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Fibonacci And Lucas Numbers Which Are Product Of Two Jacobsal-Lucas Numbers*

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Abstract

In this study, we find all Fibonacci numbers F_k and Lucas numbers L_k which are products of two Jacobsthal-Lucas numbers. More generally, taking k, m, n as nonnegative integers, we proved that

$$F_k = j_m j_n = (2^m + (-1)^m) \cdot (2^n + (-1)^n)$$

implies that

$$(k, m, n) = (1, 1, 1),\ (2, 1, 1),\ (3, 0, 1),\ (5, 1, 2),\ (9, 0, 4)$$

and $L_k = j_m j_n$ implies that

$$(k, m, n) = (3, 0, 0),\ (0, 0, 1),\ (1, 1, 1),\ (4, 1, 3).$$

As a result of this study, we showed that the largest Fibonacci number and Lucas number which can be written in the form

$$(2^m + (-1)^m) \cdot (2^n + (-1)^n)$$

are $F_9 = 34 = 2 \cdot 17 = (2^0 + (-1)^0) \cdot (2^4 + (-1)^4)$ and $L_4 = 7 = 1 \cdot 7 = (2^1 + (-1)^1) \cdot (2^3 + (-1)^3)$, respectively. Moreover the largest Fibonacci number and Lucas number which can be written in the form

$$2^n + (-1)^n$$

are $F_5 = 5 = 2^2 + (-1)^2$ and $L_4 = 7 = 2^3 + (-1)^3$, respectively. As a result, it is shown that the only Fermat numbers in the Fibonacci sequence are $F_3 = 3$ and $F_5 = 5$ and the only Fermat number in the Lucas sequence is $L_2 = 3$. The proofs depend on lower bounds for linear forms and some tools from Diophantine approximation.

1 Introduction

Let (F_n) and (L_n) be the sequences of Fibonacci numbers and Lucas numbers given by $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, respectively (sequences A000045 and A000032 in [9]). Binet formulas for Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \text{ and } L_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2} \text{ and } \beta = \frac{1 - \sqrt{5}}{2},$$

which are the roots of the characteristic equations $x^2 - x - 1 = 0$. It can be seen that $1 < \alpha < 2$, $-1 < \beta < 0$, and $\alpha\beta = -1$. The relation between n-th Fibonacci number F_n and α is given by

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \tag{1}$$

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for $n \geq 1$. Also, the relation between n-th Lucas number L_n and α is given by

$$\alpha^{n-1} \leq L_n \leq 2\alpha^n \tag{2}$$

for $n \geq 0$. The sequence of Jacobsthal-Lucas numbers (j_n) satisfies recurrence relation $j_n = j_{n-1} + 2j_{n-2}$ for $n \geq 2$ with initial conditions $j_0 = 2,\ j_1 = 1$ (sequence A014551 in [9]). j_n is called the n-th Jacobsthal-Lucas number. We have the Binet formula

$$j_n = 2^n + (-1)^n\,, \tag{3}$$

where 2 and -1 are the roots of the characteristic equation $x^2 - x - 2 = 0$. It is clear that

$$2^{n-1} \leq j_n \leq 2^{n+1} \tag{4}$$

for all $n \geq 0$. In [4], the authors have found all Fibonacci numbers or Pell numbers which are products of two numbers from the other sequence. In [10], Şiar found all Lucas numbers which are products of two balancing numbers. In [6], we have found all Fibonacci numbers or balancing numbers which are products of two numbers from the other sequence. In [7], we also solved the equations

$$F_k = J_m J_n$$

and

$$J_k = F_m F_n,$$

where (J_n) is the Jacobsthal sequence defined by $J_0 = 0, J_1 = 1$ and $J_n = J_{n-1} + 2J_{n-2}$ for $n \geq 2$ (sequence A001045 in [9]). In this study, we determine all solutions of the equations

$$F_k = j_m j_n \tag{5}$$

and

$$L_k = j_m j_n \tag{6}$$

in nonnegative integers k, m, n. In particular, we showed that the largest Fibonacci number and Lucas number which can be written in the form

$$(2^m + (-1)^m) \cdot (2^n + (-1)^n)$$

are $F_9 = 34 = 2 \cdot 17 = (2^0 + (-1)^0) \cdot (2^4 + (-1)^4)$ and $L_4 = 7 = 1 \cdot 7 = (2^1 + (-1)^1) \cdot (2^3 + (-1)^3)$, respectively. Moreover the largest Fibonacci number and Lucas number which can be written in the form

$$2^n + (-1)^n$$

are $F_5 = 5 = 2^2 + (-1)^2$ and $L_4 = 7 = 2^3 + (-1)^3$, respectively. As a result, it is shown that the only Fermat numbers in the Fibonacci sequence are $F_3 = 3$ and $F_5 = 5$ and the only Fermat number in the Lucas sequence is $L_2 = 3$.

