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Growth Of Entire Solutions Of Non-Linear Differential Equations*

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Abstract

This paper aims to study the growth of entire solutions of higher-order non-linear differential equations. The result obtained here extends and improves the previous results of the current authors [5].

1 Introduction

We shall assume that the reader is familiar with the fundamental results and the standard notations of the theory of entire functions, see [8] for more details. Let f be an entire function. Then its order $\sigma(f)$ and lower order $\lambda(f)$ are given by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r},$$

and

$$\lambda(f) = \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}$$

respectively, where $M(r,f)$ is the maximum modulus function of f on the circle $|z| = r$. In the sequel, we also need the following definitions.

Definition 1 ([9]) *Let f be a non-constant entire function. The hyper-order $\sigma_1(f)$ of f is defined as follows*

$$\sigma_1(f) = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r}.$$

Definition 2 *Let f be an entire function. The expression*

$$M_j[f] = \prod_{i=0}^{k} \left(f^{(i)}\right)^{n_{ij}},$$

where n_{ij} $(i = 0, 1, ..., k)$ are non-negative integers, is called a differential monomial generated by f of degree

$$\gamma_{M_j} = \sum_{i=0}^{k} n_{ij}$$

and weight

$$\Gamma_{M_j} = \sum_{i=0}^{k} (i+1)n_{ij}.$$

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The sum $P[f] = \sum\limits_{j=1}^{t} Q_j(z) M_j[f]$ is called a differential polynomial generated by f of degree

$$d = \max\left\{\gamma_{M_j} : 1 \leq j \leq t\right\},$$

where $Q_j(z) \not\equiv 0$ and $T(r, Q_j) = S(r, f)$ for $j = 1, 2, ..., t$.

The number k (the highest order of the derivative of f in $P[f]$) is called the order of $P[f]$. $P[f]$ is linear if $d = 1$. Otherwise, it is non-linear. Also, we denote by

$$\chi = \max\left\{\Gamma_{M_j} - \gamma_{M_j} : 1 \leq j \leq t\right\} = \max\left\{\sum_{i=1}^{k} i.n_{ij} : 1 \leq j \leq t\right\}.$$

In the study of solutions of complex differential equations, the order of growth of solutions is an important property. Li and Cao [5] proved the following result on the order of growth of entire solutions of a linear differential equation.

Theorem 1 ([5]) *Let $\beta_1 = \beta_1(z)$ and $\beta_2 = \beta_2(z)$ be two non-zero polynomials and let $\phi = \phi(z)$ be a polynomial. If f is a non-constant entire solution of the equation*

$$f^{(k)} - \beta_1 = (f - \beta_2) e^{\phi},$$

where k is a positive integer, then $\sigma_1(f) = \deg \phi$.

For $k \geq 2$, we consider the expression

$$L[f] = f^{(k)} + a_{k-1}(z) f^{(k-1)} + ... + a_1(z) f' + a_0(z) f, \tag{1}$$

where $a_0(z), a_1(z), ..., a_{k-1}(z)$ are polynomials in z.

Regarding the solutions of linear differential equations, Xu and Yang [10] obtained the following theorem.

Theorem 2 ([10]) *Let $f(z)$ and $\alpha(z)$ be two non-constant entire functions satisfying $\sigma(\alpha) < \lambda(f)$. Let $L(f)$ be defined by (1) and $\phi(z)$ be a polynomial in z such that*

$$\sigma(f) > \deg \phi + \max_{0 \leq j \leq k-1}\left\{\frac{\deg a_j - \deg a_k}{k - j}, 0\right\}.$$

If f is a non-constant entire solution of the following differential equation

$$L[f] + \beta(z) - \alpha(z) = (f - \alpha) e^{\phi(z)},$$

where $\beta(z)$ is an entire function satisfying $\sigma(\beta) < \lambda(f)$, then $\sigma_1(f) = \deg \phi$.

One can ask the following question:

Question: What can be said about the growth of solutions if the linear differential polynomial in Theorem 2 is replaced by differential polynomial $P[f]$ of f?

In 2017, Pramanik and Biswas [7] answered the above question in the form of the following theorem, which extended the result of Xu and Yang [10].

