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Generalizations Of Kantorovich's Inequality Induced By Majorization Theory*

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Abstract

In this paper, we apply Karamata's theorem combined with majorization theory to establish a new inequality for the upper bound of the product of two finite sums of convex functions. As applications, we derive some new generalizations of Kantorovich's inequality.

1 Introduction and Preliminaries

Let us start with some fundamental notations or definitions needed in this paper. The symbols $\mathbb{R}$ and $\mathbb{N}$ will denote the set of real numbers and the set of positive integers, respectively. For convenience, let

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n}$$

and

$$\mathbb{R}^n_{++} = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \ldots, n\}.$$

In particular, $\mathbb{R}^1_{++}$, simply denoted by $\mathbb{R}_{++}$, is $\mathbb{R}_{++} := (0, \infty)$.

Definition 1 ([14, 15, 16]) *A set $\Omega \subset \mathbb{R}^n$ is called convex if*

$$(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n) \in \Omega$$

for any $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ and $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \Omega$.

Definition 2 ([14, 15, 16]) *Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbols $\mathbf{x} \prec \mathbf{y}$) if*

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]} \quad \text{for } 1 \le k \le n-1,$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

where $x_{[1]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge \cdots \ge y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in descending order.

Definition 3 ([2]) *Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be two real-valued functions on an interval I. f and g are said to be*

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(*i*) *similarly ordered if*

$$(f(x)-f(y))(g(x)-g(y)) \geq 0 \quad \textit{for every } x, y \in I;$$

(*ii*) *oppositely ordered if*

$$(f(x)-f(y))(g(x)-g(y)) \leq 0 \quad \textit{for every } x, y \in I$$

or,

$$(f(x)-f(y))(g(y)-g(x)) \geq 0 \quad \textit{for every } x, y \in I.$$

The estimation of the upper bound of the product of two finite sum is a long lasting mathematical subject. Pólya-Szegö established a famous inequality as follows:

Theorem 1 (Pólya-Szegö [1, 2]) *Let* $0 < m_1 \leq a_k \leq M_1$ *and* $0 < m_2 \leq b_k \leq M_2$ $(k = 1, \ldots, n)$. *Then*

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{k=1}^{n} a_k b_k\right)^2. \tag{1}$$

In [3], an upper bound inequality equivalent to inequality (1) was introduced:

Theorem 2 (*Kantorovich inequality*) *Let* $\{x_k\}$, $k = 1, \ldots, n$ *be any real number sequence. If* $0 < m \leq x_k \leq M$, $i = 1, \ldots, n$, *then*

$$\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{x_k}\right) \leq \frac{(M+m)^2}{4Mm}.$$

Kantorovich inequality is a well-known inequality, this inequality is useful in numerical analysis and statistics, especially in the method of steepest descent. Therefore, it is valuable for its generalization and application. Over the years, various variations and extensions of this inequality have been investigated by many authors in several contexts. Reference [3]–[13] has many forms of generalizations and applications.

In 2005, Xu [5] proved the following generalized Kantorovich inequality.

Theorem 3 ([5]) *Let* $\alpha > 0$. *If* $0 < m \leq x_k \leq M$, $i = 1, \ldots, n$, *then*

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} \frac{1}{x_i^{\alpha}}\right)(M^{\alpha}-m^{\alpha})(M-m) \leq \frac{(M^{\alpha+1}-m^{\alpha+1})^2}{4(Mm)^{\alpha}}.$$

The following lemmas are important and will be used for proving our main results.

Lemma 1 ([16, 17]) *Let* $m \leq x_i \leq M$, $i = 1, \ldots, n, n \geq 2$, *and* $m \neq M$. *Then there is a unique* $l \in [m, M)$ *and unique integer* $k \in \{0, 1, \ldots, n\}$ *such that*

$$\sum_{i=1}^{n} x_i = (n-k-1)m + l + kM,$$

where l, k *is determined by*

$$(x_1, \ldots, x_n) \prec (\underbrace{M, \ldots, M}_{k}, l, \underbrace{m, \ldots, m}_{n-k-1}).$$

Remark 1 (*i*) *Because* $l = \sum_{i=1}^{n} x_i - (n-k-1)m - kM \in [m, M)$, *we see that*

$$\frac{\sum_{i=1}^{n} x_i - nm}{M-m} - 1 \leq k \leq \frac{\sum_{i=1}^{n} x_i - nm}{M-m}.$$

So we can determine k.

