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# Periodic Solutions For Third-Order Differential Evolution Equation Set On Singular Domain*

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**Abstract**

In this paper, we analyze the existence and uniqueness of periodic solutions for a third-order evolution differential equation set in singular cylindrical domain of $\mathbb{R}^{3+1}$. The right-hand term of the equation is taken in some anisotropic Hölder spaces. Our strategy is based essentially on the study of a third order abstract differential equation. In order to achieve our aims, we essentially use the well known Dunford's operational calculus and some usual interpolation space.

## 1 Motivation and Statement of the Problem

This work is dedicated to the study of a boundary-value problem for a third-order time differential equation involving the Laplace operator set on a singular cylindrical domain in $\mathbb{R}^{3+1}$. More precisely, let $x = (x_1, x_2, x_3)$ be a generic point of $\mathbb{R}^3$, and let $\Omega$ be an open set of $\mathbb{R}^3$ defined by

$$\Omega := \left\{x \in \mathbb{R}^3 : 0 < x_1 < 1,\ \varphi_1(x_1) < x_2 < \varphi_2(x_1),\ 0 < x_3 < 1\right\},$$

where:

(H1) $\varphi_1$ and $\varphi_2$ are real-valued Lipschitz continuous functions of parametrization defined on $[0, 1]$;

(H2) for all $x_1 \in\ ]0, 1[$
$$\varphi_1(x_1) < \varphi_2(x_1),$$

(H3) $(\varphi_2 - \varphi_1)$ is a bounded function on $]0, 1[$.

(H4) $\varphi_1$ is allowed to coincide with $\varphi_2$ for $x_1 = 0$ and $x_1 = 1$, more precisely

$$\varphi_1(0) = \varphi_2(0), \tag{1}$$

and

$$\varphi_1(1) = \varphi_2(1). \tag{2}$$

Let $T$ be a fixed positive real number. In this work, we are concerned with the solvability of the following equation

$$D_t^3 u(t, x) + \sum_{i=1}^{3} D_{x_i}^2 u(t, x) = f(t, x), \quad (t, x) \in Q, \tag{3}$$

where $Q$ is the cylindrical domain defined by

$$Q := [0, T] \times \Omega.$$

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The equation (3) is considered under the following homogenous Dirichlet boundary conditions

$$u|_{[0,T]\times\partial\Omega}=0, \tag{4}$$

and the following initial conditions of periodic type

$$\left.D_t^j u\right|_{\{0\}\times\Omega}=\left.D_t^j u\right|_{\{T\}\times\Omega}. \tag{5}$$

Here, $D_t:=\dfrac{\partial}{\partial t}$ and $D_{x_i}:=\dfrac{\partial}{\partial x_i}$.

Let $0<\theta<1$. We will use the anisotropic Hölder space $C^\theta\left([0,T];C^\theta(\Omega)\right)$ consisting of all $\theta-$Hölder continuous functions $f:[0,T]\to C^\theta(\Omega)$ such that

$$\|f\|_{C^\theta([0,T];C^\theta(\Omega))}:=\|f\|_{C([0,T];C^\theta(\Omega))}+\sup_{t>\tau}\frac{\|f(t)-f(\tau)\|_{C^\theta(\Omega)}}{|t-\tau|^\theta}<\infty.$$

We assume also that

$$f|_{[0,T]\times\partial\Omega}=0. \tag{6}$$

Due to the presence of a singular point at the origin expressed by (1) and (2), we must first approximate $Q$ by a sequence of regular domains $(Q_n)_{n\in\mathbb{N}}$ defined by

$$Q_n:=[0,T]\times\Omega_n,$$

where

$$\Omega_n:=\left\{x\in\mathbb{R}^3:\alpha_n<x_1<1-\alpha_n,\ \varphi_1(x_1)<x_2<\varphi_2(x_1),\ 0<x_3<1\right\},$$

with $(\alpha_n)_{\in\mathbb{N}}$ is sequence chosen such that

$$0<\alpha_n<1/2 \text{ and } \lim_{n\to+\infty}\alpha_n=0.$$

First, Let us consider the following approximate problems

$$D_t^3u_n(t,x)+\sum_{i=1}^{3}D_{x_i}^2u_n(t,x)=f_n(t,x),\quad (t,x)\in Q_n. \tag{7}$$

