Applied Mathematics E-Notes, 23(2023), 237-242 ©
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ISSN 1607-2510

Some Inequalities Related To The Power Exponential Function $a^{rb} + b^{ra}$*

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Received 20 April 2022

Abstract

In this paper, we consider a and b two nonnegative real numbers such that $a + b = 2$ and prove that the inequality $a^{rb} + b^{ra} \le 2 - ((a-b)/2)^2$ holds for $1/2 \le r \le 2$ and also prove that the reverse inequality holds for $0 < r \le 1/3$. In addition, two new conjectures are presented.

1 Introduction

Researches of the inequalities with the power exponential functions are one of the areas that have been actively studied in recent years. Many of the inequalities are different from the simplicity of appearance, and the proof is complicated. In [2], Coronel and Huancas describe the history of the inequalities, including many interesting power exponential functions, and their literature reviews (see also [1] and [12]). In this paper, we consider the inequalities including power exponential function $a^{rb} + b^{ra}$, and such inequalities appeared in [3] and [4], where Cîrtoaje proved some results about the inequality $a^{ra} + b^{rb} \ge a^{rb} + b^{ra}$, for positive real numbers a, b and r. Cîrtoaje, in [3] proved the following theorem.

Theorem 1 *(i) Let a, b and r be positive real numbers. If $a^{ra} + b^{rb} \ge a^{rb} + b^{ra}$ holds for $r = r_0$, then it holds for any $0 < r \le r_0$.*

(ii) If a and b are positive real numbers such that $\max\{a, b\} \ge 1$, then the inequality $a^{ra} + b^{rb} \ge a^{rb} + b^{ra}$ holds for any positive real number r.

(iii) If $0 < r \le 2$, then the inequality $a^{ra} + b^{rb} \ge a^{rb} + b^{ra}$ holds for all positive real numbers a and b.

(iv) If a and b are positive real numbers such that either $a \ge b \ge \frac{1}{r}$ or $\frac{1}{r} \ge a \ge b$, then the inequality $a^{ra} + b^{rb} \ge a^{rb} + b^{ra}$ holds for any positive real number $r \le e$.

(v) If $r > e$, then the inequality $a^{ra} + b^{rb} \ge a^{rb} + b^{ra}$ does not hold for all positive real numbers a and b.

A few years later, in [4], Cîrtoaje proved the following theorem.

Theorem 2 *If a and b are positive real numbers such that $0 < b \le \frac{1}{e} \le a \le 1$, then the inequality $a^{ra} + b^{rb} \ge a^{rb} + b^{ra}$ holds.*

From Theorems 1 and 2, Cîrtoaje provided a complete proof of the inequality $a^{ra} + b^{rb} \ge a^{rb} + b^{ra}$. Also, in [3] and [4], Cîrtoaje posted some conjectures on the inequality with power exponential function $a^{rb} + b^{ra}$ as follows.

Theorem 3 *Let r be a positive real number. The inequality $a^{rb} + b^{ra} \le 2$ holds for all nonnegative real numbers a and b with $a + b = 2$ if and only if $r \le 3$.*

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*Mathematics Subject Classifications: 26D07, 26D10, 26D20.

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Theorem 4 *If $a, b \in (0,1]$ and $r \in (0,e]$, then the inequality $2\sqrt{a^{ra}b^{rb}} \geq a^{rb} + b^{ra}$ holds.*

Theorem 3 is Conjecture 4.6 in [3] and Theorem 4 is Conjecture 2.1 in [4], and both theorems have been proved by Matejíčka [5], [6] and [7]. An extension of Theorem 3 for the case of $r < 0$ was recently introduced in [1]. The following theorems are known for inequalities with the power exponential function $a^{rb} + b^{ra}$ when r is a constant. For $r = 1$ Nishizawa [11] proved the following.

Theorem 5 *If a and b are nonnegative real numbers with $a + b = 2$, then the inequality*

$$2 - \left(\frac{|a-b|}{2}\right)^{\alpha} \leq a^b + b^a \leq 2 - \left(\frac{|a-b|}{2}\right)^{\beta}$$

holds, where the constants $\alpha = \ln 2 \cong 0.693147$ and $\beta = 2$ are the best possible.