(ii) According to the proof of Lemma 1 in reference [17], we know that l is a variable that depends on $x_1, \ldots, x_n$ and $m \le l < M$.

Lemma 2 ([16, 17]) *Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and g is convex function on $I \subset \mathbb{R}$. If $\mathbf{x} \prec \mathbf{y}$, then*

$$\sum_{i=1}^{n} g(x_i) \le \sum_{i=1}^{n} g(y_i).$$

Kontenovich inequality is a famous inverse inequality of the famous Cauchy-Schwarz inequality, from Theorem 2 and Theorem 3 we observe that the functions corresponding to the two sequences are respectively: x and $\frac{1}{x}$, x and $\frac{1}{x^\alpha}, (\alpha > 0)$. They have the common oppositely ordered pproperty, and $\frac{1}{x}$, $\frac{1}{x^\alpha}$ is convex function. So this tells us, for two general functions, when they are oppositely ordered and convex, there may be results similar to Theorem 2 and Theorem 3.

2 New Inequalities for Differentiable Convex Functions

In this section, we establish the following new inequalities which will be applied to established new generalizations of Kantorovich's inequality and other new results.

Theorem 4 *Let $0 < m \le x_i \le M$ for $i = 1, \ldots, n$. Let f and g be two nonnegative convex functions and have second derivatives on $[m, M]$. Suppose that*

(H1) $(f(M) - f(m))(g(m) - g(M)) \ge 0$, and

(H2) $[kf(M) + (n-k-1)f(m) + f(x)]g''(x) + [kg(M) + (n-k-1)g(m) + g(x)]f''(x) + 2f'(x)g'(x) \ge 0$ for $m \le x \le M$ and $1 \le k \le n-1$.

Then

$$\left(\sum_{i=1}^{n} f(x_i)\right)\left(\sum_{i=1}^{n} g(x_i)\right)(f(M) - f(m))(g(m) - g(M)) \le \left(\frac{n\,[f(M)g(m) - f(m)g(M)]}{2}\right)^2. \tag{2}$$

Proof. In order to prove our conclusion, we consider the following two possible cases:
Case 1. If $m = M$, then (2) is obvious.
Case 2. Suppose $m < M$. If $(f(M) - f(m))(g(m) - g(M)) = 0$, then the conclusion also holds immediately. Hence we may assume that $(f(M) - f(m))(g(m) - g(M)) > 0$. By Lemma 1, there exists $k \in \mathbb{N}$ with $1 \le k \le n-1$, such that

$$(x_1, \ldots, x_n) \prec (\underbrace{M, \ldots, M}_{k}, l, \underbrace{m, \ldots, m}_{n-k-1}).$$

According to the Remark 1, we know that l is a variable that depends on $x_1, \ldots, x_n$ and $m \le l \le M$. Because f and g are nonnegative convex functions, by Lemma 2, we have

$$\sum_{i=1}^{n} f(x_i) \le kf(M) + (n-k-1)f(m) + f(l),$$

and

$$\sum_{i=1}^{n} g(x_i) \le kg(M) + (n-k-1)g(m) + g(l).$$

Let

$$h_a(b) := [af(M) + (n-a-1)f(m) + f(b)] \cdot [ag(M) + (n-a-1)g(m) + g(b)]$$

for $(a, b) \in \mathbb{N} \times [m, M]$. So, we get

$$\sum_{i=1}^{n} f(x_i) \sum_{i=1}^{n} g(x_i) \leq h_k(l).$$

Clearly, for any fixed a, h_a is a function of the variable b. Taking the derivative of h_a with respect to the variable b, we have

$$h'_a(b) = f'(b)[ag(M) + (n-a-1)g(m) + g(b)] + g'(b)[af(M) + (n-a-1)f(m) + f(b)],$$

and

$$\begin{aligned} h''_a(b) = {} & f''(b)[ag(M) + (n-a-1)g(m) + g(b)] \\ & + g''(b)[af(M) + (n-a-1)f(m) + f(b)] + 2f'(b)g'(b) \geq 0. \end{aligned}$$