Here, $f_n:=f|_{Q_n}$ and $u_n$ is the solution of (7). We use the following change of variables

$$\begin{array}{lll}\Psi & :[0,T]\times Q_n & \to[0,T]\times\Pi_n,\\ & (t,x) & \mapsto(t,y):=\left(t,x_1,\dfrac{x_2-\varphi_1(x_1)}{\varphi_2(x_1)-\varphi_1(x_1)},x_3\right),\end{array}$$

where

$$\Pi_n:=\left\{y\in\mathbb{R}^3:\alpha_n<y_1<1-\alpha_n,0<y_2<1,0<y_1<1\right\}.$$

and $y=(y_1,y_2,y_3)$ is also a generic point of $\mathbb{R}^3$. We introduce the change of functions given by

$$v_n(t,y):=u_n(t,x)\text{ and }g_n(t,y):=f_n(t,x).$$

Denoting by $D$ the vector with $i$the component $D_i=(D_{y_i})_{1\le i\le 3}$. After substitution, the new version of (7) set in the cylinder $[0,T]\times\Pi_n$ is given by

$$D_t^3v_n(t,y)+\mathcal{A}(y,D)\,v_n(t,y)=g_n(t,y), \tag{8}$$

where

$$\mathcal{A}(y,D):=D_{y_1}^2-a(y)\,D_{y_2}+b(y)\,D_{y_2}^2+D_{y_3}^2.$$

Here,

$$a(y) = \frac{\partial}{\partial y_1}\left(\frac{\varphi_1'(y_1)+(\varphi_2'(y_1)-\varphi_1'(y_1))\,y_2}{\varphi_2(y_1)-\varphi_1(y_1)}\right) - \frac{(\varphi_1'(y_1)+(\varphi_2'(y_1)-\varphi_1'(y_1))\,y_2)}{(\varphi_2(y_1)-\varphi_1(y_1))^2}\left(\varphi_2'(y_1)-\varphi_1'(y_1)\right), \tag{9}$$

$$b(y) = \frac{1}{(\varphi_2(y_1)-\varphi_1(y_1))^2} + \left(\frac{\varphi_1'(y_1)}{\varphi_2(y_1)-\varphi_1(y_1)} + \frac{\varphi_2'(y_1)-\varphi_1'(y_1)}{\varphi_2(y_1)-\varphi_1(y_1)}y_2\right)^2, \tag{10}$$

and

$$c(y) = -2\frac{\varphi_1'(y_1)}{\varphi_2(y_1)-\varphi_1(y_1)} - 2\frac{(\varphi_2'(y_1)-\varphi_1'(y_1))}{\varphi_2(y_1)-\varphi_1(y_1)}y_2. \tag{11}$$

The boundary conditions associated to the problem (8) are given by

$$v_n|_{[0,T]\times\partial\Omega_n} = 0, \tag{12}$$

and the initial conditions associated to the problem (8) are

$$D_t^j v_n\Big|_{\{0\}\times\Pi_n} = D_t^j v_n\Big|_{\{T\}\times\Pi_n}, \quad 1 \le j \le 2. \tag{13}$$

Set $E := C^\theta(\Pi_n)$, $0 < \theta < 1$ and define the vector valued functions :

$$v_n : [0,T] \to E \ ; \ t \longrightarrow v_n(t) \ ; \quad v_n(t)(y) := v_n(t,y),$$

$$g_n : [0,T] \to E \ ; \ t \longrightarrow g_n(t) \ ; \quad g_n(t)(y) := g_n(t,y).$$

Further on, consider the operator $A$ defined by

$$\begin{cases} D(A) & := \left\{\phi \in E : \mathcal{A}(.,D)\,\phi \in E, \ \ \phi|_{\partial\Pi_n} = 0\right\}, \ 1 < p < +\infty \\ (A\phi)(y) & := \mathcal{A}(y,D)\,\phi(y). \end{cases} \tag{14}$$

Then, problem (8) can be formulated as follows:

$$\begin{cases} v_n'''(t) + Av_n(t) = g_n(t), & 0 \le t \le T, \\ v_n^{(j)}(0) = v_n^{(j)}(T), & 1 \le j \le 2. \end{cases} \tag{15}$$

**Remark 1** *It is easy to see that the functions a and b defined by (9) and (10) are in $C^\infty(\Pi_n)$, uniformly for all $y \in \Omega_n$.*

