

# Double Wijsman Strongly Deferred Cesàro Equivalence\*

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## Abstract

In this paper, we firstly presented the definitions of asymptotical Wijsman deferred Cesàro equivalence, Wijsman strongly deferred Cesàro equivalence and Wijsman strongly  $r$ -deferred Cesàro equivalence ( $0 < r < \infty$ ) for double sequences of sets. Then, we proved some theorems associated with the concept of  $W_2^\lambda[D]^r$ -equivalence, and we examined the relations between the concepts of  $W_2^\lambda[D]^r$ -equivalence and  $W_2^\lambda DS$ -equivalence for double sequences of sets.

## 1 Introduction

Long after Pringsheim [36] introduced the concept of convergence for double sequences, this concept was extended to the concept of statistical convergence by Mursaleen and Edely [26] and also to the concept of lacunary statistical convergence by Patterson and Savaş [34]. Furthermore, the concept of asymptotical equivalence for double sequences was firstly studied by Patterson [33].

The concept of Wijsman convergence which was discussed in this paper is one of the concepts of several convergence for sequences of sets (see, [4, 5, 6, 27, 38, 43, 44]). The concept of Wijsman convergence for double sequences of sets was firstly introduced by Nuray et al. [28]. Also, the concepts of Wijsman Cesàro summability, Wijsman statistical convergence and Wijsman lacunary statistical convergence for double sequences of sets were studied by Nuray et al. in [28], [31] and [29], respectively. Furthermore, some basic asymptotical equivalence definitions in the Wijsman sense for double sequences of sets were given by Nuray et al. [30] with examples.

Long after Agnew [1] introduced the concept of deferred Cesàro mean for real (or complex) sequences, the concept of deferred statistical convergence was studied by Küçükaslan and Yılmaztürk [23]. Also, the concepts of deferred Cesàro summability and deferred statistical convergence for double sequences was presented by Dağadur and Sezgek in [7] and [37]. Furthermore, some basic asymptotical deferred equivalence definitions as asymptotical deferred equivalence, strongly  $r$ -deferred Cesàro equivalence and deferred statistical equivalence for sequences were given by Koşar et al. [22].

In [2], Altınok et al. introduced the concepts of Wijsman deferred Cesàro summability and Wijsman deferred statistical convergence for sequences of sets. Also, the definitions of asymptotical deferred equivalence and deferred statistical equivalence in the Wijsman sense for sequences of sets were given by Altınok et al. [3]. Recently, Ulusu and Gülle [42] studied on the concepts of deferred Cesàro summability and deferred statistical convergence in the Wijsman sense for double sequences of sets.

The intent of this paper is to study on some new asymptotical deferred equivalence types in the Wijsman sense for double sequences of sets, which are more comprehensive than the concepts given in [30].

More information on the concepts of convergence and asymptotical equivalence in this paper can be found in [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 24, 25, 32, 35, 39, 40, 41].

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## 2 Basic Concepts

Let's start by remembering some basic definitions and notations (See, [16, 28, 30, 31, 33, 34, 42]). For a metric space  $(\mathcal{Y}, \rho)$ ,  $\mu(y, U)$  denote the distance from  $y$  to  $U$  where

$$\mu(y, U) = \inf_{u \in U} \rho(y, u) := \mu_y(U)$$

for any  $y \in \mathcal{Y}$  and any non-empty  $U \subseteq \mathcal{Y}$ .

For a non-empty set  $\mathcal{Y}$ , let a function  $f : \mathbb{N} \rightarrow 2^{\mathcal{Y}}$  (the power set of  $\mathcal{Y}$ ) be defined by  $f(k) = U_k \in 2^{\mathcal{Y}}$  for each  $k \in \mathbb{N}$ . Then, the sequence  $\{U_k\} = \{U_1, U_2, \dots\}$ , which is the codomain elements of  $f$ , is called sequences of sets.