Thus for any a, h_a is a convex function on $[m, M]$. Since $l \in [m, M]$, we see that $h_k(l) \leq h_k(m)$ or $h_k(l) \leq h_k(M)$, where

$$\begin{aligned} h_k(m) &= [kf(M) + (n-k-1)f(m) + f(m)][kg(M) + (n-k-1)g(m) + g(m)] \\ &= [kf(M) + (n-k)f(m)] \cdot [kg(M) + (n-k)g(m)] \\ &= f(M)g(M)k^2 + k(n-k)f(M)g(m) + k(n-k)f(m)g(M) + (n-k)^2 f(m)g(m) \\ &= \varphi(k), \end{aligned}$$

and

$$\varphi(k) = (f(M) - f(m))(g(M) - g(m))k^2 + n(f(M)g(m) + f(m)g(M) - 2f(m)g(m))k + n^2 f(m)g(m).$$

Next, we will find the maximum value of φ. $(f(M) - f(m))(g(m) - g(M)) \geq 0$, φ has a maximum value. Since

$$\varphi'(k) = 2(f(M) - f(m)) \cdot (g(M) - g(m))k + n(f(M)g(m) + f(m)g(M) - 2f(m)g(m)),$$

if $\varphi'(w) = 0$, then we obtain

$$w = \frac{2nf(m)g(m) - n(f(M)g(m) + f(m)g(M))}{2(f(M) - f(m))(g(M) - g(m))} = \frac{A}{2B}.$$

Hence φ has maximum value

$$\varphi(w) = \varphi\left(\frac{A}{2B}\right) = B\left(\frac{A}{2B}\right)^2 - A\frac{A}{2B} + C = \frac{4BC - A^2}{4B},$$

where

$$\begin{gathered} A = 2nf(m)g(m) - n(f(M)g(m) + f(m)g(M)), \\ B = (f(M) - f(m))(g(M) - g(m)), \\ C = n^2 f(m)g(m). \end{gathered}$$

On the other hand, since

$$\begin{aligned} h_k(M) &= [kf(M) + (n-k-1)f(m) + f(M)] \cdot [kg(M) + (n-k-1)g(m) + g(M)] \\ &= [(k+1)f(M) + (n-k-1)f(m)] \cdot [(k+1)g(M) + (n-k-1)g(m)] \\ &= h_{k+1}(m), \end{aligned}$$

the maximum value of $h_k(M)$ is the same as that of $h_k(m)$. So, we have

$$\sum_{i=1}^{n} f(x_i) \sum_{i=1}^{n} g(x_i) \leq h_k(l) \leq h_k(m) \leq \frac{4BC - A^2}{4B}.$$

Since

$$\begin{aligned}4BC &= 4(f(M)g(M) - f(M)g(m) - f(m)g(M) + f(m)g(m))n^2 f(m)g(m)\\ &= 4n^2 f(m)g(m)f(M)g(M) - 4n^2 f(m)f(M)g^2(m)\\ &- 4n^2 f^2(m)g(m)g(M) + 4n^2 f^2(m)g^2(m),\end{aligned}$$

and

$$\begin{aligned}A^2 &= 4n^2 f^2(m)g^2(m) - 4n^2 f(m)f(M)g^2(m) - 4n^2 f^2(m)g(m)g(M)\\ &+ n^2 f^2(M)g^2(m) + n^2 f^2(m)g^2(M) + 2n^2 f(m)f(M)g(m)g(M),\end{aligned}$$

we get

$$4BC - A^2 = -n^2(f(M)g(m) - f(m)g(M))^2$$

and hence

$$\frac{4BC - A^2}{4B} = \frac{-n^2(f(M)g(m) - f(m)g(M))^2}{4(f(M) - f(m))(g(M) - g(m))} \geq 0.$$

