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Location Of The Zeros Of Lacunary-Type Polynomials*

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Abstract

In this paper, by relaxing the hypothesis of well-known Eneström-Kakeya theorem, we obtain a result which is applicable to the lacunary-type of polynomials and generalizes several well-known results concerning the location of zeros of polynomials. In addition to this, we show by examples that our results presents better information about the bounds of zeros of polynomials than some known results.

1 Introduction

Various experimental observations and investigations when translated into mathematical language lead to mathematical models. The solution of these models could lead to problems of solving algebraic polynomial equations of certain degree. The study of zeros of these algebraic complex polynomials is an old theme in analytic theory of polynomials, has spawned a vast amount of research over the past millennium includes its applications both within and outside of mathematics. In addition to having numerous applications, this study has been the inspiration for much theoretical research (including being the initial motivation for modern algebra). Algebraic and analytic methods for finding zeros of a polynomial, in general, can be quite complicated, so it is desirable to put some restrictions on polynomials. This motivated the study of identifying suitable regions in the complex plane containing the zeros of a polynomial when their coefficients are restricted with special conditions. The most amusing problem of the algebra is to find the zeros of a polynomial. But as the degree of a polynomial shoots up, it is very difficult to find the zeros of a polynomial. This makes identification of regions containing zeros of a polynomial a significant problem. In 1829, Cauchy [8] gave a very simple expression for the zero-bound in terms of the coefficients of a polynomial. In fact he proved that all the zeros of a polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_n \neq 0.$$

lie in the disc

$$|z| \leq 1 + \max_{0 \leq j \leq n-1} |a_j|.$$

The remarkable property of this result is its simplicity of computations. In literature [15], there exists several results concerning the bounds for zeros of polynomials. A classical result on the location of zeros of a polynomial with restricted coefficients known as Eneström-Kakeya theorem (see section 8.3 of [19]):

Theorem 1 *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in $|z| \leq 1$.

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In literature (see [1]–[24]) there exist several generalizations of Eneström-Kakeya theorem. There is always a need for better and better results in this subject because of its application in many areas including signal processing, communication theory, cryptography, control theory, combinatorics and mathematical biology. In this paper, by using standard techniques we establish regions in which zeros of a lacunary type polynomial

$$P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \quad 1 \leq \mu \leq n, \; a_0 \neq 0$$

lie by putting certain restrictions on the real coefficients of a given lacunary- type polynomial. Joyal et al. [11] extended Theorem 1 as they dropped the restriction on the hypothesis that all the coefficients be non negative and proved the following Theorem.

Theorem 2 *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ lie in

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Aziz and Zargar [1] extended Theorem 2 in the sense as they relaxed the hypothesis of Eneström Kakeya theorem and proved some interesting result. In fact they proved the following theorem.

Theorem 3 *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that for some $k \geq 1$.*

$$ka_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ lie in

$$|z + k - 1| \leq \frac{ka_n - a_0 + |a_0|}{|a_n|}.$$

Further, W. M. Shah and Liman [24] extended Theorem 3 to the polynomials with complex coefficients by proving the following theorem.

Theorem 4 *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients such that for some real β, $|arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $j = 0, 1, 2, \ldots, n$ and $k \geq 1$,*

$$k|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $P(z)$ lie in

$$\Big|z - k - 1\Big| \leq \frac{1}{|a_n|}\Big\{(k|a_n| - |a_0|)(\cos\alpha + \sin\alpha) + |a_0| + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|\Big\}.$$

Recently, Rather et al. [20] using standard techniques and obtained the result which gives regions containing all the zeros of the polynomial with real coefficients and generalize several results concerning the generalization of Eneström Kakeya Theorem. In fact they proved the following theorem.

Theorem 5 *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients such that for some $k_j \geq 1$, $1 \leq j \leq r$ where $1 \leq r \leq n$,*

$$k_1 a_n \geq k_2 a_{n-1} \geq k_3 a_{n-2} \geq \cdots \geq k_r a_{n-r+1} \geq a_{n-r} \geq \cdots \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ lie in

$$\left|z + k_1 - 1 - (k_2 - 1)\frac{a_{n-1}}{a_n}\right|$$

$$\leq \quad \frac{1}{|a_n|}\left\{k_1a_n-(k_2-1)|a_{n-1}|+2\sum_{j=2}^{r}(k_j-1)|a_{n-j+1}|-a_0+|a_0|\right\}.$$

For $r=2$ in Theorem 5, they obtained another result which answers the question raised by Professor N. K. Govil regarding the determination of regions containing all the zeros of the polynomial at International conference held at the University of Jammu, India, in 2007. In fact they proved the following theorem.

Theorem 6 *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n with real coefficients such that for some $k_1\geq 1$, $k_2\geq 1$,*

$$k_1a_n\geq k_2a_{n-1}\geq a_{n-2}\geq\cdots\geq a_1\geq a_0,$$

then all the zeros of $P(z)$ lie in

$$\left|z+k_1-1-(k_2-1)\frac{a_{n-1}}{a_n}\right|\leq\frac{1}{|a_n|}\Big(k_1a_n-(k_2-1)|a_{n-1}|-a_0+|a_0|\Big).$$

More recently Rather et al. [21] extended the Theorem 5 to the polynomial with complex coefficients and proved the following result.