Our study can be viewed as a continuation of the previous work on this subject. We follow the approach and the method presented in [4]. In Section 2, we introduce necessary lemmas and theorems. Then in Section 3, we prove our main theorems.

2 Auxiliary Results

In [4], [10], and [6], in order to solve Diophantine equations of the form (5) and (6), the authors have used Baker's theory of lower bounds for a nonzero linear form in logarithms of algebraic numbers. Since such bounds are of crucial importance in effectively solving Diophantine equations of the similar form, we start with recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \ldots + a_d = a_0 \prod_{i=1}^{d} \left(x - \eta^{(i)}\right) \in \mathbb{Z}[x],$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the $\eta^{(i)}$'s are conjugates of η. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\left\{|\eta^{(i)}|, 1\right\}\right)\right) \tag{7}$$

is called the logarithmic height of η. In particular, if $\eta = a/b$ is a rational number with $\gcd(a,b) = 1$ and $b \geq 1$, then $h(\eta) = \log(\max\{|a|, b\})$.

The following properties of logarithmic height are found in many works stated in the references:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{8}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{9}$$

$$h(\eta^m) = |m| h(\eta). \tag{10}$$

Now we give a theorem which is deduced from Corollary 2.3 of Matveev [8] and provides a large upper bound for the subscript n in the equations (5) and (6)(also see Theorem 9.4 in [3]).

Theorem 1 *Assume that $\gamma_1, \gamma_2, ..., \gamma_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, b_2, ..., b_t$ are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 A_2 \cdots A_t\right),$$

where

$$B \geq \max\{|b_1|, ..., |b_t|\},$$

and $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ for all $i = 1, ..., t$.

The following lemma is given in [2]. This lemma is an immediate variation of the result due to Dujella and Pethő from [5], which is a version of a lemma of Baker and Davenport [1]. This lemma will be used to reduce the upper bound for the subscript n in the equations (5) and (6). Let $||x||$ denote the distance from x to the nearest integer. That is, $||x|| = \min\{|x - n| : n \in \mathbb{Z}\}$ for any real number x. Then we have

Lemma 1 *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v, and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

The proof of the following lemma is easy and will be omitted.

Lemma 2 *If the real numbers x and r satisfy $|e^x - 1| < r < \frac{3}{4}$, then $|x| < 2r$.*

3 Main Theorems

Theorem 2 *The Diophantine equation* $F_k = j_m j_n = (2^m + (-1)^m) \cdot (2^n + (-1)^n)$ *has only the solutions*

$$(k, m, n) = (1, 1, 1),\ (2, 1, 1),\ (3, 0, 1),\ (5, 1, 2),\ (9, 0, 4)$$

in nonnegative integers.

Proof. Assume that the equation $F_k = j_m j_n$ holds. We assume that $0 \leq m \leq n \leq 69$. Then, by using the Mathematica program, we see that $k \leq 200$. In this case, with the help of Mathematica program, we obtain only the solutions

$$\begin{aligned} 1 &= F_1 = F_2 = j_1 j_1 = 1 \cdot 1, \\ 2 &= F_3 = j_0 j_1 = 2 \cdot 1, \\ 5 &= F_5 = j_1 j_2 = 1 \cdot 5, \\ 34 &= F_9 = j_0 j_4 = 2 \cdot 17, \end{aligned}$$

in the range $0 \leq m \leq n \leq 69$. From now on, assume that $n \geq 70$ and $m \geq 2$. Using the inequality (1) and (4), we get the inequality

$$\alpha^{k-2} \leq F_k = j_m j_n \leq 2^{2n+2} < \alpha^{4n+4},$$

which yields to $k < 4n + 6$. On the other hand, the inequality

$$2^{n-1} < j_m j_n = F_k \leq \alpha^{k-1} < 2^k,$$

implies that $k > n - 1 > 69$. Since

$$\frac{\alpha^k - \beta^k}{\sqrt{5}} = F_k = j_m j_n = 2^{n+m} + (-1)^{n+m} + 2^n (-1)^m + 2^m (-1)^n, \tag{11}$$

it is seen that

$$\frac{\alpha^k}{\sqrt{5}} - 2^{n+m} = \frac{\beta^k}{\sqrt{5}} + 2^n (-1)^m + 2^m (-1)^n + (-1)^{n+m}.$$