Therefore we show that

$$\sum_{i=1}^{n} f(x_i) \sum_{i=1}^{n} g(x_i) \leq \frac{-n^2(f(M)g(m) - f(m)g(M))^2}{4(f(M) - f(m))(g(M) - g(m))},$$

or, equivalence,

$$\left(\sum_{i=1}^{n} f(x_i)\right)\left(\sum_{i=1}^{n} g(x_i)\right)(f(M) - f(m))(g(m) - g(M)) \leq \left(\frac{n\,(f(M)g(m) - f(m)g(M))}{2}\right)^2.$$

The proof is completed. ■

As a consequence of Theorem 4, we can obtain the following generalized Kantorovich's inequality.

Corollary 1 *Let $0 < m \leq x_i \leq M$ for $i = 1, \ldots, n$, and let f and g be two nonnegative convex functions and have second derivatives on $[m,\, M]$. If f, g are oppositely ordered on $[m,\, M]$ and $(fg)'' \geq 0$ on $[m,\, M]$, then*

$$\left(\sum_{i=1}^{n} f(x_i)\right)\left(\sum_{i=1}^{n} g(x_i)\right)(f(M) - f(m))(g(m) - g(M)) \leq \left(\frac{n\,(f(M)g(m) - f(m)g(M))}{2}\right)^2.$$

Proof. Since f, g are oppositely ordered on $[m,\, M]$, we have $(f(M)-f(m))(g(m)-g(M)) \geq 0$ and condition (H1) as in Theorem 4 is proved. We conclude that condition (H2) as in Theorem 4 holds. In fact note that

$$(fg)'' \geq 0 \quad \text{on } [m, M]$$

if and only if

$$f(x)g''(x) + f''(x)g(x) + 2f'(x)g'(x) \geq 0 \quad \text{for } x \in [m, M].$$

For $m \leq x \leq M$ and $1 \leq k \leq n-1$, we obtain

$$\begin{aligned}& [kf(M) + (n-k-1)f(m) + f(x)]g''(x)\\ & +[kg(M) + (n-k-1)g(m) + g(x)]f''(x) + 2f'(x)g'(x)\\ =\; & [kf(M) + (n-k-1)f(m)]g''(x) + [kg(M) + (n-k-1)g(m)]f''(x)\\ & +f(x)g''(x) + g(x)f''(x) + 2f'(x)g'(x) \geq 0.\end{aligned}$$

Therefore all the conditions of **Theorem 4** are satisfied and the conclusion follows from Theorem 4 immediately. ■

3 New generalizations of Kantorovich's inequality

Applying Theorem 4 or Corollary 1, we can obtain new exponential generalizations of Kantorovich's inequality.

Theorem 5 *Let $0 < m \leq x_i \leq M$ for $i = 1, \ldots, n$. If one of the following conditions is satisfied:*

(i) $\alpha = 1$ *and* $\beta < 0$*;*

(ii) $\alpha > 1$ *and* $\beta < 0$ *with* $\alpha + \beta \geq 1$ *or* $\alpha + \beta \leq 0$*,*

then

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha}\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\beta}\right)(M^{\alpha} - m^{\alpha})(m^{\beta} - M^{\beta}) \leq \left(\frac{M^{\alpha}m^{\beta} - m^{\alpha}M^{\beta}}{2}\right)^2. \tag{3}$$

