Applied Mathematics E-Notes, 23(2023), 83-89 © ISSN 1607-2510
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All Nontrivial Solutions Of A Complex Dynamical System Are Periodic With Prime Period 9 Or 18*

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Received 4 April 2023

Abstract

In this note, we show that for the complex dynamical system $z_{k+1} + z_{k-1} = |\Im(z_k)| + i|\Re(z_k)|$, all its nontrivial solutions have the prime periods 9 or 18.

1 Introduction

The study of discrete time dynamical systems in one complex variable reveals many interesting periodic and aperiodic as well as chaotic phenomena (for an elementary exposition, [1] may be consulted). In general, it is difficult to give a full account of the periodic behavior of a genuine nonlinear recurrence relation.

In this note, however, we consider one such dynamical system and show that all its nontrivial solutions are either 9- or 18-periodic.

More specifically, we consider the recurrence relation

$$z_{k-1} + z_{k+1} = F(z_k),\ k \in \mathbf{Z} = \{.., -2, -1, 0, 1, 2, ...\} \tag{1}$$

where $F : \mathbf{C} \to \mathbf{C}$ is the map defined by

$$F(z) = |\Im(z)| + i|\Re(z)|,\ i = \sqrt{-1}, \tag{2}$$

where $\Re(z)$ is the real part and $\Im(z)$ is the imaginary part of z.

Given z_0 and z_1, say, it is easily seen that we can iteratively calculate $z_2, z_3, ...,$ and $z_{-1}, z_{-2}, ...$ in a unique manner. The resulting sequence $\{z_k\}_{k\in\mathbf{Z}}$ is called a solution of (1). A quick computer simulation experiment shows that all nontrivial solutions are periodic with period 18. Furthermore, if $\Im(z_0) = \Re(z_0)$ and $\Im(z_1) = \Re(z_1)$, then 9 is the prime period and otherwise 18 is.

Our technique to prove this observation is to relate our complex dynamical system with the following *real* three term recurrence relation

$$\varphi_{k+1} + \varphi_{k-1} = |\varphi_k|,\ k \in \mathbf{Z}. \tag{3}$$

We first show that all its (real) solutions (i.e., real sequences $\{\nu_k\}_{k\in\mathbf{Z}}$ that satisfy (3)) can be 'generated' by one solution $\varphi = \{\varphi_k\}_{k\in\mathbf{Z}}$ which satisfies $(\varphi_0, \varphi_1) = (-1, 0)$, then we show that all nontrivial solutions of (3) are periodic with prime period 9. By decomposing a solution ζ of (1) into two real sequences φ and ψ which are solutions of (3), we may then show that any nontrivial solution $\zeta = \{\zeta_k\}_{k\in\mathbf{Z}}$ of (1) is periodic with period 18 and if $\Re(\zeta_k) = \Im(\zeta_k)$ and $\Re(\zeta_{k+1}) = \Im(\zeta_{k+1})$, then ζ is 9-periodic; otherwise, ζ is 18-periodic.

*Mathematics Subject Classifications: 11B37, 11B50.

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2 The Real Dynamical System

Let us first go through some elementary properties of our equation (3). First, recall that a real sequence $\varphi = \{\varphi_i\}_{i\in\mathbf{Z}}$ is said to be periodic if there is a positive integer T such that $\varphi_{m+T} = \varphi_m$ for all $m \in \mathbf{Z}$. The positive integer T is called a period of φ. If ω is the least among all periods of φ, then ω is called the least or prime period of φ and φ is said to be ω-periodic. An elementary fact is that the prime period is a factor of all periods of φ.

If φ is ω-periodic, then any (row) vector consisting of ω consecutive terms of it will be called a cycle and denoted by

$$\varphi^{[\alpha]} = (\varphi_\alpha, \varphi_{\alpha+1}, ..., \varphi_{\alpha+\omega-1})\,.$$

Clearly, if $\varphi^{[\alpha]}$ is known, then

$$\varphi_i = \varphi^{[\alpha]}_{(i+1) \bmod \omega},\ i \in \mathbf{Z}.$$

Next, a solution of (3) is a real sequence $\varphi = \{\varphi_i\}_{i\in\mathbf{Z}}$ which renders (3) into an identity after substitution. Since (3) can be rewritten as

$$\varphi_{i+1} = |\varphi_i| - \varphi_{i-1},$$

or

$$\varphi_{i-1} = |\varphi_i| - \varphi_{i+1},$$

therefore, we may see that a solution φ of (3) is uniquely determined by any pair $(\varphi_k, \varphi_{k+1})$ of two consecutive terms of φ.

