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Fuzzy Multivalued Mappings And Stability Of Fixed Point Sets*

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Abstract

In this article, we introduce multivalued fuzzy contraction mappings and use the fuzzy nearest point property to demonstrate the existence of a fixed point. We derive a result by defining fuzzy fixed point set stability for multivalued fuzzy contractions. An example is also included to demonstrate the usability of the main result.

1 Introduction

Nadler [15] proposed a multivalued version of Banach's theorem on metric spaces in 1969, which used the Hausdorff distance. Lopez et al. [11] extended this concept to fuzzy metric spaces. On the set of compact subsets of a fuzzy metric space, they considered a fuzzy Hausdorff distance. Here we utilize the definition of Lopez et al. [11] to establish a multivalued fuzzy contraction mapping principle. It should be noted that the advent of the notion of fuzzy set in Zadeh's work in 1965 [22] introduced a new tenet in mathematics. Fuzzy notions have made headway into practically every theoretical and applied mathematics discipline. Its application has been broad, particularly in mathematical analysis, of which fuzzy metric spaces are an example. Sehgal et al. [17] extended Banach's contraction mapping principle to probabilistic metric spaces, and it is also valid in the fuzzy metric space described by Kramosil et al. [9]. George and Veeramani updated this formulation [7] to produce a fuzzy metric space with a Hausdorff topology. Fixed point theory has grown significantly in this domain, with various expansions and generalizations of Sehgal et al. [17] result. One of the reasons for this evolution is the Hausdorff topology of space, which is fundamental in many metric fixed point theory problems. The fixed point theory has evolved in numerous directions (see [4, 5, 6, 8, 13] and references cited therein).

Stability is a concept in dynamical systems which has a relation to limiting behaviors. There are several notions of stability in dynamical systems [16, 21]. Here, fuzzy stability is defined for fixed point sets in fuzzy metric spaces with the help of the fuzzy Hausdorff metric. It is a fuzzy extension of the corresponding definition for the stability of fixed point sets in metric spaces. Such problems in metric spaces have been considered in a good number of papers as, for instance, in [1, 2, 3, 10, 12, 14, 18, 19, 20]. Generally, a mapping may not have a fixed point. When the mapping has a fixed point, it may not be unique. Thus the fixed point sets are not singleton sets in general. This warrants an application of setvalued analysis for the study of the stability of fixed point sets. More often than not, in the study of such stabilities, sequences of multivalued mappings appear. The main reason for this is that multivalued mappings often have more than one fixed point compared to their single-valued counterparts.

In this paper, we provide a multivalued variant of the Banach contraction mapping principle in the setting of fuzzy metric spaces. We define fuzzy stability for fixed point sets and show that the fixed point sets associated with the multivalued contractions defined in this paper are fuzzy stable. We extend the notion of contraction proposed by Sehgal et al. [17] to the multivalued domain in the context of fuzzy Hausdorff distance.

The main features of the present work are:

- We introduce the multivalued fuzzy contraction.

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- We prove a fuzzy version of the multivalued contraction mapping principle.
- We introduce the notion of fuzzy stability of fixed point sets.
- We prove that fixed point sets associated with multivalued contractions are fuzzy stable.
- We use a fuzzy Hausdorff metric in our results.
- We extend set valued analysis to some domains of fuzzy fixed point theory.

2 Preliminaries

In this section, we describe the mathematical principles that we employ to prove our theorems.

Definition 1 *The mapping* $* : [0,1] \times [0,1] \to [0,1]$ *is said to be a t-norm [7] when the following hold:*

1. $*$ *is commutative as well as associative,*
2. $1 * c_1 = c_1$ *whenever* $c_1 \in [0,1]$,
3. $c_1 * c_2 \leq c_3 * c_4$ *whenever* $c_1 \leq c_3$ *and* $c_2 \leq c_4$, *for each* $c_1, c_2, c_3, c_4 \in [0,1]$.

The t-norm $*$ *is called continuous if it is continuous as a mapping.*

Example 1 *Some examples of the t-norm are:*

- $a * b = a \cdot b$.
- $a * b = \min\{a, b\}$.