At this level, we note that the class of differential equations given by (15) have recently attracted a considerable attention from the researcher community. In fact, the abstract third-order differential equations of the above type can be viewed as an abstract version of several concrete problems useful in modeling certain problems arising in various fields of agriculture, biology, economics and physics [1]–[18]; such problems also model some problems of diffusion or heat conductivity in viscoelastic media [16]. For growth population problems, we refer the reader to [17]. According to the classification of [13], the equation (15) belongs to the class of parabolic operator differential equations. For the reader's convenience, we recall that a complete third-order operator differential equation in its standard form is given by

$$u'''(t) + Au''(t) + Bu'(t) + Cu(t) = f(t), \quad t \ge 0, \tag{16}$$

where $A$, $B$ and $C$ are variable operator coefficients in a suitable complex Banach space. Note that the Hilbertian theory for equation (16) has been studied widely. We mention here the fundamental contributions made in [15] and see also the references cited therein. Other relevant results related to the equation (16) can be founded in [2], where the authors have studied the following problem

$$u'''(t) + (3A + A_1)\, u''(t) + (3A + A_2)\, u'(t) + A^3 u(t) = f(t), \quad t \in \mathbb{R}. \tag{17}$$

Some optimal results about the existence and uniqueness of this problem have been established, provided that

- $f \in L^2(\mathbb{R}, H)$,
- $(A, D(A))$ is a self-adjoint positive definite in a Hilbert space $H$,
- $(A_1, D(A_1))$ and $(A_2, D(A_2))$ are linear operators.

In [3], the problem

$$u'''(t) + (3A + A_1)\, u''(t) + (3A + A_2)\, u'(t) + \left(A^3 + A_3\right) u(t) = f(t), \quad t \geq 0,$$

was considered under the following assumptions

- $f \in L^2(\mathbb{R}^+, H)$,
- $(A, D(A))$ is a self-adjoint positive definite in a Hilbert space $H$,
- $(A_1, D(A_1))$ and $(A_2, D(A_2))$ $(A_3, D(A_3))$ are linear operators.
- $u''(0) = u'(0) = u(0) = 0$.

Concerning the $L^p$ setting of third-order differential equations with variable operator coefficients, we found in [11] a fairly complete study of the following equation

$$u'''(t) + 3Au''(t) + 3Au'(t) + A^3 u(t) = f(t), \ t \geq 0,$$

which is needed to investigate the following problem

$$(D_t + \Delta)^3\, u(t, x) = h(t, x), \quad (t, x) \in \Pi, \tag{18}$$

where $\Pi$ is a cylindrical domain

$$\Pi = \mathbb{R}^+ \times \Omega;$$

and its base $\Omega$ is a cusp domain defined by

$$\Omega := \left\{(x, y) \in \mathbb{R}^2 : 0 < x < a, \ -\psi(x) < y < \psi(x)\right\}, \tag{19}$$

where

$$\psi(x) = x^\alpha, \quad 1 < \alpha \leq 2,$$

$a$ is a given positive real number and the right hand term $h$ of (18) belongs to the anisotropic Lebesgue space $L^p(\mathbb{R}^+; E)$, $1 < p < +\infty$, where $E$ is a suitable Banach space.

In this research study, we build explicitly a representation of the solution $v_n(t)$ of the problem (15) and study its optimal regularity. We essentially utilize the Dunford functional calculus and the methods applied in [9], [10] and [11].

## 2 The Abstract Version of Problem (8)

In order to study our abstract problem (15), we need the following useful result (if $A$ is a closed linear operator, then $\rho(A)$ denotes its resolvent set):

**Proposition 1** *Let $A$ be the operator defined by (14). If the assumptions (H1)∼(H3) hold. Then, the operator $A$ satisfies the following hypothesis*

$$\mathbb{R}^{+} \subset \rho(A) \quad \text{and } \exists C > 0, \forall z \geqslant 0 \ \left\| (A - zI)^{-1} \right\|_{L(E)} \leqslant \frac{C}{1+z}, \tag{20}$$

**Proof.** See Theorem 4.1 in [5]. ■

**Remark 2** *By using a classical argument of analytic continuation on the resolvent. The estimate (20) holds true in the sector of the form*

$$\Sigma_{\theta_0, r_0} = \{ z \in \mathbb{C}^{*} : |\arg(z)| \leqslant \theta_0 \} \cup \{ z \in \mathbb{C} : |z| \leqslant r_0 \},$$