The equation (3) has several invariance properties. First, let $\psi = \{\psi_i\}_{i\in\mathbf{Z}}$ be a real sequence, a translation of ψ is a sequence $E^j\psi$, $j \in \mathbf{Z}$, defined by

$$(E^j\psi)_m = \psi_{m-j},\ m \in \mathbf{Z},$$

(in particular, $E^0\psi = \psi$).

The following two invariance results are easily verified.

Lemma 1 *Let $\varphi = \{\varphi_k\}_{k\in\mathbf{Z}}$ be a solution of (3). Then for any $j \in \mathbf{Z}$, $E^j\varphi$ is also a solution of (3). Furthermore, if ω is the prime period of φ, then ω is also the prime period of $E^j\varphi$.*

Lemma 2 *If $\zeta = \{\zeta_i\}_{i\in\mathbf{Z}}$ is a solution of (3), then for any $\alpha \ge 0$, $\alpha\zeta$ is also a solution of (3). Furthermore, if $\xi = \{\xi_i\}_{i\in\mathbf{Z}}$ is a solution of (3) such that $\zeta_i \times \xi_i \ge 0$ for any $i \in \mathbf{Z}$, then for any $\alpha, \beta \ge 0$, $\alpha\zeta + \beta\xi$ is also a solution of (3).*

We now identify a solution that can be used to generate all other solutions. It is the solution $u = \{u_i\}_{i\in\mathbf{Z}}$ of (3) that satisfies $(u_0, u_1) = (-1, 0)$.

Lemma 3 *Let $u = \{u_i\}_{i\in\mathbf{Z}}$ be the solution of (3) that satisfies $(u_0, u_1) = (-1, 0)$. Then u is 9-periodic and*

$$u^{[0]} = (-1, 0, 1, 1, 0, -1, 1, 2, 1). \tag{4}$$

Furthermore, $u_i \times (E^5u)_i \ge 0$ for $i \in \mathbf{Z}$.

The proof is easy. Indeed, from (3) and $(u_0, u_1) = (-1, 0)$, we may calculate $u_2 = 1$, $u_3 = 1, ..., u_9 = -1, u_{10} = 0, ...$ so that 9 is a period of u. Then we may check that 1 and 3 are not periods of u. That is, the prime period of u is 9. From u, we then see that

$$(E^5u)^{[0]} = (0, -1, 1, 2, 1, -1, 0, 1, 1), \tag{5}$$

so that $u_i \times (E^5u)_i \ge 0$ for $i \in \mathbf{Z}$.

Lemma 4 *Let*

$$\varphi = \alpha u + \beta(E^5 u),$$

where α, β *are nonnegative real numbers. If* $(\alpha, \beta) = (0, 0)$, *then* $\varphi = \{0\}$; *otherwise,* φ *is a* 9*-periodic solution of (3) and*

$$\varphi^{[0]} = (-\alpha, -\beta, \alpha + \beta, \alpha + 2\beta, \beta, -\alpha - \beta, \alpha, 2\alpha + \beta, \alpha + \beta). \tag{6}$$

Notice that if $\alpha = \beta = 0$, then αu and $\beta\left(E^5 u\right)$ are both trivial solutions such that φ is also a trivial solution by direct verification. Suppose $(\alpha, \beta) \neq (0, 0)$. Then by Lemma 2, φ is a solution of (3) since $(\alpha u)_i \times (\beta E^5 u)_i \geq 0$ for all $i \in \mathbf{Z}$. Since 9 is a period of u and $E^5 u$, it is also a period of φ. Suppose φ is 1-periodic. then $-\alpha = \varphi_0 = \varphi_1 = -\beta$ and $\alpha + \beta = \varphi_2 = \varphi_1 = -\alpha$ which leads to $\alpha = \beta = 0$, a contradiction. Suppose φ is 3-periodic. Then from (6), $-\alpha = \varphi_0 = \varphi_6 = \alpha$ and $-\beta = \varphi_1 = \varphi_4 = \beta$ that leads to $\alpha = \beta = 0$ which is also a contradiction. Therefore, if $(\alpha, \beta) \neq (0, 0)$, then φ cannot be 1- nor 3-periodic. Hence φ is 9-periodic.