Taking absolute values, we obtain

$$\left| \frac{\alpha^k}{\sqrt{5}} - 2^{n+m} \right| \leq \frac{|\beta|^k}{\sqrt{5}} + 2^n + 2^m + 1.$$

Dividing both sides of this inequality by 2^{n+m}, we obtain

$$\left| \frac{\alpha^k \cdot 2^{-(n+m)}}{\sqrt{5}} - 1 \right| \leq \frac{|\beta|^k}{2^{n+m} \cdot \sqrt{5}} + \frac{1}{2^m} + \frac{1}{2^n} + \frac{1}{2^{n+m}} < \frac{2.1}{2^m}, \tag{12}$$

where we have used the fact that

$$\left(\frac{\sqrt{5} + |\beta|^k}{\sqrt{5} \cdot 2^n} + 2 \right) \leq 2.1$$

for $k > 69$ and $n \geq 70$. Now, let us apply Theorem 1 with $\gamma_1 := 1/\sqrt{5},\ \gamma_2 := \alpha,\ \gamma_3 := 2$ and $b_1 := 1,\ b_2 := k,\ b_3 := -(n+m)$. Note that the numbers γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $\mathbb{K} = \mathbb{Q}(\sqrt{5})$. It is obvious that the degree of the field $\mathbb{K}$ is 2. So $D = 2$. Now, we show that

$$\Lambda_1 := \frac{\alpha^k \cdot 2^{-(n+m)}}{\sqrt{5}} - 1$$

is nonzero. For, if $\Lambda_1 = 0$, then we get

$$\alpha^k = 2^{n+m}\sqrt{5}.$$

That is, $\alpha^{2k} \in \mathbb{Q}$, which is impossible for any $k > 0$. Since

$$h(\gamma_1) = h(1/\sqrt{5}) \leq h(\sqrt{5}) = \frac{\log 5}{2} < 0.81$$

and

$$h(\gamma_2) = \frac{\log \alpha}{2} < 0.25, h(\gamma_3) = \log 2 < 0.7$$

by (7) and (9), we can take $A_1 := 1.62$, $A_2 := 0.5$, and $A_3 := 1.4$. Also, since $k < 4n + 6$, we can take $B := \max\{1, |k|, |-(n+m)|\} = 4n + 6$. Taking into account the inequality (12) and using Theorem 1, we obtain

$$\frac{2.1}{2^m} > |\Lambda_1| > \exp\left(C(1 + \log(4n+6))(1.62)(0.5)(1.4)\right),$$

where $C = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2)$. By a simple computation, it follows that

$$m \log 2 < 1.1 \cdot 10^{12}(1 + \log(4n+6)) + \log(2.1). \tag{13}$$

Now, rearranging the equation $F_k = j_m j_n$ as

$$\frac{\alpha^k}{j_m\sqrt{5}} - 2^n = \frac{\beta^k}{j_m\sqrt{5}} + (-1)^n,$$

and taking absolute values, we obtain

$$\left|\frac{\alpha^k}{j_m\sqrt{5}} - 2^n\right| \leq \frac{|\beta|^k}{j_m\sqrt{5}} + 1.$$