Throughout the study,  $(\mathcal{Y}, \rho)$  will be considered as a separable metric space and  $U, U_{kj}$  ( $k, j \in \mathbb{N}$ ) will be considered as any non-empty closed subsets of  $\mathcal{Y}$ . The double sequence of sets  $\{U_{kj}\}$  is said to be

(i) Wijsman convergent to the set  $U$  provided that

$$\lim_{k,j \rightarrow \infty} \mu_y(U_{kj}) = \mu_y(U),$$

for each  $y \in \mathcal{Y}$  and it is denoted by  $U_{kj} \xrightarrow{W_2} U$ ;

(ii) Wijsman strongly Cesàro summable to the set  $U$  provided that

$$\lim_{m,n \rightarrow \infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |\mu_y(U_{kj}) - \mu_y(U)| = 0,$$

for each  $y \in \mathcal{Y}$  and it is denoted by  $U_{kj} \xrightarrow{W_2[C]} U$ ;

(iii) Wijsman statistically convergent to the set  $U$  provided that for every  $\varepsilon > 0$

$$\lim_{m,n \rightarrow \infty} \frac{1}{mn} \left| \left\{ (k, j) : k \leq m, j \leq n : |\mu_y(U_{kj}) - \mu_y(U)| \geq \varepsilon \right\} \right| = 0,$$

for each  $y \in \mathcal{Y}$  and and it is denoted by  $U_{kj} \xrightarrow{W_2(S)} U$ .

The deferred Cesàro mean  $D_{\psi, \varphi}$  of a double sequence  $\mathcal{U} = \{U_{kj}\}$  is defined by

$$(D_{\psi, \varphi} \mathcal{U})_{mn} = \frac{1}{\psi(m)\varphi(n)} \sum_{k=p(m)+1}^{q(m)} \sum_{j=s(n)+1}^{t(n)} \mu_y(U_{kj}) := \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m), t(n)} \mu_y(U_{kj}),$$

where  $[p(m)], [q(m)], [s(n)]$  and  $[t(n)]$  are sequences of non-negative integers satisfying following conditions:

$$p(m) < q(m), \lim_{m \rightarrow \infty} q(m) = \infty; \quad s(n) < t(n), \lim_{n \rightarrow \infty} t(n) = \infty \tag{1}$$

and

$$q(m) - p(m) = \psi(m); \quad t(n) - s(n) = \varphi(n). \tag{2}$$

Note here that the method  $D_{\psi, \varphi}$  is openly regular for any selection of the sequences  $[p(m)], [q(m)], [s(n)]$  and  $[t(n)]$ . Throughout the paper, unless otherwise specified,  $[p(m)], [q(m)], [s(n)]$  and  $[t(n)]$  are considered as sequences of non-negative integers satisfying (1) and (2).

The double sequences of sets  $\{U_{kj}\}$  is said to be

(i) Wijsman deferred Cesàro summable to the set  $U$  provided that

$$\lim_{m,n \rightarrow \infty} \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \mu_y(U_{kj}) = \mu_y(U),$$

for each  $y \in \mathcal{Y}$  and it is denoted by  $U_{kj} \xrightarrow{W_2D} U$ ;

(ii) Wijsman strongly  $r$ -deferred Cesàro summable ( $0 < r < \infty$ ) to the set  $U$  provided that

$$\lim_{m,n \rightarrow \infty} \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} |\mu_y(U_{kj}) - \mu_y(U)|^r = 0,$$

for each  $y \in \mathcal{Y}$  and it is denoted by  $U_{kj} \xrightarrow{W_2[D]^r} U$ ; where if

If  $r = 1$ , then the sequence is simply called Wijsman strongly deferred Cesàro summable to the set  $U$  and it is denoted by  $U_{kj} \xrightarrow{W_2[D]} U$ ; and

(iii) Wijsman deferred statistically convergent to the set  $U$  provided that for every  $\varepsilon > 0$

$$\lim_{m,n \rightarrow \infty} \frac{1}{\psi(m)\varphi(n)} \left| \left\{ (k, j) : p(m) < k \leq q(m), s(n) < j \leq t(n), |\mu_y(U_{kj}) - \mu_y(U)| \geq \varepsilon \right\} \right| = 0$$

and it is denoted by  $U_{kj} \xrightarrow{W_2DS} U$ .

The non-negative double sequences  $a = [a_{kj}]$  and  $b = [b_{kj}]$  are said to be asymptotically equivalent to multiple  $\lambda$  provided that

$$P - \lim_{k,j \rightarrow \infty} \frac{a_{kj}}{b_{kj}} = \lambda.$$

It is denoted by  $a \overset{\lambda}{\sim} b$ .