Proof. (i) Suppose that $\alpha = 1$ and $\beta < 0$. Let $f(x) = x$ and $g(x) = x^{\beta}$ for $x > 0$. Then $f(x)$ is an increasing convex function and $g(x)$ is a decreasing convex function. Clearly, $(f(M) - f(m))(g(m) - g(M)) \geq 0$. Let $m \leq x \leq M$ and $1 \leq k \leq n - 1$. Set $u = kM + (n - k - 1)m$. Thus $u \geq M$. Since

$$\begin{aligned}
& [kf(M) + (n-k-1)f(m) + f(x)]g''(x) \\
& +[kg(M) + (n-k-1)g(m) + g(x)]f''(x) + 2f'(x)g'(x) \\
= \; & [kM + (n-k-1)m + x]\beta(\beta-1)l^{\beta-2} + 2\beta x^{\beta-1} \\
= \; & (u+x)\beta(\beta-1)x^{\beta-2} + 2\beta x^{\beta-1} \\
= \; & -\beta x^{\beta-2}[(u+x)(1-\beta) - 2x] \\
= \; & -\beta x^{\beta-2}[u(-\beta+1) + (-\beta-1)x]
\end{aligned}$$

and $m \leq x \leq M \leq u$, we get

$$\begin{aligned}
& [kf(M) + (n-k-1)f(m) + f(x)]g''(x) \\
& + [kg(M) + (n-k-1)g(m) + g(x)]f''(x) + 2f'(x)g'(x) \geq 0.
\end{aligned}$$

Hence all the conditions of Theorem 4 are satisfied. By Theorem 4, we have

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha}\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\beta}\right)(M^{\alpha} - m^{\alpha})(m^{\beta} - M^{\beta}) \leq \left(\frac{M^{\alpha}m^{\beta} - m^{\alpha}M^{\beta}}{2}\right)^2.$$

(ii) Let $\varphi(x) = x^{\gamma}$ for $x > 0$. Then $\varphi^{''}(x) = \gamma(\gamma - 1)x^{\gamma-2}$ for $x > 0$. It is easy to see that when $\gamma \geq 1$ or $\gamma < 0$, $h(x)$ is a convex function on $\mathbb{R}_{++}$. Suppose that $\alpha > 1$and $\beta < 0$ with $\alpha + \beta \geq 1$ or $\alpha + \beta \leq 0$. Let $f(x) = x^{\alpha}$ and $g(x) = x^{\beta}$ for $x > 0$. Then $f(x)$ is an increasing convex function and $g(x)$ is a decreasing convex function. Obviously, $(f(M) - f(m))(g(m) - g(M)) \geq 0$. Direct calculation gives

$$\begin{aligned}
(f(x)g(x))'' &= f(x)g''(x) + g(x)f''(x) + 2f'(x)g'(x) \\
&= \beta(\beta-1)x^{\alpha+\beta-2} + \alpha(\alpha-1)x^{\alpha+\beta-2} + 2\alpha\beta x^{\alpha+\beta-2} \\
&= (\alpha+\beta)(\alpha+\beta-1)x^{\alpha+\beta-2} \geq 0.
\end{aligned}$$

Therefore all the conditions of Corollary 1 are satisfied and the conclusion follows immediately from Corollary 1.

The proof is completed. ■

Remark 2 *(a) Taking $\alpha = 1$ and $\beta < 0$ in Theorem 5, we can get Theorem 3.*

(b) Let $0 < m \le x_i \le M$ for $i = 1, \dots, n$. For any $\gamma > 0$, by Theorem 5, we obtain

$$A_n(\boldsymbol{x})(M-m)(M^\gamma - m^\gamma) \le \frac{(M^{\gamma+1} - m^{\gamma+1})^2}{4M^\gamma m^\gamma} H_n(\boldsymbol{x}^\gamma),$$

where $A_n(\boldsymbol{x})$ is the arithmetic mean of $x_1, \cdots, x_n$ and $H_n(\boldsymbol{x}^\gamma)$ is the harmonic mean of $x_1^\gamma, \cdots, x_n^\gamma$. In particular, if we take $\gamma = 1$ in last inequality, then

$$A_n(\boldsymbol{x}) \le \frac{(M+m)^2}{4Mm} H_n(\boldsymbol{x}).$$

Theorem 6 *Let $0 < m \le x_i \le M$ for $i = 1, \dots, n$. If one of the following conditions is satisfied:*