Next, we show that any nontrivial solution φ of (3) can be written as

$$\varphi = \alpha E^j u + \beta(E^{j+5} u)$$

for some $j \in \{0, 1, ..., 8\}$ and some $(\alpha, \beta) \in \left\{(x, y) \in \mathbf{R}^2 \mid x \geq 0, y > 0\right\}$.

To this end, we form 9 half rays of the form

$$\left\{t\left((E^j u)_0, (E^j u)_1\right) \mid t > 0\right\}, \; j = 0, 1, 2, ..., 8,$$

and create partitions of the plane based on these half rays, namely $\{\Omega_0, \Omega_1, ..., \Omega_9\}$ (see Figure 1) defined by

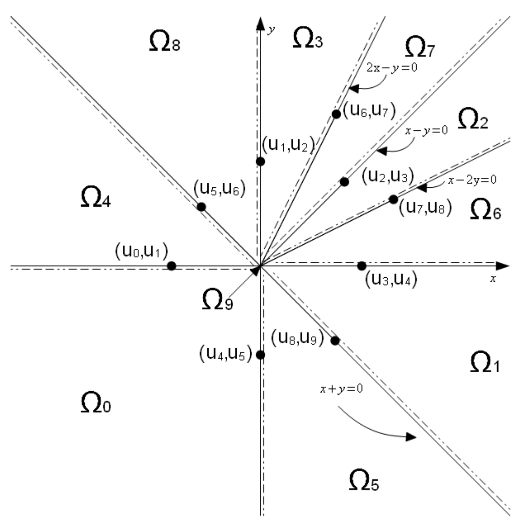

Figure 1: The partitions of $\Omega_{i \in \{0,...,9\}}$ on $\mathbf{R^2}$.

$$
\begin{aligned}
\Omega_0 &= \{(x,y) \mid x \le 0, y < 0\},\\
\Omega_1 &= \{(x,y) \mid y \le 0, x+y > 0\},\\
\Omega_2 &= \{(x,y) \mid x-y \ge 0, x-2y < 0\},\\
\Omega_3 &= \{(x,y) \mid x \ge 0, 2x-y < 0\},\\
\Omega_4 &= \{(x,y) \mid y \ge 0, x+y < 0\},\\
\Omega_5 &= \{(x,y) \mid x+y \le 0, x > 0\},\\
\Omega_6 &= \{(x,y) \mid x-2y \ge 0, y > 0\},\\
\Omega_7 &= \{(x,y) \mid 2x-y \ge 0, x-y < 0\},\\
\Omega_8 &= \{(x,y) \mid x+y \ge 0, x < 0\},\\
\Omega_9 &= \{(0,0)\}.
\end{aligned}
$$

In view of Figure 1, it is clear that for any $(s,t) \in \mathbf{R}^2 \backslash \{(0,0)\}$, there is a unique $j \in \{0,..,8\}$ and unique pair $(\alpha,\beta) \in (-\Omega_0)$ so that

$$(s,t) = \alpha\left((E^j u)_0, (E^j u)_1\right) + \beta\left((E^{j+5}u)_0, (E^{j+5}u)_1\right). \tag{7}$$

Theorem 1 *Let $\varphi = \{\varphi_k\}_{k\in\mathbf{Z}}$ be a nontrivial solution of (3). Then there is a unique 3-tuple (α,β,j) where $(\alpha,\beta) \in (-\Omega_0)$ and $j \in \{0,...,8\}$ so that*

$$\varphi = \alpha E^j u + \beta E^{j+5} u. \tag{8}$$

Indeed, if φ is nontrivial, then $(\varphi_0,\varphi_1) = (s,t) \in \mathbf{R}^2 \backslash \{(0,0)\}$. Hence (7) holds for unique $j \in \{0,1,...,8\}$ and unique $\alpha \ge 0$ and unique $\beta > 0$. By (7), we see that (8) holds since $(\varphi_0,\varphi_1) = (s,t)$.