For any non-empty closed subsets  $\{U_{kj}\}, \{V_{kj}\} \in \mathcal{Y}$  such that  $\mu_y(U_{kj}) > 0$  and  $\mu_y(V_{kj}) > 0$  for each  $y \in \mathcal{Y}$ , the double sequences  $\{U_{kj}\}$  and  $\{V_{kj}\}$  are said to be Wijsman asymptotically equivalent to multiple  $\lambda$  provided that

$$\lim_{k,j \rightarrow \infty} \frac{\mu_y(U_{kj})}{\mu_y(V_{kj})} := \lim_{k,j \rightarrow \infty} \mu_y\left(\frac{U_{kj}}{V_{kj}}\right) = \lambda,$$

for each  $y \in \mathcal{Y}$  and it is denoted by  $U_{kj} \overset{W_2^\lambda}{\sim} V_{kj}$ .

As an example to this concept, the following sequences of circles in  $\mathbb{R}^2$  can be given. Let  $\mathcal{Y} = \mathbb{R}^2$  and double sequences  $\{U_{kj}\}$  and  $\{V_{kj}\}$  be defined

$$U_{kj} := \{(u, v) : u^2 + v^2 - 2kju = 0\},$$

$$V_{kj} := \{(u, v) : u^2 + v^2 + 2kju = 0\}.$$

Then, we have

$$\lim_{k,j \rightarrow \infty} \mu_y\left(\frac{U_{kj}}{V_{kj}}\right) = 1$$

for each  $y \in \mathcal{Y}$ , i.e.  $U_{kj} \overset{W_2^\lambda}{\sim} V_{kj}$ .

**Definition 1** ([16]) For any non-empty closed subsets  $\{U_{kj}\}, \{V_{kj}\} \in \mathcal{Y}$  such that  $\mu_y(U_{kj}) > 0$  and  $\mu_y(V_{kj}) > 0$  for each  $y \in \mathcal{Y}$ , the double sequences  $\{U_{kj}\}$  and  $\{V_{kj}\}$  are said to be asymptotically Wijsman deferred statistical equivalent to multiple  $\lambda$  provided that for every  $\varepsilon > 0$ ,

$$\lim_{m,n \rightarrow \infty} \frac{1}{\psi(m)\varphi(n)} \left| \left\{ (k, j) : p(m) < k \leq q(m), s(n) < j \leq t(n), \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right| \geq \varepsilon \right\} \right| = 0,$$

for each  $y \in \mathcal{Y}$  and it is denoted by  $U_{kj} \overset{W_2^\lambda DS}{\sim} V_{kj}$ .

The class of all double sequences of sets that asymptotically Wijsman deferred statistical equivalent is denoted by  $\{W_2^\lambda DS\}$ .

A double sequence  $\theta_2 = \{(k_m, j_n)\}$  is said to be double lacunary sequence if there exist two increasing sequences  $(k_m)$  and  $(j_n)$  of integers such that

$$k_0 = 0, h_m = k_m - k_{m-1} \rightarrow \infty \text{ and } j_0 = 0, \bar{h}_n = j_n - j_{n-1} \rightarrow \infty \text{ as } m, n \rightarrow \infty.$$

### 3 New Definitions

In this section, we give the definitions of asymptotically Wijsman deferred Cesàro equivalence, Wijsman strongly deferred Cesàro equivalence and Wijsman strongly  $r$ -deferred Cesàro equivalence for double sequences of sets.

In the following definitions, we will consider that  $\mu_y(U_{kj}) > 0$  and  $\mu_y(V_{kj}) > 0$ , for each  $y \in \mathcal{Y}$  and any non-empty closed subsets  $\{U_{kj}\}, \{V_{kj}\} \in \mathcal{Y}$ .

**Definition 2** The double sequences  $\{U_{kj}\}$  and  $\{V_{kj}\}$  are said to be asymptotically Wijsman deferred Cesàro equivalent to multiple  $\lambda$  provided that

$$\lim_{m,n \rightarrow \infty} \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) = \lambda,$$

for each  $y \in \mathcal{Y}$ . In this case, the notation  $U_{kj} \overset{W_2^\lambda D}{\sim} V_{kj}$  is used and simply call these sequences asymptotically Wijsman deferred Cesàro equivalent if  $\lambda = 1$ .

**Definition 3** The double sequences  $\{U_{kj}\}$  and  $\{V_{kj}\}$  are said to be asymptotically Wijsman strongly deferred Cesàro equivalent to multiple  $\lambda$  provided that

$$\lim_{m,n \rightarrow \infty} \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right| = 0,$$

for each  $y \in \mathcal{Y}$ . In this case, the notation  $U_{kj} \overset{W_2^\lambda [D]}{\sim} V_{kj}$  is used and simply call these sequences asymptotically Wijsman strongly deferred Cesàro equivalent if  $\lambda = 1$ .

