\mathfrak{u}, \mathfrak{u}, T\mathfrak{u}) : \mathfrak{u} = -2\} = 2.$$

Hence, the parametric rectified linear unit activation function $PReLU$ fixes the disc $D^S_{0,2} = [0, 1]$ and the circle $C^S_{0,2} = \{1\}$.

Remark 6 *A parametric rectified linear unit is a kind of leaky rectified linear unit making it a parameter for the neural network to understand itself. It fixes the "dying rectified linear unit" problem, and speeds up training as it does not have zero-slope parts. This activation function improves the performance of convolutional neural networks in Image Net classification with minimum risk of overfitting. It is fascinating to see that mappings forming a fixed circle (disc) have been exploited as activation functions in neural networks and allow to choose the appropriate activation function in accordance with the required problem. Consequently, our conclusions may also be significant under a suitable environment for numerous neural networks.*

6 Solution of Integral Equation

Let $\mathcal{U} = C([0,l],\mathbb{R})$ symbolizes the collection of continuous real-valued functions on $[0,l]$. The space $\mathcal{U} = C([0,l],\mathbb{R})$ equipped with the norm $\|\mathfrak{u}\|_\infty = \max_{t\in[0,l]} |\mathfrak{u}(t)|,\ \mathfrak{u}(t) \in C([0,l],\mathbb{R})$ is a Banach space. Define $\mathcal{S} : \mathcal{U}\times\mathcal{U}\times\mathcal{U} \to \mathbb{R}^+$ as $\mathcal{S}(\mathfrak{u},\mathfrak{v},\mathfrak{w}) = |\mathfrak{u}-\mathfrak{w}| + |\mathfrak{u}+\mathfrak{w}-2\mathfrak{v}|$. $(\mathcal{U},\mathcal{S})$ is a complete S-metric space. Next, we solve the subsequent integral equation utilizing the interpolative fixed point technique.

$$\mathfrak{u}(t) = \int_0^l b(t,s)M(s,\mathfrak{u}(s))ds + g(t),\ t\in[0,l]. \tag{27}$$

Define $T : \mathcal{U} \to \mathcal{U}$ as

$$T\mathfrak{u}(t) = \int_0^l b(t,s)M(s,\mathfrak{u}(s))ds + g(t),\ t\in[0,l].$$

Consider the following hypotheses.

1. The functions $M : [0,l]\times\mathcal{U} \to \mathbb{R},\ b : [0,l]\times[0,l] \to \mathbb{R}$ and $g : [0,l] \to \mathbb{R}$ are continuous and

$$|M(s,\mathfrak{u}(s)) - M(s,\mathfrak{v}(s))| \le |\mathfrak{u}(s) - \mathfrak{v}(s)|;$$

and

$$\int_0^l M(s,\mathfrak{u}(s))ds \le \|\mathfrak{u}(s)\|_\infty.$$

Now

$$\begin{aligned}
\mathcal{S}(T\mathfrak{u},T\mathfrak{u},T\mathfrak{v}) &= 2|T\mathfrak{u} - T\mathfrak{v}| \\
&= 2\Big|\int_0^l b(t,s)M(s,\mathfrak{u}(s))ds - \int_0^l b(t,s)M(s,\mathfrak{u}(s))ds\Big| \\
&= 2\Big|\int_0^l b(t,s)[M(s,\mathfrak{u}(s)) - M(s,\mathfrak{u}(s))]ds\Big| \\
&\le 2\int_0^l |b(t,s)||M(s,\mathfrak{u}(s)) - M(s,\mathfrak{v}(s))|ds \\
&\le \int_0^l |b(t,s)|.|\mathfrak{u}(s) - \mathfrak{v}(s)|ds \\
&\le \|\mathfrak{u}(s) - \mathfrak{v}(s)\|_\infty \int_0^l |b(t,s)|ds \\
&\le \|\mathfrak{u}(s) - \mathfrak{v}(s)\|_\infty,
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{u}) &= 2|T\mathfrak{u}-\mathfrak{u}| \\
&= 2|\int_0^v b(t,s)M(s,\mathfrak{u}(s))ds + g(t) - \mathfrak{u}| \\
&\leq 2|\int_0^v b(t,s)M(s,\mathfrak{u}(s))ds| + 2|g(t)-\mathfrak{u}(t)| \\
&\leq 2\|\mathfrak{u}(s)\|_\infty + 2|g(t)-\mathfrak{u}(t)|,
\end{aligned}$$

$$\mathcal{S}(\mathfrak{v},\mathfrak{v},T\mathfrak{v}) \leq \|\mathfrak{v}(s)\|_\infty + 2|g(t)-\mathfrak{v}(t)|,$$
$$\mathcal{S}(\mathfrak{u},\mathfrak{u},T\mathfrak{v}) \leq \|\mathfrak{v}(s)\|_\infty + 2|g(t)-\mathfrak{u}(t)|,$$
$$\mathcal{S}(\mathfrak{v},\mathfrak{v},T\mathfrak{u}) \leq \|\mathfrak{u}(s)\|_\infty + 2|g(t)-\mathfrak{v}(t)|.$$

For $\alpha = \beta = \delta = \frac{1}{6}$ and $\gamma = \frac{1}{7}$, mapping T validates Theorem 2. Hence, the integral equation (27) has a solution in an S-metric space.

7 Conclusion

We have explored the geometry of the collection of fixed points via interpolative techniques in an S-metric space by establishing multiple fixed points, fixed circle, and fixed disc conclusions. Furthermore, we have excluded the possibility of an identity map in the existence of a fixed circle (disc) on S-metric spaces. To establish the significance of novel fixed circle (disc) conclusions in the neural network, which permits to choose the appropriate activation function according to the underlying problem, we have discussed the parametric rectified linear unit activation function. In the sequel, we have presented some interesting remarks to compare our results with the existing ones and demonstrate the significance of our outcomes. Investigations of multiple fixed point and fixed figure problems in metric fixed point theory have been enriched to problems formulated in terms of interpolative contractive conditions on an S-metric space which need not always arise from any metric. Consequently, more general conclusions have been established than those existing in the literature. It has been demonstrated by illustrative examples that these extensions, improvements, and generalizations are genuine. We have concluded the paper by solving an integral equation utilizing interpolative fixed point techniques. Our results provide a specific procedure and directions for further investigation in this recently developed S-metric space.

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