Definition 2 *The 3-tuple* $(X, M, *)$ *is said to be a fuzzy metric space [7] if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0,\infty)$ *satisfying the following conditions, for all* $x, y, z \in X$ *and* $t, s > 0$,

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ *if and only if* $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5. $M(x, y, \cdot) : (0,\infty) \to [0,1]$ *is continuous.*

Example 2 *Let* $X = \mathbb{R}$. *Define* $a * b = a \cdot b$ *and*

$$M(x, y, t) = \left[\exp(\frac{|x-y|}{t})\right]^{-1}$$

for all $x, y \in X$ *and* $t \in (0,\infty)$. *Then* $(X, M, *)$ *is a fuzzy metric space (see [7]).*

Definition 3 *Let* $(X, M, *)$ *be a fuzzy metric space. A sequence* $\{x_n\}$ *in* X *is said to be convergent [7] to* $x \in X$ *if* $\lim_{n\to\infty} M(x_n, x, t) = 1$ *for some* $t > 0$.

Definition 4 *A sequence* $\{x_n\}$ *in a fuzzy metric space* $(X, M, *)$ *is a Cauchy sequence [7] if and only if for each* $\epsilon > 0$, $t > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that* $M(x_n, x_m, t) > 1 - \epsilon$ *for all* $n, m \geq n_0$.

Let (X, d) be a metric space. Denote by $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$, and let M_d be the function defined on $X \times X \times (0, \infty)$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $x, y \in X$ and $t > 0$.

Then $(X, M_d, *)$ is a fuzzy metric space called the standard fuzzy metric space and M_d is called the standard fuzzy metric induced by d [7].

George and Veeramani [7] proved that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$ where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $\epsilon \in (0, 1)$ and $t > 0$.

Then, (X, τ_M) is a metrizable topological space. The convergence in Definition 3 is the same as the convergence in this topology. Moreover, if (X, d) is a metric space, then the topology generated by d coincides with the topology τ_M generated by the fuzzy metric M_d [7].

Given a fuzzy metric space $(X, M, *)$, we denote $\mathcal{K}(X)$, the set of nonempty compact subsets of (X, τ_M).

Definition 5 **([11])** *Let B be a nonempty subset of a fuzzy metric space $(X, M, *)$. For $a \in X$ and $t > 0$, define*

$$M(a, B, t) = \sup\{M(a, b, t) : b \in B\}.$$

Lemma 1 **([11])** *Let $(X, M, *)$ be a fuzzy metric space. Then, for each $a \in X$, $B \in \mathcal{K}(X)$ and $t > 0$, there is $b_0 \in B$ such that*

$$M(a, B, t) = M(a, b_0, t).$$

Definition 6 **([11])** *Let $(X, M, *)$ be a fuzzy metric space. The function H_M on $\mathcal{K}(X) \times \mathcal{K}(X) \times (0, \infty)$ given by*

$$H_M(A, B, t) = \min\left\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\right\},$$

for all $A, B \in \mathcal{K}(X)$ and $t > 0$, is called the fuzzy Hausdorff metric.

Theorem 1 **([11])** *Let $(X, M, *)$ be a fuzzy metric space. Then $(\mathcal{K}(X), H_M, *)$ is a fuzzy metric space.*

Lemma 2 **([11])** *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0, \infty)$.*

A fixed point of $T : X \to \mathcal{K}(X)$ is $z \in X$ such that $z \in T(z)$. Let $\{A_n\}$ be a sequence of compact subsets of a fuzzy metric space $(X, M, *)$. Then by $A_n \to A$, where A is also compact, we mean the convergence in $(\mathcal{K}(X), H_M, *)$, that is, for all $t > 0$

$$H_M(A_n, A, t) \to 1 \text{ as } n \to \infty.$$

A mapping $T : X \to \mathcal{K}(X)$ is continuous if $x_n \to x$ as $n \to \infty$ implies $T(x_n) \to T(x)$ as $n \to \infty$ where the convergence is in $(\mathcal{K}(X), H_M, *)$, that is, with respect to the fuzzy Hausdorff metric. The sequence of mappings $T_n : X \to \mathcal{K}(X)$ is uniformly convergent to a mapping $T : X \to \mathcal{K}(X)$ whenever

$$\inf_{x \in X} H_M(T_n(x), T(x), t) \to 1 \text{ as } n \to \infty \text{ for all } t > 0.$$

3 Main Results

In this section, we provide some results using the multivalued fuzzy contraction mapping principle. We first define the fuzzy nearest point property for a multivalued function.