It is rather easy to find the coefficients α and β and the exponent j in (8). Indeed, we let the mapping $\Gamma : \mathbf{R}^2 \longmapsto \mathbf{R}^2$ be defined by

$$\Gamma(s,t) = (\Gamma_1(s,t),\Gamma_2(s,t)) = \begin{cases} (-s,-t) & \text{if } (s,t)\in\Omega_0,\\ (-t,s+t) & \text{if } (s,t)\in\Omega_1,\\ (s-t,2t-s) & \text{if } (s,t)\in\Omega_2,\\ (s,t-2s) & \text{if } (s,t)\in\Omega_3,\\ (t,-s-t) & \text{if } (s,t)\in\Omega_4,\\ (-s-t,s) & \text{if } (s,t)\in\Omega_5,\\ (s-2t,t) & \text{if } (s,t)\in\Omega_6,\\ (2s-t,t-s) & \text{if } (s,t)\in\Omega_7,\\ (s+t,-s) & \text{if } (s,t)\in\Omega_8,\\ (0,0) & \text{if } (s,t)\in\Omega_9, \end{cases} \tag{9}$$

where $(s,t) \in \mathbf{R}^2$. By direct verification, we may easily see that

$$(\alpha,\beta) = (\Gamma_1(s,t),\Gamma_2(s,t)) \in (-\Omega_0)$$

in (7) as well as $\Gamma(0,0) = (0,0)$.

Theorem 2 *Let $\zeta = \{\zeta_k\}_{k\in\mathbf{Z}}$ be a nontrivial solution of (3) with $(\zeta_0,\zeta_1) = (s,t) \in \Omega_{j\in\{0,1,2,...,8\}}$. Then*

$$\zeta = \Gamma_1(s,t)E^j u + \Gamma_2(s,t)E^{j+5}u. \tag{10}$$

Proof. Let $\zeta = \{\zeta_k\}_{k\in\mathbf{Z}}$ be a solution of (3) with $(\zeta_0,\zeta_1) = (s,t)$. Suppose $(s,t) \in \Omega_0$. Then by (9), $(\Gamma_1(s,t),\Gamma_2(s,t)) = (-s,-t)$. Let

$$\xi = \Gamma_1(s,t)E^0 u + \Gamma_2(s,t)E^5 u.$$

Thus by (4) and (5), we have $(u_0, u_1, (E^5u)_0, (E^5u)_1) = (-1, 0, 0, -1)$. Therefore, we see that

$$\begin{aligned}(\xi_0, \xi_1) &= \left(\Gamma_1(s,t)u_0 + \Gamma_2(s,t)(E^5u)_0, \Gamma_1(s,t)u_1 + \Gamma_2(s,t)(E^5u)_1\right), \\ &= ((-1)(-s) + 0(-t), 0(-s) + (-1)(-t)), \\ &= (s,t), \\ &= (\zeta_0, \zeta_1),\end{aligned}$$

which implies $\zeta = \xi$. Suppose $(s,t) \in \Omega_1$. Then by (9), $(\Gamma_1(s,t), \Gamma_2(s,t)) = (-t, s+t)$ and let

$$\xi = \Gamma_1(s,t)E^1u + \Gamma_2(s,t)E^6u.$$

Thus, by (4) and (5), we have $((E^1u)_0, (E^1u)_1, (E^6u)_0, (E^6u)_1) = (1, -1, 1, 0)$. Therefore, we see that

$$\begin{aligned}(\xi_0, \xi_1) &= \left(\Gamma_1(s,t)(E^1u)_0 + \Gamma_2(s,t)(E^6u)_0, \Gamma_1(s,t)(E^1u)_1 + \Gamma_2(s,t)(E^6u)_1\right), \\ &= ((-t) + (s+t), (-1)(-t) + 0(s+t)), \\ &= (s,t), \\ &= (\zeta_0, \zeta_1),\end{aligned}$$

which implies $\zeta = \xi$. The other cases where $j \in \{2, ..., 8\}$ can be handled by similar manners and this completes the proof. ■

We remark that if ζ is the trivial solution, then (10) holds for $j = 9$. Since $\Gamma(0,0) = (0,0)$, by (10) and Lemma 4, we see that ζ is a trivial solution of (3).