(i) $p > 1$ and $\lambda \ge \frac{2M}{p-1}$;

(ii) $p > 1$ and $-m < \lambda \le 0$;

(iii) $p \le -1$ and $\lambda \ge 0$,

then

$$\begin{aligned} &\left(\frac{1}{n}\sum_{i=1}^n x_i^p\right)\left(\frac{1}{n}\sum_{i=1}^n \frac{1}{(\lambda + x_i)^p}\right)(M^p - m^p)\left[(\lambda+M)^p - (\lambda+m)^p\right] \\ \le\ & \frac{\left[M^p(\lambda+M)^p - m^p(\lambda+m)^p\right]^2}{4(\lambda+m)^p(\lambda+M)^p}. \end{aligned}$$

Proof. We only verify (i) and (ii), and a similar argument could be made for proving (iii). Let $f(x) = x^p$ and $g(x) = (\lambda + x)^{-p}$ for $0 < x \le M$. If $p > 1$ and $\lambda \ge \frac{2M}{p-1}$, or $p > 1$ and $-m < \lambda \le 0$, then direct calculation give

$$f'(x) = px^{p-1} > 0, \quad g'(x) = -p(\lambda+x)^{-p-1} < 0,$$
$$f''(x) = p(p-1)x^{p-2} > 0, \quad g''(x) = p(p+1)(\lambda+x)^{-p-2} > 0,$$

and

$$\begin{aligned} (f(x)g(x))'' &= f''(x)g(x) + g''(x)f(x) + 2f'(x)g'(x) \\ &= p(p-1)\frac{x^{p-2}}{(\lambda+x)^p} + p(p+1)\frac{x^p}{(\lambda+x)^{p+2}} - 2p^2\frac{x^{p-1}}{(\lambda+x)^{p+1}} \\ &= p\frac{x^{p-2}}{(\lambda+x)^{p+2}}\left[(p-1)(\lambda+x)^2 + (p+1)x^2 - 2px(\lambda+x)\right] \\ &= p\lambda\frac{x^{p-2}}{(\lambda+x)^{p+2}}[(p-1)\lambda - 2x]. \end{aligned}$$

• If $p > 1$ and $-m < \lambda \le 0$, we have

$$(\lambda+x)^{p+2} > 0,\ p\lambda \le 0 \text{ and } (p-1)\lambda - 2x < 0.$$

So, we get

$$(f(x)g(x))'' = p\lambda\frac{x^{p-2}}{(\lambda+x)^{p+2}}[(p-1)\lambda - 2x] \ge 0.$$

• If $p > 1$ and $\lambda \ge \frac{2M}{p-1}$, since $x \le M$, we obtain

$$(f(x)g(x))'' = p\lambda\frac{x^{p-2}}{(\lambda+x)^{p+2}}[(p-1)\lambda - 2x] \ge p\lambda\frac{x^{p-2}}{(\lambda+x)^{p+2}}(2M - 2x) \ge 0.$$

On the other hand, since

$$f(M) = M^p,\ f(m) = m^p,\ g(M) = \frac{1}{(\lambda + M)^p} \text{ and } g(m) = \frac{1}{(\lambda + m)^p},$$

it is not hard to show that $(f(M) - f(m))(g(m) - g(M)) \geq 0$. Therefore the desired conclusion follows immediately from Corollary 1.

The proof is completed. ■

Remark 3 *In fact, by taking $p = -1$ and $\lambda = 0$ in Theorem 6, we can obtain*

$$\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}\right) \leq \frac{(M+m)^2}{4Mm},$$

which is the original Kantorovich inequality. Hence, Theorem 6 is a real generalization of Kantorovich inequality.

4 Conclusions

In this paper, we use Karamata's theorem combined with majorization theory to establish an inequality for the upper bound of the product of two finite sums about convex function. As applications, we derive some new generalizations of Kantorovich's inequality.

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