For $r = 2$ the following inequalities are known.

Theorem 6 *If a and b are nonnegative real numbers such that $a + b = 1$, then the inequality $a^{2b} + b^{2a} \leq 1$ holds.*

Theorem 7 *If a and b are nonnegative real numbers with $a + b = c$, then the inequality $a^{2b} + b^{2a} \leq 1$ holds for $1/2 \leq c \leq 1$.*

Theorem 8 *If a and b are nonnegative real numbers such that $a + b = 1$, then the inequality $a^{2b} + b^{2a} > 6083/6144 \cong 0.990072$ holds.*

Theorem 9 *If a and b are nonnegative real numbers with $a + b = 2$, then the inequality*

$$a^{2b} + b^{2a} \leq 2 - \left(\frac{a-b}{2}\right)^2$$

holds.

Theorem 6 is Conjecture 4.8 in [3] and have been proved by Cîrtoaje [4] and Theorems 7 and 8 have been proved by Nishizawa [9] and [10]. Theorem 9 is Proposition 4.5 in [3]. If $r = 3$, then Miyagi and Nishizawa [8] proved the following result, which is Conjecture 4.7 in [3].

Theorem 10 *If a and b are nonnegative real numbers such that $a + b = 2$, then the inequality*

$$a^{3b} + b^{3a} \leq 2 - \left(\frac{a-b}{2}\right)^4$$

holds.

In this paper, we describe an inequality that compares the power exponential function $a^{rb} + b^{ra}$ and $\left(\frac{a-b}{2}\right)^2$. Our main results are the following.

Theorem 11 *If a and b are nonnegative real numbers with $a + b = 2$, then the inequality*

$$a^{rb} + b^{ra} \geq 2 - \left(\frac{a-b}{2}\right)^2$$

holds for $0 < r \leq \frac{1}{3}$.

Theorem 12 *If a and b are nonnegative real numbers with $a + b = 2$, then the inequality*

$$a^{rb} + b^{ra} \leq 2 - \left(\frac{a-b}{2}\right)^2$$

holds for $\frac{1}{2} \leq r \leq 2$.

2 Proof of Theorem 11 and Theorem 12

We will give a sequence of lemmas needed to prove Theorem 11.

Lemma 1 *For $0 < t < 1$, we have*

$$(1-t)^{-1+t} > (1+t)^{-1-t}.$$

Proof. We set

$$f(t) = \ln(1-t)^{-1+t} - \ln(1+t)^{-1-t} = (-1+t)\ln(1-t) - (-1-t)\ln(1+t)$$

and reduce the proof to get that $f(t) > 0$ on $(0,1)$. We observe that the derivatives of f are given by

$$f'(t) = 2 + \ln(1-t) + \ln(1+t)$$

and

$$f''(t) = -\frac{2t}{(1-t)(1+t)} < 0.$$

Hence f is concave on $(0,1)$ or equivalently $f(t) > (f(1) - f(0))t + f(0) = (2\ln 2)\,t > 0$ for all $t \in (0,1)$, since $f(0) = 0$ and $f(1) = \lim_{t\to 1-0} f(t) = 2\ln 2 \cong 1.38629$. ∎

Lemma 2 *For $0 < t < 1$, we have*

$$(1-t)^{\frac{1}{3}(1+t)} > (1-t)\left(1 + \frac{2}{3}t + \frac{2}{9}t^2 + \frac{t^3}{18}\right).$$

Proof. We set

$$\begin{aligned} f(t) &= \ln(1-t)^{\frac{1}{3}(1+t)} - \ln\left((1-t)\left(1 + \frac{2}{3}t + \frac{2}{9}t^2 + \frac{t^3}{18}\right)\right) \\ &= \frac{1+t}{3}\ln(1-t) - \ln(1-t) - \ln\left(1 + \frac{2}{3}t + \frac{2}{9}t^2 + \frac{t^3}{18}\right) \end{aligned}$$

and the derivatives of $f(t)$ are

$$f'(t) = \frac{1}{3}\ln(1-t) + \frac{t(-18 - 11t - 7t^2 + t^3)}{3(t-1)(18 + 12t + 4t^2 + t^3)}$$

and

$$f''(t) = \frac{t(288 + 600t + 108t^2 + 141t^3 + 70t^4 + 17t^5 + t^6)}{3(t-1)^2(18 + 12t + 4t^2 + t^3)^2}.$$