Definition 7 *A multivalued mapping $T : X \to \mathcal{K}(X)$ is said to satisfy fuzzy nearest point property if for $x \in X$, there exists $y \in T(x)$ such that $M(x, y, t) = M(x, T(x), t)$ for all $t > 0$.*

If T is single-valued, then the fuzzy nearest point property is trivially satisfied. An example of multivalued contraction satisfying fuzzy nearest point property is given in Example 3. It may be seen that the corresponding property is automatically satisfied in the non-fuzzy case. Now, we introduce the definition of multivalued fuzzy contraction.

Definition 8 *Let $(X, M, *)$ be a fuzzy metric space with minimum t-norm $*$. A multivalued mapping $T : X \to \mathcal{K}(X)$ is called a multivalued fuzzy contraction, if for all $x, y \in X$, we have*

$$H_M(T(x), T(y), \lambda t) \geq M(x, y, t),$$

where $0 < \lambda < 1$ and $t > 0$.

We begin with the following result.

Lemma 3 *Let $(X, M, *)$ be a fuzzy metric space with minimum t-norm $*$ and with the properties $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. Then every sequence $\{x_n\}$ satisfying*

$$M(x_n, x_{n+1}, \lambda t) \geq M(x_{n-1}, x_n, t), \text{ for all } t > 0, n \geq 1, \tag{1}$$

is a Cauchy sequence.

Proof. Let $\{x_n\}$ be a sequence in X satisfies (1). If possible let $\{x_n\}$ be not a Cauchy sequence, then there exists $\epsilon > 0$ and $s > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$M(x_{m(k)}, x_{n(k)}, s) < 1 - \epsilon. \tag{2}$$

From (1) it is easy to show that

$$M(x_n, x_{n+1}, \lambda^n t) \geq M(x_0, x_1, t) \quad \text{for all } t > 0, n \geq 1. \tag{3}$$

As $\lim_{t\to\infty} M(x_0, x_1, t) = 1$, there exists $b > 0$ such that

$$M(x_0, x_1, b) > 1 - \epsilon. \tag{4}$$

Now we can find k such that

$$\frac{b}{1-\lambda}\lambda^{m(k)} < s. \tag{5}$$

Again

$$\begin{aligned} & M(x_{m(k)}, x_{m(k)+1}, b\lambda^{m(k)}) * M(x_{m(k)+1}, x_{m(k)+2}, b\lambda^{m(k)+1}) * \cdots * M(x_{n(k)-1}, x_{n(k)}, b\lambda^{n(k)-1}) \\ \leq\ & M(x_{m(k)}, x_{n(k)}, b(\lambda^{m(k)} + \lambda^{m(k)+1} + \ldots + \lambda^{n(k)-1})) \\ \leq\ & M(x_{m(k)}, x_{n(k)}, \frac{b}{1-\lambda}\lambda^{m(k)}) \\ \leq\ & M(x_{m(k)}, x_{n(k)}, s) \qquad [\text{by } (5)]. \end{aligned} \tag{6}$$

From (3) and (6), we get

$$M(x_0, x_1, b) * M(x_0, x_1, b) * \ldots * M(x_0, x_1, b) \leq M(x_{m(k)}, x_{n(k)}, s).$$

Since $*$ is the minimum t-norm, we get

$$M(x_0, x_1, b) \leq M(x_{m(k)}, x_{n(k)}, s).$$

Hence using (4), we have

$$(1 - \epsilon) < M(x_{m(k)}, x_{n(k)}, s),$$

which contradicts (2). Therefore $\{x_n\}$ is a Cauchy sequence in X. ■

The following result shows that every multivalued fuzzy contraction mapping is continuous.