*with some small $\theta_0 > 0$, and $r_0 > 0$.*

Following the same strategy as in [6], [7], [8], [9], [10] and [12] the formal solution of our abstract problem (15) is given using the Dunford integral by

$$v_n(t) := -\frac{1}{2\pi i} \int_{\Gamma} \int_{0}^{T} \mathcal{K}_z(t,s)(A - zI)^{-1} g_n(s)\, ds\, dz, \tag{21}$$

where $\Gamma$ denotes the sectorial boundary curve of $\Sigma_{\theta_0, r_0}$ oriented positively, remaining in $\rho(A) \backslash \mathbb{R}^{+}$, and $\mathcal{K}_z$ is the Green kernel given by

$$\mathcal{K}_z(t,s) := \begin{cases} \mathcal{C}_z(t,s), & 0 \leq s \leq t \leq T, \\ \mathcal{S}_z(t,s), & 0 \leq t \leq s \leq T, \end{cases}$$

where $\mathcal{C}_z$ and $\mathcal{S}_z$ are complex valued functions given by

$$\begin{aligned} \mathcal{C}_z(t,s) \ : \ =& \ \frac{2e^{-\frac{z}{2}(t-s)} \sinh\left(-\frac{\sqrt{3}}{2} z(t-s) - \frac{\pi}{6}\right)}{\Phi_T(z)} \\ & - \frac{2e^{-\frac{z}{2}(t-s)} e^{-\frac{z}{2}T} \sinh\left(-\frac{\sqrt{3}}{2} z(t-s-T) - \frac{\pi}{6}\right)}{\Phi_T(z)} \\ & + \frac{2e^{-\frac{z}{2}(s-t)}}{\Psi_T(z)}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_z(t,s) \ : \ =& \ \frac{2e^{-\frac{z}{2}(t+T-s)} \sinh\left(-\frac{\sqrt{3}}{2} z(t+T-s) - \frac{\pi}{6}\right)}{\Phi_T(z)} \\ & - \frac{2e^{-\frac{z}{2}T} e^{-\frac{z}{2}(t+T-s)} \sinh\left(-\frac{\sqrt{3}}{2} z(t-s) - \frac{\pi}{6}\right)}{\Phi_T(z)} \\ & + \frac{2e^{-\frac{z}{2}(s-t-T)}}{\Psi_T(z)}. \end{aligned}$$

Here, $\Phi_z$ and $\Psi_z$ are two functions of complex variables given by

$$\Phi_T(z) := 3z^2\left(1+e^{-zT}\right) - 2e^{-\frac{z}{2}T}\cosh\left(-\frac{\sqrt{3}}{2}zT\right), \tag{22}$$

$$\Psi_T(z) := 3z^2\left(1-e^{-zT}\right). \tag{23}$$

For the sake of simplicity, set

$$v_n(t) := v_{n,1}(t) + v_{n,2}(t),$$

where

$$v_{n,1}(t) := -\frac{1}{2\pi i}\int_\Gamma\int_0^t \mathcal{C}_z(t,s)(A-zI)^{-1}g_n(s)\,ds\,dz,$$

and

$$v_{n,2}(t) := -\frac{1}{2\pi i}\int_\Gamma\int_t^T \mathcal{S}_z(t,s)(A-zI)^{-1}g_n(s)\,ds\,dz.$$

**Remark 3** *If not stated otherwise, the symbol $C$ denotes henceforth a generic positive real constant.*

Let us show that our representation formula (21) is proper. We need the following auxiliary results:

**Lemma 1** *Let $\Phi_T$ and $\Psi_T$ be the functions defined by (22) and (23). Then, for all $z \in \Sigma_{\theta_0,r_0}$, we have*

$$|\Phi_z(T)| = \mathcal{O}\left(|z|^2\right),$$

*and*

$$|\Psi_T(z)| = \mathcal{O}\left(|z|^2\right).$$

**Proof.** The consideration is similar for both functions, $\Phi_T$ and $\Psi_T$. First of all, it is easy to see that

$$\begin{aligned} |\Phi_z(T)| &= \left|3z^2\left(1+e^{-zT}\right) - 2e^{-\frac{z}{2}T}\cosh\left(-\frac{\sqrt{3}}{2}zT\right)\right| \\ &\geq 3|z|^2\left|1+e^{-zT}\right| - 2\left|e^{-\frac{z}{2}T}\cosh\left(-\frac{\sqrt{3}}{2}zT\right)\right| \\ &\geq 3|z|^2\left|1+e^{-zT}\right|. \end{aligned}$$