We note that $f''(t) > 0$ for $0 < t < 1$. Thus, $f'(t)$ is strictly increasing for $0 < t < 1$ and $f'(t) > f'(0) = 0$. From $f(t)$ is strictly increasing for $0 < t < 1$ and $f(t) > f(0) = 0$, we have $f(t) > 0$ for $0 < t < 1$. ∎

Lemma 3 *For $0 < t < 1$, we have*

$$(1+t)^{\frac{1}{3}(1-t)} > 1 + \frac{t}{3} - \frac{4}{9}t^2 + \frac{t^3}{18} + \frac{t^4}{18}.$$

Proof. By Taylor expansion, if $0 < t < 1$ and $0 < p < 1$, then there is u in $(0,t)$ such that we have

$$(1+t)^p = 1 + pt + \frac{1}{2}p(p-1)t^2(1+u)^{p-2} \geq 1 + pt + \frac{1}{2}p(p-1)t^2\,.$$

Therefore, the following inequality holds.

$$(1+t)^{\frac{1}{3}(1-t)} \geq 1 + \frac{t}{3}(1-t) - \frac{t^2}{18}(1-t)(2+t) = 1 + \frac{t}{3} - \frac{4}{9}t^2 + \frac{t^3}{18} + \frac{t^4}{18}.$$

■

Proof of Theorem 11. If $(a, b) = (1, 1)$, $(0, 2)$, $(2, 0)$, then the equality occurs, so we consider the case of $0 < a < 1 < b < 2$. Without loss of generality, we may assume that $a = 1 - t$ and $b = 1 + t$, then we have

$$a^{rb} + b^{ra} = (1-t)^{r(1+t)} + (1+t)^{r(1-t)} = f(r, t),$$

where $0 < t < 1$. The derivatives of $f(r, t)$ by r are

$$\frac{\partial f}{\partial r}(r, t) = (1-t)^{r(1+t)}(1+t)\ln(1-t) + (1-t)(1+t)^{r(1-t)}\ln(1+t)$$

and

$$\frac{\partial^2 f}{\partial r^2}(r, t) = (1-t)^{r(1+t)}(1+t)^2\left(\ln(1-t)\right)^2 + (1-t)^2(1+t)^{r(1-t)}\left(\ln(1+t)\right)^2.$$

It is clear that $\frac{\partial^2 f}{\partial r^2}(r, t)$ is positive for any positive r and any $t \in (0, 1)$. Therefore, $\frac{\partial f}{\partial r}(r, t)$ is strictly increasing for $r > 0$ and we have

$$\begin{aligned}\frac{\partial f}{\partial r}(r, t) \leq \frac{\partial f}{\partial r}\left(\frac{1}{2}, t\right) &= (1-t)^{\frac{1}{2}(1+t)}(1+t)\ln(1-t) + (1-t)(1+t)^{\frac{1}{2}(1-t)}\ln(1+t) \\ &= (1-t)(1+t)\left((1-t)^{\frac{1}{2}(-1+t)}\ln(1-t) + (1+t)^{\frac{1}{2}(-1-t)}\ln(1+t)\right)\end{aligned}$$

for $0 < r < \frac{1}{2}$. From Lemma 1 and $0 < \ln(1+t) < t$ and $\ln(1-t) < -t < 0$ for $0 < t < 1$, we have

$$\frac{\partial f}{\partial r}(r, t) \leq \frac{\partial f}{\partial r}\left(\frac{1}{2}, t\right) \leq (1-t)(1+t)t\left(-(1-t)^{\frac{-1+t}{2}} + (1+t)^{\frac{-1-t}{2}}\right) < 0.$$