Lemma 4 *Let $(X, M, *)$ be a fuzzy metric space and $T : X \to \mathcal{K}(X)$ be a multivalued fuzzy contraction mapping. Then T is continuous.*

Proof. Let $\{x_n\}$ be a sequence in X converging to a point x in X. Then for all

$$\lim_{n\to\infty} M(x_n, x, t) = 1.$$

Now $H_M(T(x_n), T(x), t) \geq M(x_n, x, t/\lambda) \to 1$ as $n \to \infty$. This implies $T(x_n) \to T(x)$ in the Hausdorff fuzzy metric. Hence T is continuous. ■

Theorem 2 *Let $(X, M, *)$ be a complete fuzzy metric space with minimum t-norm $*$ and with the property that $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. Then every multivalued fuzzy contraction mapping T defined on X with fuzzy nearest point property has a fixed point, that is, there exists $x \in X$ such that $x \in T(x)$.*

Proof. Let $x_0 \in X$ and $x_1 \in T(x_0)$. Then for all $t > 0$,

$$H_M(T(x_0), T(x_1), t) = \min\left\{\inf_{x\in T(x_0)} M(x, T(x_1), t), \inf_{x\in T(x_1)} M(T(x_0), x, t)\right\}.$$

Now there are two possibilities.

Case I: Consider

$$\min\left\{\inf_{x\in T(x_0)} M(x, T(x_1), t), \inf_{x\in T(x_1)} M(T(x_0), x, t)\right\} = \inf_{x\in T(x_0)} M(x, T(x_1), t).$$

Then for all $t > 0$,

$$\begin{aligned} H_M(T(x_0), T(x_1), t) &= \min\left\{\inf_{x\in T(x_0)} M(x, T(x_1), t), \inf_{x\in T(x_1)} M(T(x_0), x, t)\right\} \\ &\leq \inf_{x\in T(x_0)} M(x, T(x_1), t) \\ &\leq M(x_1, T(x_1), t). \end{aligned} \tag{7}$$

By fuzzy nearest point property, there exists $x_2 \in T(x_1)$ such that

$$M(x_1, T(x_1), t) = M(x_1, x_2, t) \text{ for all } t > 0. \tag{8}$$

From (7) and (8)

$$H_M(T(x_0), T(x_1), t) \leq M(x_1, x_2, t) \text{ for all } t > 0. \tag{9}$$

Case II: Consider

$$\min\left\{\inf_{x\in T(x_0)} M(x, T(x_1), t), \inf_{x\in T(x_1)} M(T(x_0), x, t)\right\} = \inf_{x\in T(x_1)} M(T(x_0), x, t).$$

On the similar lines of the above case, we get

$$H_M(T(x_0), T(x_1), t) \leq M(x_1, x_2, t) \text{ for all } t > 0. \tag{10}$$

Also since T is a multivalued fuzzy contraction, for all $t > 0$

$$H_M(T(x_0), T(x_1), \lambda t) \geq M(x_0, x_1, t). \tag{11}$$

Using (9) and (11) we have for all $t > 0$

$$M(x_1, x_2, \lambda t) \geq M(x_0, x_1, t). \tag{12}$$

Similarly we can get $x_3 \in T(x_2)$ such that

$$M(x_2, x_3, \lambda t) \geq M(x_1, x_2, t) \text{ for all } t > 0. \tag{13}$$

Continuing this process we construct a sequence $\{x_n\}$ in X for which $x_{n+1} \in T(x_n)$ and for all $n \geq 1$, $t > 0$

$$M(x_n, x_{n+1}, \lambda t) \geq M(x_{n-1}, x_n, t).$$

By Lemma 3, $\{x_n\}$ is a Cauchy sequence. As $(X, M, *)$ is complete, $x_n \to x$ (say) for some $x \in X$.

By continuity of T, $T(x_n) \to T(x)$ in $(\mathcal{K}(X), H_M, *)$. Moreover, since $x_{n+1} \in T(x_n)$ and $T(x)$ is closed, we have $x \in T(x)$. This completes the proof. ■

Example 3 *Let $(X, M, *)$ be a complete fuzzy metric space, where $X = \mathbb{R}$, $M(x, y, t) = e^{-\frac{|x-y|}{t}}$ and $*(a, b) = \min\{a, b\}$.*

Let $T : X \to \mathcal{K}(X)$ be defined as

$$Tx = \begin{cases} [-x/2, x/2], & \text{if } x \geq 0, \\ [x/2, -x/2], & \text{if } x < 0. \end{cases}$$

Then T is a multivalued fuzzy contraction mapping, and Theorem 2 applies to this example.

Another trivial example of multivalued contraction mapping is $Tz = X$ for all $z \in X$ where X is compact in the topology induced by the fuzzy metric M for a minimum t norm. Theorem 2 is also applicable to this example and all $z \in X$ are fixed points. The purpose of citing this example is to show that the fixed point of a multivalued fuzzy contraction may not be unique.