As an example to this concept, we can give the following sequences of circles in  $\mathbb{R}^2$ .

**Example 1** Let  $\mathcal{Y} = \mathbb{R}^2$  and double sequences  $\{U_{kj}\}$  and  $\{V_{kj}\}$  be defined as following:

$$U_{kj} := \begin{cases} \left\{ (u, v) \in \mathbb{R}^2 : (u + 1)^2 + (v - 1)^2 = \frac{1}{kj} \right\} & \text{if } p(m) < k \leq q(m), s(n) < j \leq t(n) \\ & \text{and } k, j \text{ are square integers,} \\ \{(-e, -e)\} & \text{otherwise.} \end{cases}$$

and

$$V_{kj} := \begin{cases} \left\{ (u, v) \in \mathbb{R}^2 : (u - 1)^2 + (v + 1)^2 = \frac{1}{kj} \right\} & \text{if } p(m) < k \leq q(m), \quad s(n) < j \leq t(n) \\ & \text{and } k, j \text{ are square integers,} \\ \{(-e, -e)\} & \text{otherwise.} \end{cases}$$

Then, we have

$$\frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - 1 \right| \rightarrow 0, \quad (m, n \rightarrow \infty)$$

for each  $y \in \mathcal{Y}$  (i.e.  $U_{kj} \overset{W_2^\lambda[D]}{\sim} V_{kj}$ ,  $\lambda = 1$ ).

**Definition 4** The double sequences  $\{U_{kj}\}$  and  $\{V_{kj}\}$  are said to be asymptotically Wijsman strongly  $r$ -deferred Cesàro equivalent to multiple  $\lambda$  provided that

$$\lim_{m,n \rightarrow \infty} \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r = 0,$$

for each  $y \in \mathcal{Y}$ , where  $0 < r < \infty$ . In this case, the notation  $U_{kj} \overset{W_2^\lambda[D]^r}{\sim} V_{kj}$  is used and simply call these sequences asymptotically Wijsman strongly  $r$ -deferred Cesàro equivalent if  $\lambda = 1$ .

The class of all double sequences of sets that asymptotically Wijsman strongly  $r$ -deferred Cesàro equivalent will be denoted by  $\{W_2^\lambda[D]^r\}$ .

**Remark 1**

(i) For  $p(m) = 0, q(m) = m$  and  $s(n) = 0, t(n) = n$ , the concepts of asymptotical Wijsman deferred Cesàro equivalence, Wijsman strongly deferred Cesàro equivalence and Wijsman strongly  $r$ -deferred Cesàro equivalence coincides with the concepts of asymptotical Wijsman Cesàro equivalence, Wijsman strongly Cesàro equivalence and Wijsman strongly  $r$ -Cesàro equivalence ( $W_2^\lambda[C]^r$ ) for double sequences of sets in [30], respectively.

(ii) For  $p(m) = k_{m-1}, q(m) = k_m$  and  $s(n) = j_{n-1}, t(n) = j_n$  where  $\{(k_m, j_n)\}$  is a double lacunary sequence, the concepts of asymptotical Wijsman deferred Cesàro equivalence, Wijsman strongly deferred Cesàro equivalence and Wijsman strongly  $r$ -deferred Cesàro equivalence coincides with the concepts of asymptotical Wijsman lacunary equivalence, Wijsman strongly lacunary equivalence and Wijsman strongly  $r$ -lacunary equivalence for double sequences of sets in [30], respectively.

### 4 Main Theorems

In this section, we firstly prove some theorems associated with the concept of  $W_2^\lambda[D]^r$ -equivalence for double sequences of sets. Let  $\{T_{kj}\}, \{U_{kj}\}$  and  $\{V_{kj}\}$  be double sequences of non-empty closed subsets of  $\mathcal{Y}$ .