Theorem 7 *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n with complex coefficients such that for some real β, $|arg\ a_j-\beta|\leq\alpha\leq\frac{\pi}{2}$, $j=0,1,2,\ldots,n$ and $k\geq 1$, $a_{n-j}\neq 0$, $j=0,1,2,\ldots,r$ where $1\leq r\leq n-1$,*

$$k_0|a_n|\geq k_1|a_{n-1}|\geq k_2|a_{n-2}|\geq\cdots\geq k_r|a_{n-r}|\geq\cdots\geq|a_1|\geq|a_0|,$$

then all the zeros of $P(z)$ lie in

$$\begin{aligned}&\left|z+k_0-1-(k_1-1)\frac{a_{n-1}}{a_n}\right|\\ \leq\ &\frac{1}{|a_n|}\Bigg\{(k_0|a_n|-|a_0|)(\cos\alpha-\sin\alpha)+2\sin\alpha\left(\sum_{j=1}^{r}k_j|a_{n-j}|+\sum_{j=r+1}^{n}|a_{n-j}\right)\\ &-(k_1-1)|a_{n-1}|+2\sum_{j=1}^{r}(k_j-1)|a_{n-j}|+|a_0|\Bigg\}.\end{aligned}$$

2 Main Results

Although Theorems 5, 6 and 7 are applicable to the larger class of polynomials as compared to all other Eneström-Kakeya type results, but are not applicable to the polynomials whose one or two coefficients are zero. For instance, if we consider the polynomial $P(z)=5z^5+4z^4+3z^3+0z^2+0z+1$, then one can note that all Eneström-Kakeya type results including Theorems 5, 6 and 7 are not applicable to this polynomial. So it is interesting to look for the results applicable to such class of polynomials. Motivated by this, here we establish the following results applicable to such class of polynomials of the type $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j, 1\leq\mu\leq n, a_0\neq 0$. In this paper, we extend Theorem 5 and Theorem 7 to the lacunary-type polynomials with real coefficients and thereby, obtain a result with relaxed hypothesis that give zero bound of the lacunary-type polynomials with real coefficients. In fact, we prove the following theorem.

Theorem 8 *Let $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\leq\mu\leq n$, $a_0\neq 0$ is a polynomial of degree n with real coefficients such that some $k_j\geq 1$, $j=\mu,\ \mu+1,\ldots,\ \mu+r-1$, where $\mu\leq r\leq n$,*

$$k_\mu a_n\geq k_{\mu+1}a_{n-1}\geq k_{\mu+2}a_{n-2}\geq\cdots\geq k_{\mu+r-1}a_{n-r+1}\geq a_{n-r}\geq\cdots\geq a_\mu\geq a_0.$$

Then all the zeros of $P(z)$ *lie in*

$$\begin{aligned}&\Big|(z+k_\mu-1)-(k_{\mu+1}-1)a_{n-1}/a_n\Big|\\ \leq\ & \frac{1}{|a_n|}\left(k_\mu a_n-(k_{\mu+1}-1)|a_{n-1}|+2\sum_{j=\mu+1}^{\mu+r-1}(k_j-1)|a_{n+\mu-j}|-a_\mu+|a_\mu|+2|a_0|\right).\end{aligned}$$

We may apply Theorem 8 to the polynomial $P(tz)$ to obtain the following result:

Corollary 1 *Let* $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\leq\mu\leq n$, $a_0\neq 0$ *is a polynomial of degree* n *with real coefficients such that some* $t>0$ *and* $k_j\geq 1$, $j=\mu,\ \mu+1,\ldots,\ \mu+r-1$, *where* $\mu\leq r\leq n$,

$$\begin{aligned}k_\mu t^n a_n &\geq k_{\mu+1}t^{n-1}a_{n-1}\geq k_{\mu+2}t^{n-2}a_{n-2}\geq\cdots\geq k_{\mu+r-1}t^{n-r+1}a_{n-r+1}\\ &\geq t^{n-r}a_{n-r}\geq\cdots\geq a_\mu t^\mu\geq a_0.\end{aligned}$$

Then all the zeros of $P(z)$ *lie in*

$$\begin{aligned}&\Big|(z+k_\mu-1)t^\mu-(k_{\mu+1}-1)a_{n-1}/a_n\Big|\\ \leq\ & \frac{1}{|a_n|}\Bigg(k_\mu t^\mu a_n-(k_{\mu+1}-1)|a_{n-1}|+2\sum_{j=\mu+1}^{\mu+r-1}(k_j-1)|a_{n+\mu-j}|/t^{j-\mu-1}-a_\mu/t^{n-\mu-1}\\ &+|a_\mu|/t^{n-\mu}+|a_0|/t^{n-\mu}+|a_0|/t^n\Bigg).\end{aligned}$$

Taking $r=2$ and $a_0\geq 0$ in Corollary 1, we get the following result:

Corollary 2 *Let* $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\leq\mu\leq n$, $a_0\neq 0$ *is a polynomial of degree* n *with real coefficients such that some* $t>0$ *and* $k_\mu\geq 1$, $k_{\mu+1}\geq 1$,

$$k_\mu t^n a_n\geq k_{\mu+1}t^{n-1}a_{n-1}\geq t^{n-2}a_{n-2}\geq\cdots\geq a_\mu t^\mu\geq a_0\geq 0.$$

Then all the zeros of $P(z)$ *lie in*

$$\Big|(z+k_\mu-1)t^\mu-(k_{\mu+1}-1)a_{n-1}/a_n\Big|\leq k_\mu t^\mu-(k_{\mu+1}-1)a_{n-1}/a_n+|a_0|/t^{n-\mu}+|a_0|/t^n.$$

Taking $t=1$ in Corollary 2, we get following interesting result:

Corollary 3 *Let* $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j, 1\leq\mu\leq n$, $a_0\neq 0$ *is a polynomial of degree* n *with real coefficients such that some* $k_\mu\geq 1$, $k_{\mu+1}\geq 1$,

$$k_\mu a_n\geq k_{\mu+1}a_{n-1}\geq a_{n-2}\geq\cdots\geq a_\mu\geq a_0\geq 0.$$

Then all the zeros of $P(z)$ *lie in*

$$\Big|(z+k_\mu-1)-(k_{\mu+1}-1)a_{n-1}/a_n\Big|\leq k_\mu-(k_{\mu+1}-1)a_{n-1}/a_n+2|a_0|.$$

Taking $k_{\mu+1}=1$ in Corollary 3, we get following interesting result:

Corollary 4 *Let* $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j, 1\leq\mu\leq n, a_0\neq 0$ *is a polynomial of degree* n *with real coefficients such that some* $k_\mu\geq 1$,

$$k_\mu a_n\geq a_{n-1}\geq a_{n-2}\geq\cdots\geq a_\mu\geq a_0\geq 0.$$

Then all the zeros of $P(z)$ *lie in*

$$\Big|z+k_\mu-1\Big|\leq k_\mu+2|a_0|.$$

Since the results discussed above are applicable to a small class of lacunary-type polynomials , so it is interesting to look for the results applicable to the large class of polynomials. Next, we extend Theorem 8 to the polynomials with complex coefficients and thereby, obtain a result with relaxed hypothesis that gives zero bounds of the polynomials with complex coefficients. In fact, we prove the following theorem.

Theorem 9 *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$, $a_0 \ne 0$ is a polynomial of degree n with complex coefficients such that for some real β, $|arg\ a_j - \beta| \le \alpha \le \frac{\pi}{2}$, $j = 0, 1, 2, \ldots, n$ and $k_j \ge 1$, $j = \mu - 1$, μ, $\mu + 1, \ldots, \mu + r - 1$, where $\mu \le r \le n$,*

$$k_{\mu-1}|a_n| \ge k_\mu |a_{n-1}| \ge k_{\mu+1}|a_{n-2}| \ge \cdots \ge k_{\mu+r-2}|a_{n-r+1}| \ge k_{\mu+r-1}|a_{n-r}| \ge \cdots \ge |a_\mu| \ge |a_0|. \quad (1)$$

Then all the zeros of $P(z)$ lie in

$$\begin{aligned}\left|(z + k_{\mu-1} - 1) - (k_\mu - 1)\frac{a_{n-1}}{a_n}\right| \le \frac{1}{a_n}\Bigg[&(k_{\mu-1}|a_n| - |a_\mu|)(\cos\alpha + \sin\alpha) \\ &+ 2\sin\alpha\left(\sum_{j=\mu}^{\mu+r-1} k_j|a_{n+\mu-j-1}| + \sum_{j=\mu+r}^{n-1} |a_{n+\mu-j-1}|\right) \\ &-(k_\mu - 1)|a_{n-1}| + 2\sum_{j=\mu}^{\mu+r-1}(k_j - 1)|a_{n+\mu-j-1}| + |a_\mu| + 2|a_0|\Bigg].\end{aligned}$$

We may apply Theorem 9 to the polynomial $P(tz)$ to obtain the following result:

Corollary 5 *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, 1 \le \mu \le n, a_0 \ne 0$ is a polynomial of degree n with complex coefficients such that for some real β, $|arg\ a_j - \beta| \le \alpha \le \frac{\pi}{2}$, $j = 0, 1, 2, \ldots, n$ and $t > 0$, $k_j \ge 1$, $j = \mu - 1, \mu, \mu + 1, \ldots, \mu + r - 1$, where $\mu \le r \le n$,*

$$\begin{aligned}k_{\mu-1}t^n|a_n| \ge k_\mu t^{n-1}|a_{n-1}| \ge k_{\mu+1}t^{n-2}|a_{n-2}| \ge \cdots \ge k_{\mu+r-2}t^{n-r+1}|a_{n-r+1}| \\ \ge k_{\mu+r-1}t^{n-r}|a_{n-r}| \ge \cdots \ge t^\mu|a_\mu| \ge |a_0|.\end{aligned}$$