Thanks to [4, Lemma 3], it follows that there exist $C > 0$ such that, for every $z \in \Sigma_{\delta_0,\varepsilon_0}$, we have:

$$\left|1+e^{-zT}\right| \geqslant C.$$

This confirms that

$$|\Phi_z(T)| \geq C|z|^2.$$

■

**Remark 4** *Observe that, thanks to (20) and Lemma 1, all integrals appearing in the formula (21) are absolutely convergent.*

Now, we will establish some regularity results for the problem (15). First, we recall that a strict solution $v_n$ to (15) is any vectorial function $v_n$ such that

$$v \in C^3([0,T]\,;E) \cap C([0,T]\,;D(A)),$$

and (15) is satisfied, where

$$E := C^\theta\left(\Pi_n\right).$$

The following propositions are needed for proving some optimal regularity results of the solution (21):

**Proposition 2** *Let $g_n \in C^\theta\left([0,T]\,;E\right)$ with $0<\theta<1$. Then, we have $v_n \in D(A)$.*

**Proof.** Since $v_n(t) = v_{n,1}(t) + v_{n,2}(t)$, it suffices to prove the absolute convergence of the following integrals

$$\Delta_1 := -\frac{1}{2\pi i}\int\limits_\Gamma \int\limits_0^t \mathcal{C}_z\left(t,s\right) A(A-zI)^{-1} g_n(s)\,ds\,dz,$$

and

$$\Delta_2 := -\frac{1}{2\pi i}\int\limits_\Gamma \int\limits_t^T \mathcal{S}_z\left(t,s\right) A(A-zI)^{-1} g_n(s)\,ds\,dz.$$

We restrict ourselves to the study of $\Delta_1$; we can analogously consider the integral $\Delta_2$. We write the first integral in the form

$$\Delta_1 = I_1 + I_2 + I_3,$$

where

$$I_1 := -\frac{1}{2\pi i}\int\limits_\Gamma \int\limits_0^t \frac{2e^{-\frac{z}{2}(t-s)}\sinh\left(-\frac{\sqrt{3}}{2}z\left(t-s\right)-\frac{\pi}{6}\right)A(A_\lambda - zI)^{-1}g_n(s)\,ds\,dz}{\Phi_T\left(z\right)},$$

$$I_2 := \frac{1}{2\pi i}\int\limits_\Gamma \int\limits_0^t \frac{2e^{-\frac{z}{2}(t-s)}e^{-\frac{z}{2}T}\sinh\left(-\frac{\sqrt{3}}{2}z\left(t-s-T\right)-\frac{\pi}{6}\right)}{\Phi_T\left(z\right)}A(A_\lambda - zI)^{-1}g_n(s)\,ds\,dz,$$

$$I_3 := -\frac{1}{2\pi i}\int\limits_\Gamma \int\limits_0^t \frac{2e^{-\frac{z}{2}(s-t)}}{\Psi_T\left(z\right)}A(A_\lambda - zI)^{-1}g_n(s)\,ds\,dz.$$

Concerning the first integral $I_1$, it is easy to see that the estimate (20) implies

$$\left\|A(A-zI)^{-1}\right\|_{L(E)},$$

is bounded. On the other hand, Lemma 1 allows us to write

$$\begin{aligned}
\left\|I_1\right\|_E &\leq C\int\limits_\Gamma \int\limits_0^t \frac{e^{-\frac{\operatorname{Re} z}{2}(t-s)}\cosh\left(-\frac{\sqrt{3}}{2}\operatorname{Re} z\left(t-s\right)-\frac{\pi}{6}\right)}{\left|\Phi_T\left(z\right)\right|}ds\ \left|dz\right| \left\|g_n\right\|_{C^\theta([0,T];E)} \\
&\leq C\int\limits_\Gamma \left(\int\limits_0^t \frac{e^{-\frac{\operatorname{Re} z}{2}(t-s)}\cosh\left(-\frac{\sqrt{3}}{2}\operatorname{Re} z\left(t-s\right)-\frac{\pi}{6}\right)}{\left|z\right|^2}ds\right)\left|dz\right| \left\|g_n\right\|_{C^\theta([0,T];E)},
\end{aligned}$$