**Theorem 1** Let  $T_{kj} \subseteq V_{kj}$  for all  $k, j \in \mathbb{N}$ , then

$$U_{kj} \overset{W_2^\lambda[D]^r}{\sim} V_{kj} \Rightarrow U_{kj} \overset{W_2^\lambda[D]^r}{\sim} T_{kj}.$$

**Proof.** Assume that  $T_{kj} \subseteq V_{kj}$  for all  $k, j \in \mathbb{N}$  and  $U_{kj} \overset{W_2^\lambda[D]^r}{\sim} V_{kj}$ . For all  $k, j \in \mathbb{N}$ ,

$$\begin{aligned} T_{kj} \subseteq V_{kj} &\Rightarrow \mu_y(V_{kj}) \leq \mu_y(T_{kj}) \\ &\Rightarrow \left| \mu_y\left(\frac{U_{kj}}{T_{kj}}\right) - \lambda \right| \leq \left| \mu_y\left(\frac{U_{kj}}{V_{kj}}\right) - \lambda \right| \end{aligned}$$

is hold for each  $y \in \mathcal{Y}$ . Thus, for  $0 < r < \infty$  we have

$$\frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y\left(\frac{U_{kj}}{T_{kj}}\right) - \lambda \right|^r \leq \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y\left(\frac{U_{kj}}{V_{kj}}\right) - \lambda \right|^r.$$

For  $m, n \rightarrow \infty$ , by our assumption, we get that  $U_{kj} \overset{W_2^\lambda[D]^r}{\sim} T_{kj}$ . ■

**Theorem 2** Let  $U_{kj} \subseteq T_{kj}$  for all  $k, j \in \mathbb{N}$ , then

$$U_{kj} \overset{W_2^\lambda[D]^r}{\sim} V_{kj} \Rightarrow T_{kj} \overset{W_2^\lambda[D]^r}{\sim} V_{kj}.$$

**Proof.** Assume that  $U_{kj} \subseteq T_{kj}$  for all  $k, j \in \mathbb{N}$  and  $U_{kj} \overset{W_2^\lambda[D]^r}{\sim} V_{kj}$ . For all  $k, j \in \mathbb{N}$ ,

$$\begin{aligned} U_{kj} \subseteq T_{kj} &\Rightarrow \mu_y(T_{kj}) \leq \mu_y(U_{kj}) \\ &\Rightarrow \left| \mu_y\left(\frac{T_{kj}}{V_{kj}}\right) - \lambda \right| \leq \left| \mu_y\left(\frac{U_{kj}}{V_{kj}}\right) - \lambda \right| \end{aligned}$$

is hold for each  $y \in \mathcal{Y}$ . Thus, for  $0 < r < \infty$  we have

$$\frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y\left(\frac{T_{kj}}{V_{kj}}\right) - \lambda \right|^r \leq \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y\left(\frac{U_{kj}}{V_{kj}}\right) - \lambda \right|^r.$$

For  $m, n \rightarrow \infty$ , by our assumption, we get that  $T_{kj} \overset{W_2^\lambda[D]^r}{\sim} V_{kj}$ . ■

Considering that  $V_{kj} \cap T_{kj} \subseteq V_{kj}$  and  $U_{kj} \subseteq U_{kj} \cup T_{kj}$  for all  $k, j \in \mathbb{N}$ , we give the following corollary.

**Corollary 1** Let  $U_{kj} \overset{W_2^\lambda[D]^r}{\sim} V_{kj}$ . Then

$$U_{kj} \overset{W_2^\lambda[D]^r}{\sim} V_{kj} \cap T_{kj} \quad \text{and} \quad U_{kj} \cup T_{kj} \overset{W_2^\lambda[D]^r}{\sim} V_{kj}.$$

In the following theorem, we will give the relation between the concepts of  $W_2^\lambda[D]^r$ -equivalence and  $W_2^\lambda[C]^r$ -equivalence for double sequences of sets.

**Theorem 3** Let  $\left(\frac{p(m)}{\psi(m)}\right)$  and  $\left(\frac{s(n)}{\varphi(n)}\right)$  are bounded. Then, double sequences  $\{U_{kj}\}$  and  $\{V_{kj}\}$  are  $W_2^\lambda[D]^r$ -equivalent if these sequences are  $W_2^\lambda[C]^r$ -equivalent.