4 Stability

In this section, we give some results on the fuzzy stability of fixed point sets. We denote by $F(T)$, the set of all fixed points of T. If $T : X \to \mathcal{K}(X)$ is continuous, then it can be trivially seen that the set of its fixed points $F(T)$ is closed in $(X, M, *)$. We next define the fuzzy stability of fixed point sets.

Definition 9 *Let $\{T_n : X \to \mathcal{K}(X),\ n = 1, 2, 3, \dots\}$, where $\mathcal{K}(X)$ is the set of compact subsets of the fuzzy metric space $(X, M, *)$, be such that T_n tends to T uniformly as $n \to \infty$, where T is a mapping from $T : X \to \mathcal{K}(X)$. The fixed point sets of $\{T_n\}$ are said to be fuzzy stable if $F(T_n) \to F(T)$ as $n \to \infty$, that is $\lim_{n\to\infty} H_M(F(T_n), F(T), t) = 1$ for all $t > 0$.*

Theorem 3 *Let $(X, M, *)$ be a complete fuzzy metric space with minimum t-norm $*$ with the condition that $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and $T_1, T_2 : X \to \mathcal{K}(X)$ are multivalued contractions with the same contraction constant λ, satisfy fuzzy nearest point property. Let $F(T_1), F(T_2)$ be compact. Then for all $t > 0$*

$$H_M(F(T_1), F(T_2), \frac{t}{1-\lambda}) \geq \inf_{x\in X} H_M(T_1(x), T_2(x), t). \tag{14}$$

Proof. By Theorem 2 both $F(T_1)$ and $F(T_2)$ are nonempty. Let $x_0 \in F(T_1)$, then for all $t > 0$,

$$\begin{aligned} H_M(T_1(x_0), T_2(x_0), t) &= \min\left\{ \inf_{x\in T_1(x_0)} M(x, T_2(x_0), t), \inf_{x\in T_2(x_0)} M(T_1(x_0), x, t) \right\} \\ &\leq \inf_{x\in T_1(x_0)} M(x, T_2(x_0), t). \end{aligned}$$

On the similar lines of Theorem 2, we get

$$H_M(T_1(x_0), T_2(x_0), t) \leq M(x_0, T_2(x_0), t). \tag{15}$$

By fuzzy nearest point property, there exists $x_1 \in T_2(x_0)$ such that

$$M(x_0, T_2(x_0), t) \leq M(x_0, x_1, t), \text{ for all } t > 0. \tag{16}$$

From (15) and (16), for all $t > 0$, we have

$$H_M(T_1(x_0), T_2(x_0), t) \leq M(x_0, x_1, t). \tag{17}$$

Now for all $t > 0$,

$$\begin{aligned} H_M(T_2(x_0), T_2(x_1), t) &= \min\left\{\inf_{x \in T_2(x_0)} M(x, T_2(x_1), t), \inf_{x \in T_2(x_1)} M(T_2(x_0), x, t)\right\} \\ &\leq \inf_{x \in T_2(x_0)} M(x, T_2(x_1), t) \\ &\leq M(x_1, T_2(x_1), t). \end{aligned} \tag{18}$$

By fuzzy nearest point property, there exists $x_2 \in T_2(x_1)$ such that

$$M(x_1, T_2(x_1), t) \leq M(x_1, x_2, t), \text{ for all } t > 0. \tag{19}$$

From (18) and (19)

$$H_M(T_2(x_0), T_2(x_1), t) \leq M(x_1, x_2, t), \text{ for all } t > 0. \tag{20}$$

As T_2 is multivalued contraction with contraction constant λ, we have for all $t > 0$,

$$H_M(T_2(x_0), T_2(x_1), \lambda t) \geq M(x_0, x_1, t). \tag{21}$$

From (20) and (21), for all $t > 0$,

$$M(x_1, x_2, \lambda t) \geq M(x_0, x_1, t). \tag{22}$$

Similarly we can get $x_3 \in T_2(x_2)$ such that for all $t > 0$,

$$M(x_2, x_3, \lambda t) \geq M(x_1, x_2, t). \tag{23}$$

Continuing this process we construct a sequence $\{x_n\}$ in X for which $x_{n+1} \in T_2(x_n)$ and for all $n \geq 1$, $t > 0$

$$M(x_n, x_{n+1}, \lambda t) \geq M(x_{n-1}, x_n, t).$$

By Lemma 3, $\{x_n\}$ is a Cauchy sequence in X. As $(X, M, *)$ is complete, $\{x_n\}$ has a limit x(say) in X.