and

$$\begin{aligned}
\|I_1\|_E &\leq C\int_\Gamma \left(\int_0^t \frac{e^{\left(-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)\operatorname{Re} z(t-s)+\frac{\pi}{6}\right)}}{|z|^2}\,ds\right)|dz|\,\|g_n\|_{C^\theta([0,T];E)} \\
&\leq C\int_\Gamma \left(\frac{1}{|z|^2}\int_0^t e^{-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)\operatorname{Re} z(t-s)}\,ds\right)|dz|\,\|g_n\|_{C^\theta([0,T];E)} \\
&\leq C\left(\int_\Gamma \frac{1-e^{-\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)\operatorname{Re} zt}}{\operatorname{Re} z}\frac{d|z|}{|z|^2}\right)\|g_n\|_{C^\theta([0,T];E)} \\
&\leq C\left(\int_\Gamma \frac{1}{\operatorname{Re} z}\frac{|dz|}{|z|^2}\right)\|g_n\|_{C^\theta([0,T];E)},
\end{aligned}$$

which implies that the integral $I_1$ is finite. Similarly we can estimate $I_2$ and $I_3$, which implies the required result. ■

**Proposition 3** *Let $g_n \in C^\theta([0,T];E)$ with $0<\theta<1$. Then, we have*

$$g_n(.) - Av_n(.) \in C^\theta([0,T];E).$$

**Proof.** The proof of this result is based on a cumbersome calculus and we will only present here the main details. Let $0<t_1<t_2<T$ . Then we have:

$$\begin{aligned}
&(g_n(t_2)-Av_n(t_2))-(g_n(t_1)-Av_n(t_1)) \\
=\;& g_n(t_2)-g_n(t_1)-(Av_n(t_2)-Av_n(t_1)) \\
=\;& g_n(t_2)-g_n(t_1)+\Lambda.
\end{aligned}$$

First, we set

$$\Lambda(z) := A(A-zI)^{-1}.$$

Then

$$\begin{aligned}
&Av_n(t_2)-Av_n(t_1) \\
=\;& -\frac{1}{2\pi i}\int_\Gamma\int_0^{t_1}(\mathcal{C}_z(t_2,s)-\mathcal{C}_z(t_1,s))\,\Lambda(z)\,g_n(s)\,ds\,dz \\
&-\frac{1}{2\pi i}\int_\Gamma\int_{t_1}^{t_2}(\mathcal{C}_z(t_2,s)-\mathcal{S}_z(t_1,s))\,\Lambda(z)\,g_n(s)\,ds\,dz \\
&-\frac{1}{2\pi i}\int_\Gamma\int_{t_2}^{T}(\mathcal{S}_z(t_2,s)-\mathcal{S}_z(t_1,s))\,\Lambda(z)\,g_n(s)\,ds\,dz \\
:\;& =\sum_{i=1}^{3} I_3.
\end{aligned}$$

These three integrals can be treated similarly. For instance, the first integral can be written as follows

$$I_1 := \sum_{k=1}^{3} I_{1k},$$

where

$$I_{11} := -\frac{1}{\pi i}\int\limits_{\Gamma}\int\limits_{0}^{t_1}\frac{e^{-\frac{z}{2}(s-t_2)}-e^{-\frac{z}{2}(s-t_1)}}{\Psi_T(z)}\Lambda(z)\,g_n(s)\,ds\,dz,$$

$$\begin{aligned} I_{12} \quad : \quad &= -\frac{1}{\pi i}\int\limits_{\Gamma}\int\limits_{0}^{t_1}\frac{e^{-\frac{z}{2}(t_2-s)}e^{-\frac{z}{2}T}\sinh\left(-\frac{\sqrt{3}}{2}z\,(t_2-s-T)-\frac{\pi}{6}\right)}{\Phi_T(z)}\Lambda(z)\,g_n(s)\,ds\,dz \\ &+\frac{1}{\pi i}\int\limits_{\Gamma}\int\limits_{0}^{t_1}\frac{e^{-\frac{z}{2}(t_1-s)}e^{-\frac{z}{2}T}\sinh\left(-\frac{\sqrt{3}}{2}z\,(t_1-s-T)-\frac{\pi}{6}\right)}{\Phi_T(z)}\Lambda(z)\,g_n(s)\,ds\,dz, \end{aligned}$$