Dividing both sides of the above inequality by 2^n, it is seen that

$$\left|\frac{2^{-n} \cdot \alpha^k}{j_m\sqrt{5}} - 1\right| < \frac{|\beta|^k}{2^n j_m\sqrt{5}} + \frac{1}{2^n} < \frac{1.1}{2^n}, \tag{14}$$

where we used the fact that

$$\left(\frac{|\beta|^k}{j_m\sqrt{5}} + 1\right) < 1.1$$

for $k > 69$. Let $\gamma_1 := \alpha$, $\gamma_2 := 2$, $\gamma_3 := j_m\sqrt{5}$, and $b_1 := k$, $b_2 := -n$, $b_3 := -1$. Then, we can apply Theorem 1. The numbers γ_1, γ_2, and γ_3 are positive real numbers and elements of the field $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ and so $D = 2$. In a similar manner, one can verify that

$$\Lambda_2 = \alpha^k 2^{-n}/j_m\sqrt{5} - 1 \neq 0.$$

Since $h(\gamma_1) = \dfrac{\log \alpha}{2} < 0.25$ and $h(\gamma_2) = \log 2 < 0.7$ by (7), we can take $A_1 := 0.5$ and $A_2 := 1.4$. Using the properties (7), (9), and (10), it is seen that

$$\begin{aligned} h(\gamma_3) = h(j_m\sqrt{5}) &\leq h(j_m) + h(\sqrt{5}) \\ &\leq \log 2^{m+1} + \frac{\log 5}{2} < 1.5 + m \log 2, \end{aligned}$$

by (4). So we can take $A_3 := 3+2m\log 2$. Since $k < 4n+6$, it follows that $B := 4n+6 > \max\{|k|, |-n|, |-1|\}$. Thus, taking into account the inequality (14) and using Theorem 1, we obtain

$$\frac{1.1}{2^n} > |\Lambda_2| > \exp\left(C(1 + \log(4n+6))(0.5)(1.4)(3 + 2m\log 2)\right),$$

or

$$n\log 2-\log 1.1<6.79\cdot 10^{11}(1+\log(4n+6))(3+2m\log 2), \tag{15}$$

where $C=-1.4\cdot 30^6\cdot 3^{4.5}\cdot 2^2\cdot(1+\log 2)$. Inserting the inequality (13) into the last inequality, a computer search with Mathematica gives us that $n<9.63\cdot 10^{27}$.

Now, let us try to reduce the upper bound on n by applying Lemma 1. Let

$$z_1:=k\log\alpha-(n+m)\log 2+\log(1/\sqrt{5}).$$

Then

$$|e^{z_1}-1|<\frac{2.1}{2^m}$$

by (12). As $m\geq 2$, we get $|e^{z_1}-1|<\frac{2.1}{2^m}<3/4$ and therefore it is seen that

$$|z_1|<\frac{4.2}{2^m}$$

by Lemma 2. That is,

$$0<\left|k\log\alpha-(n+m)\log 2+\log(1/\sqrt{5})\right|<\frac{4.2}{2^m}.$$

Dividing this inequality by $\log 2$, we get

$$0<\left|k\left(\frac{\log\alpha}{\log 2}\right)-(n+m)+\left(\frac{\log(1/\sqrt{5})}{\log 2}\right)\right|<6.06\cdot 2^{-m}. \tag{16}$$

Let $\gamma:=\dfrac{\log\alpha}{\log 2}\notin\mathbb{Q}$ and $M:=3.86\cdot 10^{28}$. Then we found that $M>4n+6>k$ and q_{67}, the denominator of the 67 th convergent of γ exceeds $6M$. Now take

$$\mu:=\frac{\log(1/\sqrt{5})}{\log 2}.$$

In this case, a quick computation with Mathematica gives us the inequality

$$\epsilon=||\mu q_{67}||-M||\gamma q_{67}||>0.11.$$

Let $A:=6.06,\ B:=2$, and $w:=m$ in Lemma 1. Thus, with the help of Mathematica, we can say that the inequality (16) has no solution for

$$m\geq 103.5>\frac{\log(Aq_{67}/\epsilon)}{\log B}.$$

So $m\leq 103$. Substituting this upper bound for m into (15), we obtain $n<5.52\cdot 10^{15}$.