Then all the zeros of $P(z)$ lie in

$$\begin{aligned}\left|z + (k_{\mu-1} - 1)t - (k_\mu - 1)\frac{a_{n-1}}{a_n}\right| \le \frac{1}{a_n}\Bigg[&\left(k_{\mu-1}|a_n| - \frac{|a_\mu|}{t^{n-\mu}}\right)(\cos\alpha + \sin\alpha) \\ &+ 2\sin\alpha\left(\sum_{j=\mu}^{\mu+r-1} k_j\frac{|a_{n+\mu-j-1}|}{t^{j-\mu+1}} + \sum_{j=\mu+r}^{n-1}\frac{|a_{n+\mu-j-1}|}{t^{j-\mu+1}}\right) \\ &-(k_\mu - 1)\frac{|a_{n-1}|}{t} + 2\sum_{j=\mu}^{\mu+r-1}(k_j - 1)\frac{|a_{n+\mu-j-1}|}{t^{j-\mu+1}} + \frac{|a_\mu|}{t^{n-\mu}} + 2\frac{|a_0|}{t^n}\Bigg].\end{aligned}$$

Taking $r = 1$ in Corollary 5, we get the following result:

Corollary 6 *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$, $a_0 \ne 0$ is a polynomial of degree n with complex coefficients such that for some real β, $|arg\ a_j - \beta| \le \alpha \le \frac{\pi}{2}$, $j = 0, 1, 2, \ldots, n$ and $t > 0$, $k_j \ge 1$, $j = \mu - 1, \mu$,*

$$k_{\mu-1}t^n|a_n| \ge k_\mu t^{n-1}|a_{n-1}| \ge \cdots \ge t^\mu|a_\mu| \ge |a_0|.$$

Then all the zeros of $P(z)$ lie in

$$\left|z + (k_{\mu-1} - 1)t - (k_\mu - 1)\frac{a_{n-1}}{a_n}\right| \le \frac{1}{a_n}\left[\left(k_{\mu-1}|a_n| - \frac{|a_\mu|}{t^{n-\mu}}\right)(\cos\alpha + \sin\alpha)\right.$$

$$+2\sin\alpha\left(k_\mu\frac{|a_{n-1}|}{t}+\sum_{j=\mu+1}^{n-1}\frac{|a_{n+\mu-j-1}|}{t^{j-\mu+1}}\right)$$
$$-(k_\mu-1)\frac{|a_{n-1}|}{t}+\frac{|a_\mu|}{t^{n-\mu}}+2\frac{|a_0|}{t^n}\Bigg].$$

Taking $t=1$ in Corollary 6, we get the following result:

Corollary 7 *Let $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\le\mu\le n$, $a_0\ne 0$ is a polynomial of degree n with complex coefficients such that for some real β, $|arg\ a_j-\beta|\le\alpha\le\frac{\pi}{2}$, $j=0,1,2,\dots,n$ and $k_j\ge 1$, $j=\mu-1,\mu$,*

$$k_{\mu-1}|a_n|\ge k_\mu|a_{n-1}|\ge\cdots\ge|a_\mu|\ge|a_0|.$$

Then all the zeros of $P(z)$ lie in

$$\left|z+(k_{\mu-1}-1)-(k_\mu-1)\frac{a_{n-1}}{a_n}\right| \le \frac{1}{a_n}\Bigg[(k_{\mu-1}|a_n|-|a_\mu|)(\cos\alpha+\sin\alpha)$$
$$+2\sin\alpha\left(k_\mu|a_{n-1}|+\sum_{j=\mu+1}^{n-1}|a_{n+\mu-j-1}|\right)$$
$$-(k_\mu-1)|a_{n-1}|+|a_\mu|+2|a_0|\Bigg].$$

Taking $k_\mu=1$ and $k_{\mu-1}=k$ in Corollary 7, we get following interesting result:

Corollary 8 *Let $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\le\mu\le n$, $a_0\ne 0$ is a polynomial of degree n with complex coefficients such that for some real β, $|arg\ a_j-\beta|\le\alpha\le\frac{\pi}{2}$, $j=0,1,2,\dots,n$ and $k\ge 1$,*

$$k|a_n|\ge|a_{n-1}|\ge\cdots\ge|a_\mu|\ge|a_0|.$$

Then all the zeros of $P(z)$ lie in

$$\Big|z+(k-1)\Big|\le\frac{1}{a_n}\left[(k|a_n|-|a_\mu|)(\cos\alpha+\sin\alpha)+2\sin\alpha\left(\sum_{j=\mu}^{n-1}|a_{n+\mu-j-1}|\right)+|a_\mu|+2|a_0|\right].$$

Remark 1 *For $\mu=1$ in Theorems 8 and 9, we get the Theorems 5 and 7 respectively.*

3 Computations and Analysis

In this section, we present some examples of a polynomial to show that Theorem 8 gives better information about the location of zeros than Cauchy's Theorem. It is worth mentioning that all existing Eneström-Kakeya type results are not applicable for these polynomials.