$$\begin{aligned} I_{13} \quad : \quad &= -\frac{1}{\pi i}\int\limits_{\Gamma}\int\limits_{0}^{t_1}\frac{e^{-\frac{z}{2}(t_2-s)}\sinh\left(-\frac{\sqrt{3}}{2}z\,(t_2-s)-\frac{\pi}{6}\right)}{\Phi_T(z)}\Lambda(z)\,g_n(s)\,ds\,dz \\ &+\frac{1}{\pi i}\int\limits_{\Gamma}\int\limits_{0}^{t_1}\frac{e^{-\frac{z}{2}(t-s)}\sinh\left(-\frac{\sqrt{3}}{2}z\,(t_1-s)-\frac{\pi}{6}\right)}{\Phi_T(z)}\Lambda(z)\,g_n(s)\,ds\,dz. \end{aligned}$$

Taking into account (20) and using the Lagrange mean value theorem, we conclude that

$$\|I_{1k}\|_E = O\left(|t_2-t_1|^{\theta}\right), \quad k\in\{1,2,3\}.$$

Summing up, we deduce that

$$|Av_n(t_2)-Av_n(t_1)\|_E = O\left(|t_2-t_1|^{\theta}\right).$$

This completes the proof of proposition. ■

The following useful remark clarifies the anisotropic character of the Hölder continuos spaces:

**Remark 5** *We should not identify the space $C^{\upsilon}(\mathbb{R}^+; C^{\upsilon}(\Omega))$ and the space $C^{\upsilon}(\Pi)$, $0<\upsilon<1$. In fact, we have:*

$$C^{\upsilon}(\Pi) = L^{\infty}([0,T]; C^{\upsilon}(\Omega)) \cap C^{\upsilon}([0,T]; C(\Omega)), \tag{24}$$

*For more details, we refer the reader to [14].*

This explains why we must describe the smoothness of (21), when the right-hand term of the problem (15) has a spatial smoothness; that is, for all $t\in[0,T]$

$$g_n \in L^{\infty}([0,T]; D_A(\sigma,+\infty)),\ 0<2\sigma<1,$$

where $D_A(\theta,+\infty)$ denotes a real Banach interpolation spaces between $D(A)$ and $E$ defined by

$$D_A(\theta,+\infty) := \left\{\psi\in E : \sup_{r>0}\left\|r^{\sigma}A\,(A-rI)^{-1}\psi\right\|_E < \infty\right\}.$$

We continue by stating the following result:

**Proposition 4** *Suppose that $g_n\in L^{\infty}([0,T], D_A(\sigma,+\infty))$, $0<2\sigma<1$. Then, under assumptions (20) and (6), we have*

$$v_n(.)\in L^{\infty}([0,T]; D_A(\sigma,+\infty)).$$

**Proof.** The required result is obtained by proving the convergence of the following term

$$\sup_{t\in[0,T]}\sup_{r>0}\left\|r^{\sigma}A(A-rI)^{-1}v_n(t)\right\|_E .$$

Using the identity

$$A(A-rI)^{-1}(A-zI)^{-1}=\frac{A(A-zI)^{-1}}{z-r}-\frac{A(A-rI)^{-1}}{z-r},$$

one has

$$\begin{aligned} r^{\sigma}A(A-rI)^{-1}v_n(t) &= -\frac{r^{\sigma}}{2i\pi}\int_{\Gamma_1}\int_0^T \mathcal{K}_z(t,s)\,\frac{A(A-zI)^{-1}}{z-r}g_n(s)\,ds\,dz \\ &\quad -\frac{r^{\sigma}}{2i\pi}\int_{\Gamma}\int_0^T \mathcal{K}_z(t,s)\,\frac{A(A-rI)^{-1}}{z-r}g_n(s)\,ds\,dz \\ : &= I_1+I_2. \end{aligned}$$

It is easy to see that by integrating to the left of $\Gamma$, we get

$$I_2=0.$$

Concerning the term $I_1$, we have:

$$\|I_1\|_E\leqslant r^{\sigma}\int_{\Gamma}\left(\int_t^{t+T}|\mathcal{K}_z(t,s)|\,ds\right)\frac{|dz|}{|z-r|\,|z|^{\sigma}}\,\|g_n\|_{D_A(\sigma,+\infty)},$$

since

$$\int_{\Gamma}\frac{|dz|}{|z-r|\,|z|^{\sigma}}=O\Big(\frac{1}{r^{\sigma}}\Big).$$