Now, let

$$z_2:=k\log\alpha-n\log 2+\log\left(\frac{1}{\sqrt{5}j_m}\right).$$

Then

$$|e^{z_2}-1|<\frac{1.1}{2^n}$$

by (14). Since $(1.1)/2^n<3/4$ for $n\geq 1$, by Lemma 2, we get

$$|z_2|<\frac{2.2}{2^n}.$$

That is,

$$0<\left|k\log\alpha-n\log 2+\log\left(\frac{1}{\sqrt{5}j_m}\right)\right|<\frac{2.2}{2^n}.$$

Dividing both sides of the above inequality by $\log 2$, we get

$$0 < \left| k\left(\frac{\log\alpha}{\log 2}\right) - n + \frac{\log\left(\frac{1}{\sqrt{5}j_m}\right)}{\log 2} \right| < 3.1 \cdot 2^{-n}. \tag{17}$$

Putting $\gamma := \dfrac{\log\alpha}{\log 2}$ and taking $M := 2.21 \cdot 10^{16}$, we found that $M > 4n + 6 > k$ and q_{41}, the denominator of the 41 st convergent of γ exceeds $6M$. Taking

$$\mu := \frac{\log\left(\frac{1}{\sqrt{5}j_m}\right)}{\log 2}$$

and considering the fact that $2 \le m \le 103$ by (15), a quick computation with Mathematica gives us the inequality

$$\epsilon = ||\mu q_{41}|| - M||\gamma q_{41}|| > 0.01.$$

Let $A := 3.1$, $B := 2$, and $w := n$ in Lemma 1. Thus, with the help of Mathematica, we can say that the inequality (17) has no solution for

$$n \ge 68.23 > \frac{\log(Aq_{41}/\epsilon)}{\log B}.$$

Therefore $n \le 68$. This contradicts our assumption that $n \ge 70$. Now let us consider the case $m = 0$ and $m = 1$ for $n \ge 70$. If we repeat the argument following (11), we find that $n \le 54$, a contradiction. This completes the proof of the theorem. ■

Theorem 3 *The Diophantine equation $L_k = j_m j_n$ has only the solutions*

$$(k, m, n) = (3, 0, 0),\ (0, 0, 1),\ (1, 1, 1),\ (4, 1, 3)$$

in nonnegative integers.

Proof. Assume that $L_k = j_m j_n = (2^m + (-1)^m) \cdot (2^n + (-1)^n)$ for some nonnegative integers k, m, n. Now assume that $0 \le m \le n \le 99$. Then, by using the Mathematica program, we see that $k \le 285$. In this case, with the help of Mathematica program, we obtain

$$\begin{aligned}4 &= L_3 = j_0 j_0 = 2 \cdot 2,\\ 2 &= L_0 = j_0 j_1 = 2 \cdot 1,\\ 1 &= L_1 = j_1 j_1 = 1 \cdot 1,\\ 7 &= L_4 = j_1 j_3 = 1 \cdot 7,\end{aligned}$$

in the range $0 \le m \le n \le 99$. From now on, we assume that $n \ge 100$. Since

$$\alpha^{k-1} \le L_k = j_m j_n \le 2^{2n+2} < \alpha^{4n+4}$$

by (2) and (4), it is seen that $k < 4n + 5$. On the other hand, the inequality

$$2^{n-1} \le j_m j_n = L_k \le 2\alpha^k < 2^{k+1},$$

implies that $k \ge n - 1 \ge 99$. Since $L_k = j_m j_n$, we get

$$\alpha^k + \beta^k = L_k = j_m j_n = 2^{n+m} + (-1)^{n+m} + 2^n (-1)^m + 2^m (-1)^n \tag{18}$$

or

$$\alpha^k - 2^{n+m} = -\beta^k + (-1)^{n+m} + 2^n (-1)^m + 2^m (-1)^n .$$

Taking absolute values, it is seen that

$$\left|\alpha^k - 2^{n+m}\right| \le |\beta|^k + 2^n + 2^m + 1.$$

Dividing both sides of this inequality by 2^{n+m}, we obtain

$$\left|\alpha^k \cdot 2^{-(n+m)} - 1\right| \le \frac{|\beta|^k}{2^{n+m}} + \frac{1}{2^m} + \frac{1}{2^n} + \frac{1}{2^{n+m}} < \frac{2.1}{2^m}, \tag{19}$$

where we used the fact that

$$\left(\frac{1+|\beta|^k}{2^n} + 2\right) \le 2.1$$

for $k \ge 99$ and $n \ge 100$. Let $\gamma_1 := \alpha$, $\gamma_2 := 2$, $b_1 := k$, $b_2 := -(n+m)$. The numbers γ_1, γ_2 are real numbers and elements of the field $\mathbb{K} = \mathbb{Q}(\sqrt{5})$. So $D = 2$. Now we show that