Example 1 *Let $P(z)=10z^3+10z^2+1$. By taking $r=2$, $k_\mu=16/15$, $k_{\mu+1}=8/7$ in Theorem 8, it follows that all the zeros of $P(z)$ lie in the disc $|z-\frac{1}{15}|\le 1.2$. Whereas if we use Cauchy's Theorem, it follows that all the zeros of $P(z)$ lie in the disc $|z|\le 2$.*

Example 2 *Let $P(z)=3z^4+2.8z^3+2.6z^2+1$. By taking $r=2$, $k_\mu=6/5$, $k_{\mu+1}=3/2$ in Theorem 8, it follows that all the zeros of $P(z)$ lie in disc $|z-0.26|\le 1.06$. Whereas if we use Cauchy's Theorem, it follows that all the zeros of $P(z)$ lie in the disc $|z|\le 2.06$.*

From the above examples, it is evident that our results give better bound than the bound obtained by using Cauchy's Theorem.

4 Lemma

For the proofs of the above results, we need the following lemma which is due to Govil and Rahman [10].

Lemma 1 *If for some real* β,

$$|arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \qquad a_j \neq 0,$$

then, for any positive real numbers t_1 *and* t_2,

$$|t_1 a_j - t_2 a_{j-1}| \leq \big|t_1|a_j| - t_2|a_{j-1}|\big|\cos\alpha + (t_1|a_j| + t_2|a_{j-1}|)\sin\alpha.$$

5 Proof of Main Results

Proof of Theorem 8. Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$, $1 \leq \mu \leq n$, $a_0 \neq 0$ is a polynomial of degree n with real coefficients such that some $k_j \geq 1$, $j = \mu, \mu+1, \ldots, \mu+r-1$, where $\mu \leq r \leq n$. Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_{n-r} - a_{n-r-1})z^{n-r} + \ldots \\ &\quad +(a_{\mu+1} - a_\mu)z^{\mu+1} + a_\mu z^\mu - a_0 z + a_0 \\ &= -a_n z^{n+1} + (k_\mu a_n - k_{\mu+1}a_{n-1} - (k_\mu - 1)a_n + (k_{\mu+1} - 1)a_{n-1})z^n \\ &\quad +(k_{\mu+1}a_{n-1} - k_{\mu+2}a_{n-2} - (k_{\mu+1} - 1)a_{n-1} + (k_{\mu+2} - 1)a_{n-2})z^{n-1} + \ldots \\ &\quad +(k_{\mu+r-2}a_{n-r+2} - k_{\mu+r-1}a_{n-r+1} - (k_{\mu+r-2} - 1)a_{n-r+2} + (k_{\mu+r-1} - 1)a_{n-r+1})z^{n-r+2} \\ &\quad +(k_{\mu+r-1}a_{n-r+1} - a_{n-r} - (k_{\mu+r-1} - 1)a_{n-r+1})z^{n-r+1} + (a_{n-r} - a_{n-r+1})z^{n-r} + \ldots \\ &\quad +(a_{\mu+1} - a_\mu)z^{\mu+1} + a_\mu z^\mu - a_0 z + a_0, \end{aligned}$$

which implies that

$$\begin{aligned} \big|F(z)\big| &= \big| - a_n z^{n+1} - (k_\mu - 1)a_n z^n + (k_\mu a_n - k_{\mu+1}a_{n-1})z^n + (k_{\mu+1} - 1)a_{n-1}z^n \\ &\quad +(k_{\mu+1}a_{n-1} - k_{\mu+2}a_{n-2})z^n - (k_{\mu+1} - 1)a_{n-1}z^{n-1} + (k_{\mu+2} - 1)a_{n-2}z^{n-1} + \ldots \\ &\quad +(k_{\mu+r-2}a_{n-r+2} - k_{\mu+r-1}a_{n-r+1})z^{n-r+2} - (k_{\mu+r-2} - 1)a_{n-r+2}z^{n-r+2} \\ &\quad +(k_{\mu+r-1} - 1)a_{n-r+1}z^{n-r+2} + (k_{\mu+r-1}a_{n-r+1} - a_{n-r})z^{n-r+1} - (k_{\mu+r-1} - 1)a_{n-r+1}z^{n-r+1} \\ &\quad +(a_{n-r} - a_{n-r-1})z^{n-r} + \cdots + (a_{\mu+1} - a_\mu)z^{\mu+1} + a_\mu z^\mu - a_0 z + a_0\big|, \end{aligned}$$