Thus, $f(r, t)$ is strictly decreasing for $0 < r < \frac{1}{2}$. By Lemmas 2 and 3, we have

$$\begin{aligned}f(r, t) \geq f\left(\frac{1}{3}, t\right) &= (1-t)^{\frac{1}{3}(1+t)} + (1+t)^{\frac{1}{3}(1-t)} \\ &\geq (1-t)\left(1 + \frac{2}{3}t + \frac{2}{9}t^2 + \frac{t^3}{18}\right) + 1 + \frac{t}{3} - \frac{4}{9}t^2 + \frac{t^3}{18} + \frac{t^4}{18} \\ &= 2 - \frac{8}{9}t^2 - \frac{t^3}{9} > 2 - \frac{8}{9}t^2 - \frac{t^2}{9} = 2 - t^2.\end{aligned}$$

Thus, we obtain $f(r, t) \geq 0$ for $0 < r \leq \frac{1}{3}$ and $0 < t < 1$. Therefore, the proof of Theorem 11 is complete. ■

Proof of Theorem 12. If $(a, b) = (1, 1)$, $(0, 2)$, $(2, 0)$, then the equality occurs, so we consider the case of $0 < a < 1 < b < 2$. Without loss of generality, we may assume that $a = 1 - t$ and $b = 1 + t$, then we have $a^{rb} + b^{ra} = (1-t)^{r(1+t)} + (1+t)^{r(1-t)} = f(r, t)$, where $0 < t < 1$. The derivatives of $f(r, t)$ by r are

$$\frac{\partial f}{\partial r}(r, t) = (1-t)^{r(1+t)}(1+t)\ln(1-t) + (1-t)(1+t)^{r(1-t)}\ln(1+t)$$

and

$$\frac{\partial^2 f}{\partial r^2}(r, t) = (1-t)^{r(1+t)}(1+t)^2\left(\ln(1-t)\right)^2 + (1-t)^2(1+t)^{r(1-t)}\left(\ln(1+t)\right)^2.$$

We observe that $\frac{\partial^2 f}{\partial r^2}(r, t)$ is positive for any positive r and any $t \in (0, 1)$. Therefore, for fixed t in $(0, 1)$, $f(r, t)$ is convex. From $(1+t)^p \leq 1 + pt$ and $(1-t)^p \leq 1 - pt$ for $0 < t \leq 1$ and $0 \leq p \leq 1$, we have

$$f\left(\frac{1}{2}, t\right) = (1-t)^{\frac{1}{2}(1+t)} + (1+t)^{\frac{1}{2}(1-t)} \leq 1 - \frac{1}{2}t(1+t) + 1 + \frac{1}{2}t(1-t) = 2 - t^2.$$

Also, by Theorem 9, we have $f(2, t) \leq 2 - t^2$. Therefore, we obtain $f(r, t) \leq 2 - t^2$ for $0 < t < 1$ and $\frac{1}{2} \leq r \leq 2$. Hence, the proof of Theorem 12 is complete. ■

3 Conjectures

We will present two conjectures regarding the inequalities with power exponential function $a^{rb} + b^{ra}$. Thus, for a positive real number r close to 3 and greater than 3, the same inequality as that in Theorem 11 may hold. It is also known that Theorem 10 holds for $r = 3$. The same inequality may hold not only for $r = 3$, but also for an interval of r that contains 3. Based on careful analysing of the two-variable functions

$$(a, r) \mapsto a^{r(2-a)} + (2-a)^{ra} - 2 + (a-1)^2$$

and

$$(a, r) \mapsto a^{r(2-a)} + (2-a)^{ra} - 2 + (a-1)^4$$

running over Maple Software (version 2021.0) we guess that:

Conjecture 1 *If a and b are nonnegative real numbers with $a + b = 2$, then the inequality*

$$a^{rb} + b^{ra} \geq 2 - \left(\frac{a-b}{2}\right)^2$$

holds for $r \geq \frac{2}{\ln 2} \cong 2.88539$.

Conjecture 2 *If a and b are nonnegative real numbers with $a + b = 2$, then the inequality*

$$a^{rb} + b^{ra} \leq 2 - \left(\frac{a-b}{2}\right)^4$$

holds for $\frac{1}{2} \leq r \leq 3$.

Acknowledgment. We would like to thank the referees for careful reading of the manuscript and for the constructive suggestions.

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