$$\Lambda_3 = \alpha^k \cdot 2^{-(n+m)} - 1$$

is nonzero. If it were, then $\alpha^k = 2^{n+m}$, which is impossible for any $k > 0$. It can be seen that

$$h(\gamma_1) = h(\alpha) < 0.25, \ h(\gamma_2) = h(2) < 0.7.$$

Thus, we can take $A_1 := 0.5, A_2 := 1.4$, and $B := 4n + 5 \ge \max\{|k|, |-(n+m)|\}$. Therefore, taking into account the inequality (19) and using Theorem 1, we obtain

$$\frac{2.1}{2^m} > |\Lambda_3| > \exp\left((C \cdot (1 + \log(4n+5)) \cdot (0.5) \cdot (1.4)\right),$$

and so

$$m \log 2 < 3649404749 \cdot (1 + \log(4n+5)) + \log(2.1), \tag{20}$$

where $C = -1.4 \cdot 30^5 \cdot 2^{4.5} \cdot 2^2 \cdot (1 + \log 2)$. Now, writing the equation $L_k = j_m j_n$ as

$$\frac{\alpha^k}{j_m} - 2^n = -\frac{\beta^k}{j_m} + (-1)^n,$$

and taking absolute values, we get

$$\left|\frac{\alpha^k}{j_m} - 2^n\right| \le \frac{|\beta|^k}{j_m} + 1.$$

Dividing both sides of this inequality by 2^n, we obtain

$$\left|\frac{\alpha^k \cdot 2^{-n}}{j_m} - 1\right| < \frac{|\beta|^k}{2^n \cdot j_m} + \frac{1}{2^n} < \frac{1.1}{2^n}, \tag{21}$$

where we used the fact that

$$\left(\frac{|\beta|^k}{j_m} + 1\right) \le 1.1$$

for $k \ge 99$. Take $\gamma_1 := \alpha$, $\gamma_2 := 2$, $\gamma_3 := j_m$, $b_1 := k$, $b_2 := -n$, $b_3 := -1$. Clearly, the numbers γ_1, γ_2, γ_3 are real numbers and elements of the field $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ and so $D = 2$. It can be seen that

$$\Lambda_4 = \frac{\alpha^k \cdot 2^{-n}}{j_m} - 1$$

is nonzero. On the other hand,

$$h(\gamma_1) = h(\alpha) < 0.25, h(\gamma_2) = h(2) < 0.7,$$

$$h(\gamma_3) = \log j_m \leq \log 2^{m+1} < 0.7 + m \log 2.$$

We can take $A_1 := 0.5$, $A_2 := 1.4$ and $A_3 := 1.4 + 2m \log 2$. Since $k < 4n + 5$, we can take $B := 4n + 5 \geq \max\{k, |-n|, |-1|\}$. Using the inequality (21) and Theorem 1, we get

$$\frac{1.1}{2^n} > |\Lambda_4| > \exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1 + \log 2)(1 + \log(4n + 5)) \cdot (0.5) \cdot (1.4) \cdot (1.4 + 2m \log 2)\right)$$

or

$$n \log 2 - \log 1.1 < 6.79 \times 10^{11} \cdot (1 + \log(4n + 5)) \cdot (1.4 + 2m \log 2). \tag{22}$$

Inserting the inequality (20) into the last inequality, a computer search with Mathematica gives us that $n < 2.66 \cdot 10^{25}$. Now we reduce this bound to a size that can be easily dealt. In order to do this, we use Lemma 1 again. Let

$$z_3 = k \log \alpha - (n + m) \log 2.$$

Then from the inequality (19), it follows that

$$|e^{z_3} - 1| < \frac{2.1}{2^m}.$$

Assume that $m \geq 2$. Then $|e^{z_3} - 1| < 3/4$ and this implies that

$$|z_3| < \frac{4.2}{2^m}$$

by Lemma 2. That is,

$$|k \log \alpha - (n + m) \log 2| < \frac{4.2}{2^m}.$$

If we divide this inequality by $(n + m) \log \alpha$, we get

$$0 < \left| \frac{k}{n + m} - \frac{\log 2}{\log \alpha} \right| < \frac{8.73}{(n + m) \cdot 2^m}. \tag{23}$$