that is

$$\begin{aligned} \big|F(z)\big| &\geq |z|^n\Bigg\{|(z + k_\mu - 1)a_n - (k_{\mu+1} - 1)a_{n-1}| - \Bigg(|k_\mu a_n - k_{\mu+1}a_{n-1}| + |k_{\mu+1}a_{n-1} - k_{\mu+2}a_{n-2}|/|z| \\ &\quad +|k_{\mu+1} - 1||a_{n-1}|/|z| + |k_{\mu+2} - 1||a_{n-2}|/|z| + \cdots + |k_{\mu+r-2}a_{n-r+2} - k_{\mu+r-1}a_{n-r+1}|/|z|^{r-2} \\ &\quad +|k_{\mu+r-2} - 1||a_{n-r+2}|/|z|^{r-2} + |k_{\mu+r-1}a_{n-r+1} - a_{n-r}|/|z|^{r-1} + |k_{\mu+r-1} - 1||a_{n-r+1}|/|z|^{r-1} \\ &\quad +|a_{n-r} - a_{n-r+1}|/|z|^r + \cdots + |a_{\mu+1} - a_\mu|/|z|^{n-\mu-1} \\ &\quad +|a_\mu|/|z|^{n-\mu} + |a_0|/|z|^{n-1} + |a_0|/|z|^n\Bigg)\Bigg\}. \end{aligned}$$

By using the hypothesis, we have for $|z| > 1$,

$$\begin{aligned} \big|F(z)\big| &\geq |z|^n\Bigg\{|(z + k_\mu - 1)a_n - (k_{\mu+1} - 1)a_{n-1}| - \Bigg(k_\mu a_n - k_{\mu+1}a_{n-1} + k_{\mu+1}a_{n-1} - k_{\mu+2}a_{n-2} \\ &\quad +(k_{\mu+1} - 1)|a_{n-1}| + (k_{\mu+2} - 1)|a_{n-2}| + \cdots + k_{\mu+r-2}a_{n-r+2} - k_{\mu+r-1}a_{n-r+1} \end{aligned}$$

$$+(k_{\mu+r-2}-1)|a_{n-r+2}|+k_{\mu+r-1}a_{n-r+1}-a_{n-r}+(k_{\mu+r-1}-1|a_{n-r+1}|$$
$$\left.\left.+a_{n-r}-a_{n-r+1}+\cdots+a_{\mu+1}-a_{\mu}+|a_{\mu}|+2|a_0|\right)\right\},$$

that is,

$$\begin{aligned}\big|F(z)\big| &\geq |a_n||z|^n\Bigg\{|(z+k_\mu-1)-(k_{\mu+1}-1)a_{n-1}/a_n|-\frac{1}{|a_n|}\Bigg(k_\mu a_n-(k_{\mu+1}-1)|a_{n-1}|\\ &\quad+2\sum_{j=\mu+1}^{\mu+r-1}(k_j-1)|a_{n+\mu-j}|-a_\mu+|a_\mu|+2|a_0|\Bigg)\Bigg\}>0,\end{aligned}$$

if

$$\begin{aligned}\Big|(z+k_\mu-1)-(k_{\mu+1}-1)a_{n-1}/a_n\Big| &> \frac{1}{|a_n|}\Bigg(k_\mu a_n-(k_{\mu+1}-1)|a_{n-1}|\\ &\quad+2\sum_{j=\mu+1}^{\mu+r-1}(k_j-1)|a_{n+\mu-j}||-a_\mu+|a_\mu|+2|a_0|\Bigg).\end{aligned}$$

This shows that those zeros of $F(z)$ whose modulus is greater than 1 lie in

$$\begin{aligned}\Big|(z+k_\mu-1)-(k_{\mu+1}-1)a_{n-1}/a_n\Big| &\leq \frac{1}{|a_n|}\Bigg(k_\mu a_n-(k_{\mu+1}-1)|a_{n-1}|\\ &\quad+2\sum_{j=\mu+1}^{\mu+r-1}(k_j-1)|a_{n+\mu-j}|-a_\mu+|a_\mu|+2|a_0|\Bigg).\end{aligned}$$

But those zeros of $F(z)$, whose modulus is less than or equal to 1 already lie in this region. Hence it follows that all the zeros of $F(z)$ and therefore of $P(z)$ lie in

$$\begin{aligned}\Big|(z+k_\mu-1)-(k_{\mu+1}-1)a_{n-1}/a_n\Big| &\leq \frac{1}{|a_n|}\Bigg(k_\mu a_n-(k_{\mu+1}-1)|a_{n-1}|\\ &\quad+2\sum_{j=\mu+1}^{\mu+r-1}(k_j-1)|a_{n+\mu-j}|-a_\mu+|a_\mu|+2|a_0|\Bigg).\end{aligned}$$

This completes the proof of Theorem 8. ■

Proof of Theorem 9. Let $P(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$, $1\leq\mu\leq n$, $a_0\neq 0$ is a polynomial of degree n with real coefficients such that some $k_j\geq 1$, $j=\mu-1,\ \mu+1,\dots,\ \mu+r-1$, where $\mu\leq r\leq n$. Consider the polynomial