By continuity of T_2, $T_2(x_n) \to T_2(x)$ as $n \to \infty$ in the Hausdorff fuzzy metric. Moreover, since $x_{n+1} \in T_2(x_n)$, and since $T_2(x)$, is closed, we have $x \in T_2(x)$. Now, for all $t > 0$,

$$\begin{aligned} & M(x_0, x_1, t) * M(x_1, x_2, \lambda t) * M(x_2, x_3, \lambda^2 t) * ..... * M(x_n, x_{n+1}, \lambda^n t) \\ \leq\ & M(x_0, x_{n+1}, (1 + \lambda + \lambda^2 + ... + \lambda^n)t). \end{aligned}$$

This implies, for all $t > 0$,

$$\begin{aligned} & M(x_0, x_1, t) * M(x_0, x_1, t) * M(x_0, x_1, t) * ..... * M(x_0, x_1, t) \\ \leq\ & M(x_0, x_{n+1}, (1 + \lambda + \lambda^2 + ... + \lambda^n)t). \end{aligned} \tag{24}$$

As $*$ is minimum t-norm, we get

$$M(x_0, x_1, t) \leq M(x_0, x_{n+1}, (1 + \lambda + \lambda^2 + ... + \lambda^n)t).$$

Taking limit as $n \to \infty$, by virtue of Lemma 2, for all $t > 0$, we get

$$M(x_0, x_1, t) \leq M(x_0, x, \frac{t}{1 - \lambda}). \tag{25}$$

From (15), for all $t > 0$,

$$M(x_0, x_1, t) \geq H_M(T_1(x_0), T_2(x_0), t) \geq \inf_{x \in X} H_M(T_1(x), T_2(x), t). \tag{26}$$

From (25) and (26) we have for all $t > 0$,

$$M(x_0, x, \frac{t}{1-\lambda}) \geq \inf_{x \in X} H_M(T_1(x), T_2(x), t). \tag{27}$$

Further, (27) and $x \in T_2(x)$ implies that for all $t > 0$,

$$M(x_0, F(T_2), \frac{t}{1-\lambda}) \geq \inf_{x \in X} H_M(T_1(x), T_2(x), t).$$

Since $x_0 \in F(T_1)$ is arbitrary, the above inequality is true for all $x_0 \in F(T_1)$, hence, for all $t > 0$,

$$\inf_{y \in F(T_1)} M(y, F(T_2), \frac{t}{1-\lambda}) \geq \inf_{x \in X} H_M(T_1(x), T_2(x), t). \tag{28}$$

By symmetry, for all $t > 0$, we have

$$\inf_{y \in F(T_2)} M(F(T_1), y, \frac{t}{1-\lambda}) \geq \inf_{x \in X} H_M(T_1(x), T_2(x), t). \tag{29}$$

From (28) and (29) we have

$$H_M(F(T_1), F(T_2), \frac{t}{1-\lambda}) \geq \inf_{x \in X} H_M(T_1(x), T_2(x), t).$$

Hence the result. ■

To establish our next theorem, we first prove the following lemma.

Lemma 5 *Let $\{T_i : X \to \mathcal{K}(X),\ i = 1, 2, 3, \ldots\}$ be a sequence of multivalued fuzzy contractions defined on the fuzzy metric space $(X, M, *)$ with the same contraction constant λ. Let $T : X \to \mathcal{K}(X)$ be such that $T_i \to T$ as $i \to \infty$. Then T is a multivalued fuzzy contraction.*

Proof. Since T_i's are multivalued fuzzy contractions with the same contraction constant λ, for all $i = 1, 2, 3, \ldots$

$$H_M(T_i(x), T_i(y), t) \geq M(x, y, t/\lambda), \quad \text{for all } t > 0.$$

Letting $i \to \infty$, observing that $(\mathcal{K}(X), H_M, *)$ is a fuzzy metric space, and by virtue of Lemma 2, we have

$$H_M(T(x), T(y), t) \geq M(x, y, t/\lambda), \quad \text{for all } t > 0.$$

This completes the proof of the lemma. ■

The following theorem is a fuzzy stability result for fixed point sets of multivalued contractions.