Assume that $m \geq 97$. Then it can be seen that

$$\frac{2^m}{17.46} > 9.07 \cdot 10^{27} > 2n \geq n + m,$$

and so we have

$$\left| \frac{k}{n + m} - \frac{\log 2}{\log \alpha} \right| < \frac{8.73}{(n + m) \cdot 2^m} < \frac{1}{2 \cdot (n + m)^2}.$$

From the known properties of continued fraction, it can be seen that the rational number $\frac{k}{n+m}$ is a convergent to $\gamma = \frac{\log 2}{\log \alpha}$. Now let $\dfrac{p_r}{q_r}$ be r-th convergent of the continued fraction of γ. Assume that $\frac{k}{n+m} = \dfrac{p_t}{q_t}$ for some t. Then we have $q_{57} > 7 \cdot 10^{25} > 2n \geq n + m$. Thus $t \in \{0, 1, 2, ..., 56\}$. Furthermore, $a_M = \max\{a_i | i = 0, 1, 2, ..., 56\} = 134$. Again, from the known properties of continued fraction, we get

$$\left| \gamma - \frac{p_t}{q_t} \right| > \frac{1}{(a_M + 2)(n + m)^2} \geq \frac{1}{136 \cdot (n + m)^2},$$

Thus, from (23), we obtain

$$\frac{8.73}{(n + m) \cdot 2^m} > \frac{1}{136 \cdot (n + m)^2}.$$

This shows that

$$\frac{5.51}{10^{29}} > \frac{8.73}{2^m} > \frac{1}{136 \cdot (n+m)} > \frac{1}{9.52 \cdot 10^{27}},$$

a contradiction. Therefore $m \leq 96$. Substituting this value of m into (22), we get $n < 5.08 \cdot 10^{15}$. Now, let

$$z_4 := k \log \alpha - n \log 2 + \log(1/j_m).$$

Then, from (21), we can write

$$|e^{z_4} - 1| < \frac{1.1}{2^n}.$$

Since $(1.1)/2^n < 3/4$ for $n \geq 1$, we get

$$|z_4| < \frac{2.2}{2^n}$$

by Lemma 2. That is,

$$|k \log \alpha - n \log 2 + \log(1/j_m)| < \frac{2.2}{2^n}.$$

Dividing both sides of this inequality by $\log 2$, we get

$$0 < \left| k \frac{\log \alpha}{\log 2} - n + \frac{\log(1/j_m)}{\log 2} \right| < 3.18 \cdot 2^{-n}. \tag{24}$$

Now, we apply Lemma 1. Let $\gamma := \log \alpha / \log 2$, $\mu := \log(1/j_m)/\log 2$, $A := 3.18$, $B := 2$, $w := n$, and $M := 2.1 \cdot 10^{16}$. It is seen that $M > 4n+5 > k$ and q_{53}, the denominator of the 53 rd convergent of γ exceeds $6M$. In this case, a quick computation with Mathematica gives us the inequality

$$\epsilon := ||\mu q_{53}|| - M||\gamma q_{53}|| > 0.$$

Thus, with the help of Mathematica, we can say that the inequality (24) has no solution for

$$n \geq 97.16 > \frac{\log(A q_{53}/\epsilon)}{\log B}.$$

Therefore, $n \leq 97$. But this contradicts the assumption that $n \geq 100$. Now let us consider the case $m = 0$ and $m = 1$ for $n \geq 100$. If we repeat the argument following (18), we find that $n \leq 13$ and $n \leq 12$, respectively, a contradiction. This completes the proof of the theorem. ■

Corollary 1 *The largest Fibonacci number and Lucas number which can be written in the form*

$$(2^m + (-1)^m) \cdot (2^n + (-1)^n)$$

are $F_9 = 34 = 2 \cdot 17 = (2^0 + (-1)^0) \cdot (2^4 + (-1)^4)$ *and* $L_4 = 7 = 1 \cdot 7 = (2^1 + (-1)^1) \cdot (2^3 + (-1)^3)$, *respectively.*

Corollary 2 *The largest Fibonacci number and Lucas number which can be written in the form*

$$2^n + (-1)^n$$

are $F_5 = 5 = 2^2 + (-1)^2$ *and* $L_4 = 7 = 2^3 + (-1)^3$, *respectively.*

Corollary 3 *The only Fermat numbers in the Fibonacci sequence are* $F_3 = 3$ *and* $F_5 = 5$ *and the only Fermat number in the Lucas sequence is* $L_2 = 3$.

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