This implies

$$\|I_1\|_E\leqslant C\,\|g_n\|_{D_A(\sigma,+\infty)}.$$

and ends the proof. ■

In our framework, the assumption (6) imply

$$D_A(\sigma,+\infty)=\left\{\psi\in C^{\sigma}(\Omega_n):\,\psi|_{\partial\Omega_n}=0\right\},\ 0<2\sigma<1.$$

This allow us to conclude that the following result holds good:

**Proposition 5** *Suppose that $g_n\in L^{\infty}([0,T];D_A(\sigma,+\infty))$, $0<2\sigma<1$. Then, under assumptions (20) and (6), we have*

$$v_n(.)\in L^{\infty}([0,T];C^{\sigma}(\Pi_n)).$$

Taking into account the preceding result, our main result concerning the abstract problem (15) is formulated as follows:

**Theorem 1** *Let*

$$g_n\in C^{\theta}([0,T];E)\cap L^{\infty}([0,T];C^{\sigma}(\Pi_n)),\quad 0<\theta<1,\quad 0<2\sigma<1.$$

*Then, under assumptions (20) and (6), the problem (15) admits a unique strict solution $v_n$ given by (21). Furthermore, there exists a real constant $C>0$ independent of $n$ such that*

$$\max_{0\le t\le T}\|v_n(t)\|_{C(\Pi_n)}\le C. \tag{25}$$

## 3 Going Back to the Singular Domain

Observe that the estimate (25) and an elementary argumentation enables one to extract a convergent subsequence $(\alpha_{nj})$ from $(\alpha_n)$ with

$$\lim_{n\to+\infty} \alpha_{nj} = 0.$$

After that, it is easy to show that there exists $v$ such that

$$\lim_{n\to+\infty} v_{nj} = \lim_{n\to+\infty} v_{nj}(t,y) = v.$$

On the other hand, we have

$$\lim_{n\to+\infty} \Omega_n = \Omega,$$

and

$$\lim_{n\to+\infty} g_n(t,y) = g(t,y), \quad (t,y) \in [0,T] \times \Omega.$$

This allow us to conclude that the following result holds good:

**Theorem 2** *Let* $g \in C^{\theta}([0,T]; C^{\min(\theta,\sigma)}(\Omega)), \quad 0 < \theta < 1,\ 0 < 2\sigma < 1,$ *such that*

$$g|_{[0,T]\times\partial\Omega} = 0.$$

*Then, the problem*

$$\begin{cases} D_t^3 v + \mathcal{A}(y,D)v = g, \\ v|_{[0,T]\times\partial\Omega} = 0, \\ D_t^j v\Big|_{\{0\}\times\Omega} = D_t^j v\Big|_{\{T\}\times\Omega}, \ 1 \le j \le 2, \end{cases}$$

*admits a unique strict solution* $v$ *such that*

$$D_t^3 v,\ \mathcal{A}(y,D)v \in C^{\theta}([0,T]; C^{\min(\theta,\sigma)}(\Omega)).$$

The inverse change of variables is given by

$$\begin{array}{lll} \Psi^{-1} & : [0,T] \times \Pi & \to [0,T] \times Q, \\ & (t,y) & \mapsto (t,x) := (t, y_1, (\varphi_2(y_1) - \varphi_1(y_1)) y_2 + \varphi_1(y_1), y_3). \end{array}$$

Using the argumentation from [6], we finally get the main result of this work:

**Theorem 3** *Let*

$$f \in C^{\theta}([0,T]; C^{\min(\theta,\sigma)}(\Omega)), \quad 0 < \theta < 1,\ 0 < 2\sigma < 1,$$

*such that*

$$f|_{[0,T]\times\partial\Omega} = 0.$$

*Then, the problem*

$$\begin{cases} D_t^3 u(t,x) + \sum_{i=1}^{3} D_{x_i}^2 u(t,x) = f(t,x), & (t,x) \in Q, \\ u|_{[0,T]\times\partial\Omega} = 0, \\ D_t^j u\Big|_{\{0\}\times\Omega} = D_t^j u\Big|_{\{T\}\times\Omega}, \end{cases}$$

*admits a unique strict solution* $u$ *such that*

$$D_t^3 u,\ \sum_{i=1}^{3} D_{x_i}^2 u \in C^{\theta}([0,T]; C^{\min(\theta,\sigma)}(\Omega)).$$

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