**Proof.** Assume that  $\left(\frac{p(m)}{\psi(m)}\right)$  and  $\left(\frac{s(n)}{\varphi(n)}\right)$  are bounded, and  $U_{kj} \stackrel{W_2^\lambda[C]^r}{\sim} V_{kj}$ . Here, we can write

$$\begin{aligned} \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r &= \frac{1}{\psi(m)\varphi(n)} \left[ \sum_{k,j=1,1}^{q(m),t(n)} - \sum_{k,j=1,1}^{p(m),t(n)} - \sum_{k,j=1,1}^{q(m),s(n)} + \sum_{k,j=1,1}^{p(m),s(n)} \right] \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \\ &= \frac{q(m)t(n)}{\psi(m)\varphi(n)} \left( \frac{1}{q(m)t(n)} \sum_{k,j=1,1}^{q(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \right) \\ &\quad - \frac{p(m)t(n)}{\psi(m)\varphi(n)} \left( \frac{1}{p(m)t(n)} \sum_{k,j=1,1}^{p(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \right) \\ &\quad - \frac{q(m)s(n)}{\psi(m)\varphi(n)} \left( \frac{1}{q(m)s(n)} \sum_{k,j=1,1}^{q(m),s(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \right) \\ &\quad + \frac{p(m)s(n)}{\psi(m)\varphi(n)} \left( \frac{1}{p(m)s(n)} \sum_{k,j=1,1}^{p(m),s(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \right), \end{aligned}$$

for each  $y \in \mathcal{Y}$ . From here, it can be clearly seen that the asymptotical Wijsman strongly  $r$ -deferred Cesàro equivalence of the sequences  $\{U_{kj}\}$  and  $\{V_{kj}\}$  is equal to the linear combination of the asymptotically Wijsman strongly  $r$ -Cesàro equivalence of these sequences. Here, this linear combination can be considered as a matrix transformation. For this matrix transformation to be regular (which is desirable), the sequence

$$\left\{ \frac{(p(m) + q(m))(s(n) + t(n))}{\psi(m)\varphi(n)} \right\}$$

must be bounded that similar situation was shown in [37]. Thus, the proof is completed. ■

Now, we will give some relations between the concepts of  $W_2^\lambda[D]^r$ -equivalence and  $W_2^\lambda DS$ -equivalence for double sequences of sets.

**Theorem 4** *If double sequences  $\{U_{kj}\}$  and  $\{V_{kj}\}$  are  $W_2^\lambda[D]^r$ -equivalent, then these sequences are  $W_2^\lambda DS$ -equivalent.*

**Proof.** Assume that  $U_{kj} \stackrel{W_2^\lambda[D]^r}{\sim} V_{kj}$  and  $0 < r < \infty$ . For every  $\varepsilon > 0$ , we can write

$$\begin{aligned} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r &\geq \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \\ &\quad \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|_{\geq \varepsilon} \\ &\geq \varepsilon^r \left| \left\{ (k, j) : p(m) < k \leq q(m), s(n) < j \leq t(n), \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right| \geq \varepsilon \right\} \right|, \end{aligned}$$

for each  $y \in \mathcal{Y}$  and so we have

$$\begin{aligned} & \frac{1}{\varepsilon^r} \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \\ & \geq \frac{1}{\psi(m)\varphi(n)} \left| \left\{ (k, j) : p(m) < k \leq q(m), s(n) < j \leq t(n), \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right| \geq \varepsilon \right\} \right|. \end{aligned}$$

For  $m, n \rightarrow \infty$ , by our assumption, we get that  $U_{kj} \overset{W_2^\lambda DS}{\sim} V_{kj}$ . ■

The sequence  $\{U_{kj}\}$  is said to be bounded if  $\sup_{k,j} \{\mu_y(U_{kj})\} < \infty$  for each  $y \in \mathcal{Y}$ . Also, we denote the space of all bounded double set sequences by  $L_\infty^2$ . The converse of Theorem 4 is provided when the sequences  $\{U_{kj}\}$  and  $\{V_{kj}\}$  are bounded. Otherwise it is not provided. We can explain this situation with the following example:

**Example 2** Let  $\mathcal{Y} = \mathbb{R}^2$  and double sequences  $\{U_{kj}\}$  and  $\{V_{kj}\}$  be defined as following:

$$U_{kj} := \begin{cases} \{(u, v) \in \mathbb{R}^2 : (u - k)^2 + (v - j)^2 = 1\} & \text{if } p(m) < k \leq q(m), s(n) < j \leq t(n) \\ & \text{and } k, j \text{ are square integers,} \\ \{(-\pi, \pi)\} & \text{otherwise,} \end{cases}$$

and

$$V_{kj} := \begin{cases} \{(u, v) \in \mathbb{R}^2 : (u + k)^2 + (v + j)^2 = 1\} & \text{if } p(m) < k \leq q(m), s(n) < j \leq t(n) \\ & \text{and } k, j \text{ are square integers,} \\ \{(-\pi, \pi)\} & \text{otherwise.} \end{cases}$$