$$\begin{aligned}F(z) &= (1-z)P(z)\\ &= -a_nz^{n+1}+(a_n-a_{n-1})z^n+\cdots+(a_{n-r}-a_{n-r-1})z^{n-r}+\dots\\ &\quad+(a_{\mu+1}-a_\mu)z^{\mu+1}+a_\mu z^\mu-a_0z+a_0\\ &= -a_nz^{n+1}+(k_{\mu-1}a_n-k_\mu a_{n-1}-(k_{\mu-1}-1)a_n+(k_\mu-1)a_{n-1})z^n\\ &\quad+(k_\mu a_{n-1}-k_{\mu+1}a_{n-2}-(k_\mu-1)a_{n-1}+(k_{\mu+1}-1)a_{n-2})z^{n-1}+\dots\\ &\quad+(k_{\mu+r-2}a_{n-r+1}-k_{\mu+r-1}a_{n-r}-(k_{\mu+r-2}-1)a_{n-r+1}+(k_{\mu+r-1}-1)a_{n-r})z^{n-r+1}\\ &\quad+(k_{\mu+r-1}a_{n-r}-a_{n-r-1}-(k_{\mu+r-1}-1)a_{n-r})z^{n-r}+(a_{n-r-1}-a_{n-r-2})z^{n-r-1}+\dots\\ &\quad+(a_{\mu+1}-a_\mu)z^{\mu+1}+a_\mu z^\mu-a_0z+a_0,\end{aligned}$$

which implies that

$$\begin{aligned}\big|F(z)\big| &= \big| - a_nz^{n+1} - (k_{\mu-1} - 1)a_nz^n + (k_{\mu-1}a_n - k_\mu a_{n-1})z^n + (k_\mu - 1)a_{n-1}z^n \\ &+(k_\mu a_{n-1} - k_{\mu+1}a_{n-2})z^n - (k_\mu - 1)a_{n-1}z^{n-1} + (k_{\mu+1} - 1)a_{n-2}z^{n-1} + \dots \\ &+(k_{\mu+r-2}a_{n-r+1} - k_{\mu+r-1}a_{n-r})z^{n-r+1} - (k_{\mu+r-2} - 1)a_{n-r+1}z^{n-r+1} \\ &+(k_{\mu+r-1} - 1)a_{n-r}z^{n-r+1} + (k_{\mu+r-1}a_{n-r} - a_{n-r-1})z^{n-r} - (k_{\mu+r-1} - 1)a_{n-r}z^{n-r} \\ &+(a_{n-r-1} - a_{n-r-2})z^{n-r-1} + \dots + (a_{\mu+1} - a_\mu)z^{\mu+1} + a_\mu z^\mu - a_0z + a_0\big|,\end{aligned}$$

that is,

$$\begin{aligned}\big|F(z)\big| &\geq |z|^n\Bigg\{|(z + k_{\mu-1} - 1)a_n - (k_\mu - 1)a_{n-1}| - \Bigg(|k_{\mu-1}a_n - k_\mu a_{n-1}| + |k_\mu a_{n-1} - k_{\mu+1}a_{n-2}|/|z| \\ &+|k_\mu - 1||a_{n-1}|/|z| + |k_{\mu+1} - 1||a_{n-2}|/|z| + \dots + |k_{\mu+r-2}a_{n-r+1} - k_{\mu+r-1}a_{n-r}|/|z|^{r-1} \\ &+|k_{\mu+r-2} - 1||a_{n-r+1}|/|z|^{r-1} + |k_{\mu+r-1} - 1||a_{n-r}|/|z|^{r-1} + |k_{\mu+r-1}a_{n-r} - a_{n-r-1}|/|z|^r \\ &+|k_{\mu+r-1} - 1||a_{n-r}|/|z|^r + |a_{n-r-1} - a_{n-r}|/|z|^{r+1} + \dots + |a_{\mu+1} - a_\mu|/|z|^{n-\mu-1} \\ &+|a_\mu|/|z|^{n-\mu} + |a_0|/|z|^{n-1} + |a_0|/|z|^n\Bigg)\Bigg\}.\end{aligned}$$

Let $|z| > 1$ so that $1/|z| < 1$. Then we have

$$\begin{aligned}\big|F(z)\big| &\geq |z|^n\Bigg\{|(z + k_{\mu-1} - 1)a_n - (k_\mu - 1)a_{n-1}| - \Bigg(|k_{\mu-1}a_n - k_\mu a_{n-1}| + |k_\mu a_{n-1} - k_{\mu+1}a_{n-2}| \\ &+|k_\mu - 1||a_{n-1}| + |k_{\mu+1} - 1||a_{n-2}| + \dots + |k_{\mu+r-2}a_{n-r+1} - k_{\mu+r-1}a_{n-r}| \\ &+|k_{\mu+r-2} - 1||a_{n-r+1}| + |k_{\mu+r-1} - 1||a_{n-r}| + |k_{\mu+r-1}a_{n-r} - a_{n-r-1}| + |k_{\mu+r-1} - 1||a_{n-r}| \\ &+|a_{n-r-1} - a_{n-r-2}| + \dots + |a_{\mu+1} - a_\mu| + |a_\mu| + 2|a_0|\Bigg)\Bigg\}.\end{aligned}$$