Corollary 1 *All nontrivial solutions of (3) are* 9*-periodic.*

Indeed, if ζ is a nontrivial solution of (3), then by Theorem 1, we have

$$E^{-j}\zeta = \alpha u + \beta E^5 u$$

for some unique $j \in \{0, 1, ..., 8\}$ and $(\alpha, \beta) \in (-\Omega_0)$. Hence, by the invariance of translations and Lemma 4, it follows that $E^{-j}u$ is 9-periodic which implies ζ is also 9-periodic.

3 The Complex Dynamical System

First of all, for ease of presentation, we recall the complex dynamical system of (1) as

$$z_{k-1} + z_{k+1} = F(z_k), \ k \in \mathbf{Z}, \tag{11}$$

where $F(z) = |\Im(z)| + i|\Re(z)|$. Notice that a solution $\zeta = \{\zeta_k\}_{k\in\mathbf{Z}}$ of (11) is a complex sequence which renders (11) into an identity after substitution. Given ζ_0 and ζ_1, we may iteratively calculate $\zeta_2, \zeta_3, ...$, and $\zeta_{-1}, \zeta_{-2}, ...$ by (11) in a unique manner and hence we see that ζ is uniquely determined by its two consecutive terms.

It's obvious to see that the trivial solution of (11) is 1-periodic. Conversely, if a solution $\zeta = \{\zeta_k\}_{k\in\mathbf{Z}}$ of (11) is 1-periodic, then $\zeta_k = \zeta_0$ for all $k \in \mathbf{Z}$ and therefore, by (11), it's found that

$$2\zeta_0 = \zeta_{k-1} + \zeta_{k+1} = F(\zeta_k) = F(\zeta_0)$$

such that

$$2\Re(\zeta_0) = |\Im(\zeta_0)| \text{ and } 2\Im(\zeta_0) = |\Re(\zeta_0)|.$$

Since $|\Re(\zeta_0)|, |\Im(\zeta_0)| \geq 0$, we have $\Re(\zeta_0), \Im(\zeta_0) \geq 0$ and by the previous discussions, it follows that

$$\Re(\zeta_0) = \frac{1}{4}\Re(\zeta_0) \text{ and } \Im(\zeta_0) = \frac{1}{4}\Im(\zeta_0)$$

which leads to $\Re(\zeta_0) = 0$ as well as $\Im(\zeta_0) = 0$. Thus, by similar arguments, we may see that $\zeta_1 = 0$ and hence ζ is the trivial solution of (11).

The complex dynamical system (11) and the (real) dynamical system (3) are related in the following manner.

Lemma 5 *If $\varphi = \{\varphi_k\}_{k\in\mathbf{Z}}$ and $\psi = \{\psi_k\}_{k\in\mathbf{Z}}$ are solutions of (3), then the complex sequence $\{z_k\}_{k\in\mathbf{Z}}$ defined by*

$$z_k = \begin{cases} \varphi_k + i\psi_k & \text{if } k \equiv 0 \mod 2, \\ \psi_k + i\varphi_k & \text{if } k \equiv 1 \mod 2, \end{cases} \tag{12}$$

is a solution of (11). Conversely, if $w = \{w_k\}_{k\in\mathbf{Z}}$ is a solution of (11), then the sequences $\varphi = \{\varphi_k\}_{k\in\mathbf{Z}}$ and $\psi = \{\psi_k\}_{k\in\mathbf{Z}}$ defined respectively by

$$\varphi_k = \begin{cases} \Re(w_k) & \text{if } k \equiv 0 \mod 2, \\ \Im(w_k) & \text{if } k \equiv 1 \mod 2, \end{cases} \tag{13}$$

and

$$\psi_k = \begin{cases} \Im(w_k) & \text{if } k \equiv 0 \mod 2, \\ \Re(w_k) & \text{if } k \equiv 1 \mod 2, \end{cases} \tag{14}$$

are solutions of (3).