It is obvious that both double sequences are not bounded. Also, for every  $\varepsilon > 0$  we have

$$\begin{aligned} & \frac{1}{\psi(m)\varphi(n)} \left| \left\{ (k, j) : p(m) < k \leq q(m), s(n) < j \leq t(n), \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - 1 \right| \geq \varepsilon \right\} \right| \\ & \leq \frac{\sqrt{\psi(m)\varphi(n)}}{\psi(m)\varphi(n)} \rightarrow 0, \quad (m, n \rightarrow \infty) \end{aligned}$$

for each  $y \in \mathcal{Y}$  (i.e.  $U_{kj} \overset{W_2^\lambda DS}{\sim} V_{kj}$ ,  $\lambda = 1$ ). On the other hand,

$$\frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - 1 \right| \not\rightarrow 0, \quad (m, n \rightarrow \infty)$$

for each  $y \in \mathcal{Y}$  (i.e.  $U_{kj} \overset{W_2^\lambda [D]}{\sim} V_{kj}$ ,  $\lambda = 1$ ).

**Theorem 5** If double sequences  $\{U_{kj}\}, \{V_{kj}\} \in L_\infty^2$  are  $W_2^\lambda DS$ -equivalent, then these sequences are  $W_2^\lambda [D]^r$ -equivalent.

**Proof.** Assume that  $\{U_{kj}\}, \{V_{kj}\} \in L_\infty^2$  and  $U_{kj} \overset{W_2^\lambda DS}{\sim} V_{kj}$ . Since  $\{U_{kj}\}, \{V_{kj}\} \in L_\infty^2$ , there is an  $\mathcal{M} > 0$  such that

$$\left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right| \leq \mathcal{M}$$



for all  $k, j \in \mathbb{N}$  and each  $y \in \mathcal{Y}$ . Thus, for every  $\varepsilon > 0$ , we have

$$\begin{aligned} & \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \\ &= \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(n),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \\ & \quad + \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \\ & \leq \frac{\mathcal{M}^r}{\psi(m)\varphi(n)} \left| \left\{ (k, j) : p(m) < k \leq q(m), s(n) < j \leq t(n), \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right| \geq \varepsilon \right\} \right| + \varepsilon^r \end{aligned}$$

for each  $y \in \mathcal{Y}$ . For  $m, n \rightarrow \infty$ , by our assumption, we get that  $U_{kj} \overset{W_2^\lambda[D]^r}{\sim} V_{kj}$ . ■

As a result, combining Theorem 4 and 5 yields the following corollary.

**Corollary 2**  $\{W_2^\lambda[D]^r\} \cap L_\infty^2 = \{W_2^\lambda DS\} \cap L_\infty^2$ .

The following theorems will be considered under the restrictions:

$$p(m) \leq p'(m) < q'(m) \leq q(m) \quad \text{and} \quad s(n) \leq s'(n) < t'(n) \leq t(n)$$

for all  $m, n \in \mathbb{N}$ , where all of these are sequences of non-negative integers.

**Theorem 6** Let  $\left( \frac{\psi'(m)\varphi'(n)}{\psi(m)\varphi(n)} \right) \rightarrow L \in \mathbb{R}$ . Then

$$U_{kj} \overset{W_2^\lambda[D]^r}{\sim}_{[\psi, \varphi]} V_{kj} \Rightarrow U_{kj} \overset{W_2^\lambda[D]^r}{\sim}_{[\psi', \varphi']} V_{kj}.$$

**Proof.** Assume that  $\left( \frac{\psi'(m)\varphi'(n)}{\psi(m)\varphi(n)} \right) \rightarrow L \in \mathbb{R}$  and  $U_{kj} \overset{W_2^\lambda[D]^r}{\sim}_{[\psi, \varphi]} V_{kj}$ . Here, we can write

$$\begin{aligned} \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m),t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r &\geq \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p'(m)+1 \\ j=s'(n)+1}}^{q'(m),t'(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \\ &\geq \frac{\psi'(m)\varphi'(n)}{\psi(m)\varphi(n)} \left( \frac{1}{\psi'(m)\varphi'(n)} \sum_{\substack{k=p'(m)+1 \\ j=s'(n)+1}}^{q'(m),t'(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \right) \end{aligned}$$

for each  $y \in \mathcal{Y}$ . Then, for  $m, n \rightarrow \infty$ , by our assumption, we get that  $U_{kj} \overset{W_2^\lambda[D]^r}{\sim}_{[\psi', \varphi']} V_{kj}$ . ■