Theorem 4 *Let $(X, M, *)$ be a complete fuzzy metric space with minimum t-norm $*$ and $\{T_i : X \to \mathcal{K}(X),$ $i = 1, 2, 3, \ldots\}$ be a sequence of setvalued mappings satisfying fuzzy nearest point property and*

$$H_M(T_i(x), T_i(y), \lambda t) \geq M(x, y, t)$$

for all $x, y \in X$ and $t > 0$, where $0 < \lambda < 1$, $i = 0, 1, 2, \ldots$. Further, suppose that $T_i \to T$ uniformly as $i \to \infty$ where $T : X \to \mathcal{K}(X)$ is a mapping. Let $F(T_i)$s and $F(T)$ be compact sets. Then $F(T_i) \to F(T)$ as $i \to \infty$, that is, fixed point sets of T_i are fuzzy stable.

Proof. Since $T_i \to T$ uniformly, for $\epsilon > 0$ and $t > 0$, we can choose N such that

$$\inf_{x \in X} H_M(T_i(x), T(x), (1-\lambda)t) > 1 - \epsilon,$$

for all $i \geq N$. Using Lemma 5, we get T is a multivalued contraction. From Theorem 3, we have

$$H_M(F(T_i), F(T), t) > 1 - \epsilon,$$

for all $i \geq N$. Hence $\lim_{i\to\infty} H_M(F(T_i), F(T), t) = 1$. This completes the proof. ■

In the following theorem, the compactness of a fixed point set is not an assumption.

Theorem 5 *Let $(X, M, *)$ be a compact fuzzy metric space with minimum t-norm $*$ and $\{T_i : X \to \mathcal{K}(X), i = 1, 2, 3, \ldots\}$ be a sequence of setvalued mappings satisfying fuzzy nearest point property and*

$$H_M(T_i(x), T_i(y), \lambda t) \geq M(x, y, t),$$

for all $x, y \in X$ and $t > 0$, where $0 < \lambda < 1$, $i = 0, 1, 2, \ldots$. Further, suppose that $T_i \to T$ uniformly as $i \to \infty$ where a mapping $T : X \to \mathcal{K}(X)$. Then $F(T_i) \to F(T)$ as $i \to \infty$, that is, fixed point sets of T_i are fuzzy stable.

Proof. It follows from Lemma 5 that T is a multivalued fuzzy contraction. Then by the continuity of T, we have that $F(T)$ is closed. Also, by the same observation $F(T_i)$ is closed for all $i = 1, 2, 3, \ldots$. Since X is compact, $F(T_i)$'s and $F(T)$ are also compact. The theorem follows by applying Theorem 4. ■

Remark 1

- *The results obtained in the paper here are also valid for functions which are closed setvalued rather than compact setvalued. This requires a modification in the definition of the fuzzy Hausdorff metric, as is done in the case of the Hausdorff distance in metric spaces in works like [2]. Such modifications are not presented here.*

- *It may also be possible to extend the results to the fuzzy metric space defined by Kramosil et al. [9]. This will require a modification in the definition of the fuzzy Hausdorff metric, which strongly depends on certain features of the space defined in [7].*

Conclusion. The assumption of minimum t-norm, the strongest t-norm, was used to prove our theorems. More work is needed to determine what other requirements must be met before our theorems may be applied to other t-norms. The nearest fixed point attribute in fuzzy sets is used here. This method can be used when the set-valued analysis is used to clarify the robustness of fixed-point sets. It is also noted that when stability is given, sequences of multivalued mappings occur. Also noteworthy is the fact that, unlike with single valued mappings, we can obtain many fixed points in the case of multivalued mappings. This is why we believe our methods are superior to others. Stability analysis is, in fact, an integral aspect of any state-of-the-art dynamical system. Therefore, our method may be used for nonlinear analysis, piquing the curiosity of the current researchers.

Open Problem. By imposing additional constraints, various weak t-norms can be employed in place of the minimum t-norm. In this study, we focus on the Banach contraction, and we find that contraction mappings of the Kannan, Ciric, and Chatterjee types yield intriguing results. Integral and differential equations may be solved using our methods as well.

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