Applying Lemma 1, we have for $|z| > 1$,

$$\begin{aligned}\big|F(z)\big| &\geq |a_n||z|^n\Bigg\{\Big|(z + k_{\mu-1} - 1) - (k_\mu - 1)\frac{a_{n-1}}{a_n}\Big| - \frac{1}{a_n}\Bigg[\Big(\big||k_{\mu-1}|a_n| - k_\mu|a_{n-1}|\big| \\ &+\big|k_\mu|a_{n-1}| - k_{\mu+1}|a_{n-2}|\big| \dots + \big|k_{\mu+r-2}|a_{n-r+1}| - k_{\mu+r-1}|a_{n-r}|\big| + \big|k_{\mu+r-1}|a_{n-r}| - |a_{n-r-1}|\big| \\ &+\big||a_{n-r-1}| - |a_{n-r-2}|\big| \dots \big||a_{\mu+1}| - |a_\mu|\big|\Big)\cos\alpha + \Big(k_{\mu-1}|a_n| + k_\mu|a_{n-1}| + k_\mu|a_{n-1}| \\ &+k_{\mu+1}|a_{n-2}| + \dots + k_{\mu+r-2}|a_{n-r+1}| + k_{\mu+r-1}|a_{n-r}| + k_{\mu+r-1}|a_{n-r}| + |a_{n-r-1}| + |a_{n-r-1}| \\ &+|a_{n-r-2}| + \dots + |a_{\mu+1}| + |a_\mu|\Big)\sin\alpha - (k_\mu - 1)|a_{n-1}| \\ &+2\sum_{j=\mu}^{\mu+r-1}(k_j - 1)|a_{n+\mu-j-1}| + |a_\mu| + 2|a_0|\Bigg]\Bigg\},\end{aligned}$$

which in view of (1), yields,

$$\begin{aligned}\big|F(z)\big| &\geq |a_n||z|^n\Bigg\{\Big|(z + k_{\mu-1} - 1) - (k_\mu - 1)\frac{a_{n-1}}{a_n}\Big| - \frac{1}{a_n}\Bigg[(k_{\mu-1}|a_n| - |a_\mu|)(\cos\alpha + \sin\alpha) \\ &+2\sin\alpha\left(\sum_{j=\mu}^{\mu+r-1}k_j|a_{n+\mu-j-1}| + \sum_{j=\mu+r}^{n-1}|a_{n+\mu-j-1}|\right)\end{aligned}$$

$$
\left. -(k_\mu - 1)|a_{n-1}| + 2\sum_{j=\mu}^{\mu+r-1}(k_j-1)|a_{n+\mu-j-1}| + |a_\mu| + 2|a_0|\Big]\right\} > 0
$$

if

$$
\begin{aligned}
\left|(z+k_{\mu-1}-1)-(k_\mu-1)\frac{a_{n-1}}{a_n}\right| > \frac{1}{a_n}\Bigg[&(k_{\mu-1}|a_n|-|a_\mu|)(\cos\alpha+\sin\alpha)\\
&+2\sin\alpha\left(\sum_{j=\mu}^{\mu+r-1}k_j|a_{n+\mu-j-1}| + \sum_{j=\mu+r}^{n-1}|a_{n+\mu-j-1}|\right)\\
&-(k_\mu-1)|a_{n-1}| + 2\sum_{j=\mu}^{\mu+r-1}(k_j-1)|a_{n+\mu-j-1}| + |a_\mu| + 2|a_0|\Bigg].
\end{aligned}
$$

This shows that those zeros of $F(z)$ whose modulus is greater than 1 lie in

$$
\begin{aligned}
\left|(z+k_{\mu-1}-1)-(k_\mu-1)\frac{a_{n-1}}{a_n}\right| \le \frac{1}{a_n}\Bigg[&(k_{\mu-1}|a_n|-|a_\mu|)(\cos\alpha+\sin\alpha)\\
&+2\sin\alpha\left(\sum_{j=\mu}^{\mu+r-1}k_j|a_{n+\mu-j-1}| + \sum_{j=\mu+r}^{n-1}|a_{n+\mu-j-1}|\right)\\
&-(k_\mu-1)|a_{n-1}| + 2\sum_{j=\mu}^{\mu+r-1}(k_j-1)|a_{n+\mu-j-1}| + |a_\mu| + 2|a_0|\Bigg].
\end{aligned}
$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already lie in this region. Hence it follows that all the zeros $F(z)$ and therefore of $P(z)$ lie in

$$
\begin{aligned}
\left|(z+k_{\mu-1}-1)-(k_\mu-1)\frac{a_{n-1}}{a_n}\right| \le \frac{1}{a_n}\Bigg[&(k_{\mu-1}|a_n|-|a_\mu|)(\cos\alpha+\sin\alpha)\\
&+2\sin\alpha\left(\sum_{j=\mu}^{\mu+r-1}k_j|a_{n+\mu-j-1}| + \sum_{j=\mu+r}^{n-1}|a_{n+\mu-j-1}|\right)\\
&-(k_\mu-1)|a_{n-1}| + 2\sum_{j=\mu}^{\mu+r-1}(k_j-1)|a_{n+\mu-j-1}| + |a_\mu| + 2|a_0|\Bigg].
\end{aligned}
$$

This completes the proof of Theorem 9. ■

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