**Theorem 7** Let the sets  $K_1 = \{k : p(m) < k \leq p'(m)\}$ ,  $K_2 = \{k : q'(m) < k \leq q(m)\}$ ,  $J_1 = \{j : s(n) < j \leq s'(n)\}$ ,  $J_2 = \{j : t'(n) < j \leq t(n)\}$  are finite for all  $m, n \in \mathbb{N}$  and  $\{U_{kj}\}, \{V_{kj}\} \in L^2_\infty$ . Then,

$$U_{kj} \overset{W_2^\lambda[D]^r_{[\psi', \varphi']}}{\sim} V_{kj} \Rightarrow U_{kj} \overset{W_2^\lambda[D]^r_{[\psi, \varphi]}}{\sim} V_{kj}.$$

**Proof.** Let the sets  $K_1, K_2, J_1, J_2$  are finite for all  $m, n \in \mathbb{N}$ . Assume that  $\{U_{kj}\}, \{V_{kj}\} \in L^2_\infty$  and  $U_{kj} \overset{W_2^\lambda[D]^r_{[\psi', \varphi']}}{\sim} V_{kj}$ . Since  $\{U_{kj}\}, \{V_{kj}\} \in L^2_\infty$ , there is an  $\mathcal{M} > 0$  such that

$$\left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right| \leq \mathcal{M}$$

for all  $k, j \in \mathbb{N}$  and each  $y \in \mathcal{Y}$ . Thus, we have

$$\begin{aligned} \frac{1}{\psi(m)\varphi(n)} \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{q(m), t(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r &= \frac{1}{\psi(m)\varphi(n)} \left( \sum_{\substack{k=p(m)+1 \\ j=s(n)+1}}^{p'(m), s'(n)} + \sum_{\substack{k=p(m)+1 \\ j=s'(n)+1}}^{p'(m), t'(n)} + \sum_{\substack{k=p(m)+1 \\ j=t'(n)+1}}^{p'(m), t(n)} \right) \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \\ &+ \frac{1}{\psi(m)\varphi(n)} \left( \sum_{\substack{k=p'(m)+1 \\ j=s(n)+1}}^{q'(m), s'(n)} + \sum_{\substack{k=p'(m)+1 \\ j=s'(n)+1}}^{q'(m), t'(n)} + \sum_{\substack{k=p'(m)+1 \\ j=t'(n)+1}}^{q'(m), t(n)} \right) \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \\ &+ \frac{1}{\psi(m)\varphi(n)} \left( \sum_{\substack{k=q'(m)+1 \\ j=s(n)+1}}^{q(m), s'(n)} + \sum_{\substack{k=q'(m)+1 \\ j=s'(n)+1}}^{q(m), t'(n)} + \sum_{\substack{k=q'(m)+1 \\ j=t'(n)+1}}^{q(m), t(n)} \right) \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r \\ &\leq \frac{1}{\psi'(m)\varphi'(n)} \sum_{\substack{k=p'(m)+1 \\ j=s'(n)+1}}^{q'(m), t'(n)} \left| \mu_y \left( \frac{U_{kj}}{V_{kj}} \right) - \lambda \right|^r + 8 \frac{\mathcal{M}^r}{\psi'(m)\varphi'(n)} \end{aligned}$$

for each  $y \in \mathcal{Y}$ . Then, for  $m, n \rightarrow \infty$ , by our assumption, we get that  $U_{kj} \overset{W_2^\lambda[D]^r_{[\psi, \varphi]}}{\sim} V_{kj}$ . ■

## 5 Conclusion

In this study, we introduced some new asymptotical deferred Cesàro equivalence concepts in the Wijsman sense for double sequences of sets, which are more comprehensive than the concepts given in [30]. Also, we proved some theorems associated with the concept of asymptotical Wijsman strongly  $r$ -deferred Cesàro equivalence ( $0 < r < \infty$ ), and we showed that for bounded double set sequences,  $\{W_2^\lambda[D]^r\} = \{W_2^\lambda DS\}$ . Furthermore, these defined concepts can be extended to new concepts for double sequences of sets with the help of the concepts of ideal convergence, invariant mean.

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