The proof is quite straightforward. Suppose φ and ψ are solutions of (3). We first note from (12) that $w_0 = \varphi_0 + i\psi_0$, $w_1 = \psi_1 + i\varphi_1$ and $w_2 = \varphi_2 + i\psi_2$. Then

$$w_2 + w_0 = \varphi_2 + \varphi_0 + i(\psi_2 + \psi_0) = |\varphi_1| + i|\psi_1| = |\Im(w_1)| + i|\Re(w_1)| = F(w_1).$$

Similarly, we can then show that (11) holds. Conversely, suppose $w = \{w_k\}_{k\in\mathbf{Z}}$ is a solution of (11). Then

$$\begin{aligned} \varphi_2 + \varphi_0 + i(\psi_2 + \psi_0) &= \Re(w_2) + \Re(w_0) + i(\Im(w_2) + \Im(w_0)) \\ &= w_2 + w_0 = F(w_1) = |\Im(w_1)| + i|\Re(w_1)| = |\varphi_1| + i|\psi_1| \end{aligned}$$

so that

$$\varphi_2 + \varphi_0 = |\varphi_1| \text{ and } \psi_2 + \psi_0 = |\psi_1|.$$

Similarly, we may show that φ and ψ are solutions of (3).

Theorem 3 *Any solution of (11) is periodic with period 18. Furthermore, for any nontrivial solution $w = \{w_k\}_{k\in\mathbf{Z}}$ of (11), if $\Re(w_0) = \Im(w_0)$ and $\Re(w_1) = \Im(w_1)$, then w is 9-periodic, otherwise w is periodic with least period 18.*

Proof. Let $w = \{w_k\}_{k\in\mathbf{Z}}$ be a solution of (11). Then by Corollary 1 and Lemma 5, the sequences φ and ψ defined by (13) and (14), respectively, are solutions of (3) with period 9. Therefore, for any odd j,

$$w_{9j} = \psi_{9j} + i\varphi_{9j} = \psi_0 + i\varphi_0 = i\left(\varphi_0 - i\psi_0\right) = i\overline{(\varphi_0 + i\psi_0)} = i\overline{w}_0.$$

Hence, for any $m \in \mathbf{Z}$,

$$\begin{aligned} w_{m+18} &= i\overline{w}_{m+9} = i\overline{i\overline{w}_m} = i\overline{i\left(\Re(w_m) - i\Im(w_m)\right)} = i\overline{(\Im(w_m) + i\Re(w_m))} \\ &= i\left(\Im(w_m) - i\Re(w_m)\right) = \Re(w_m) + i\Im(w_m) = w_m, \end{aligned}$$

that is, 18 is a period of w.

Next, if w is nontrivial, then by (13) and (14), φ or ψ is nontrivial. We assert that 6 is not a period of w. Indeed, suppose the contrary holds, then $w_0 = w_6$ and $w_1 = w_7$. Hence

$$(\Re(w_0), \Im(w_1)) = (\Re(w_6), \Im(w_7)) \text{ which implies } (\varphi_0, \varphi_1) = (\varphi_6, \varphi_7)$$

as well as

$$(\Im(w_0), \Re(w_1)) = (\Im(w_6), \Re(w_7)) \text{ which leads to } (\psi_0, \psi_1) = (\psi_6, \psi_7)\,.$$

Hence, φ and ψ have the period 6 which is contrary to Corollary 1. Note that w is not periodic with period 6 which implies 3 is neither a period of w. Therefore, the prime period of w, being a factor of 18, can only be 9 or 18. If $\Re(w_0) = \Im(w_0)$ and $\Re(w_1) = \Im(w_1)$, then by the previous discussions,

$$w_9 = i\overline{w}_0 = \Im(w_0) + i\Re(w_0) = \Re(w_0) + i\Im(w_0) = w_0$$

as well as

$$w_{10} = i\overline{w}_1 = \Im(w_1) + i\Re(w_1) = \Re(w_1) + i\Im(w_1) = w_1$$

which shows that w is 9-periodic; otherwise, $w_9 \neq w_0$ or $w_{10} \neq w_1$ so that w is 18-periodic. The proof is complete. ■

References

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