Theorem 3 ([7]) *Let $f(z)$ and $\alpha(z)$ be two non-constant entire functions satisfying $\sigma(\alpha) < \lambda(f)$. Let $P[f]$ be a non-constant differential polynomial and $\phi(z)$ be a polynomial. If f is a non-constant entire solution of the following differential equation*

$$P[f] + \beta - \alpha = (f^d - \alpha) e^{\phi(z)}, \tag{2}$$

where $\beta(z)$ is an entire function satisfying $\sigma(\beta) < \lambda(f)$, then $\sigma_1(f) = \deg \phi$.

The factor $e^{\phi(z)}$ in (2) is a non-vanishing entire function. So an obvious question arises at this point.

Question: What can be said about the growth of solutions of the following differential equation

$$P[f] - \beta = (f^d - \alpha)Q, \tag{3}$$

where $Q(z)$ is any entire function?

In this paper, we investigate the above question and answer it in the following theorem:

Theorem 4 . *Let $f(z)$, $\alpha(z)$, $\beta(z)$ be non-constant entire functions satisfying $\max\{\sigma(\alpha), \sigma(\beta)\} < \lambda(f)$ and $Q(z)$ be any entire function. If f is a non-constant entire solution of (3), then either $\sigma_1(f) = 0$ or $\sigma_1(f) \leq \sigma(Q)$. Furthermore, if $\sigma(Q) < +\infty$, then either $\sigma_1(f) = 0$ or $\sigma_1(f) = \sigma(Q)$.*

Example 1 *Consider the differential equation*

$$ff' - \alpha = f^2 - \alpha,$$

where $\alpha(z)$ is any polynomial function. Comparing it with the equation (3), we see that $P[f] = ff'$ is of degree $d = 2$, $\beta(z) = \alpha(z)$ and $Q(z) = 1$. Obviously, for the entire function $f(z) = e^z$, the conditions $\sigma(\alpha) = 0 < 1 = \lambda(f)$ and $\sigma(\beta) = 0 < 1 = \lambda(f)$ hold and $f(z) = e^z$ with $\sigma_1(f) = 0$ is a solution of the given equation.

Example 2 *Consider the differential equation*

$$f' + f - 2e^z = (f - e^z)(e^z + 1).$$

Comparing it with the equation (3), we see that $P[f] = f' + f$ is of degree $d = 1$, $\beta(z) = -2e^z$, $\alpha(z) = e^z$ and $Q(z) = e^z + 1$. Obviously, for the entire function $f(z) = e^z + e^{e^z}$, the condition $\max\{\sigma(\alpha), \sigma(\beta)\} = 1 < \infty = \lambda(f)$ holds. One can easily verify that $f(z) = e^z + e^{e^z}$ with $\sigma_1(f) = 1 = \sigma(Q)$ is a solution of the given equation.

2 Preparatory Lemmas

We now recall a well-known definition. The linear measure of a set $E \subset [0, +\infty)$ is defined as $m(E) = \int_E dt$ and the logarithmic measure of a set $F \subset [1, +\infty)$ is defined by $lm(F) = \int_F \frac{dt}{t}$. Our proof depends mainly upon the following lemmas.

Lemma 1 **([3, 4])** *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function of finite order $\sigma(f) < +\infty$,*

$$\mu(r) = \max\left\{|a_k| r^k, k = 0, 1, ...\right\}$$

be the maximum term of f and

$$\nu(r, f) = \max\{m : \mu(r) = |a_m| r^m\}$$

be the central index of f. Then

$$\limsup_{r \to +\infty} \frac{\log \nu(r, f)}{\log r} = \sigma(f)$$

and if f is a transcendental entire function of hyper-order $\sigma_1(f)$, then

$$\limsup_{r \to +\infty} \frac{\log \log \nu(r, f)}{\log r} = \sigma_1(f).$$

Lemma 2 **([2, 4, 8])** *Let $f(z)$ be a transcendental entire function, $\nu(r,f)$ be the central index of $f(z)$. Then there exists a set $E \subset (1,+\infty)$ with finite logarithmic measure such that for z satisfying $|z| = r \notin [0,1] \cup E$ and $|f(z)| = M(r,f)$, we get*

$$\frac{f^{(i)}(z)}{f(z)} = \left\{ \frac{\nu(r,f)}{z} \right\}^i (1 + o(1)), \text{ for } i \in \mathbb{N}.$$

Lemma 3 **([1])** *Let $g(z)$ be a meromorphic function with finite order $\sigma(g)$. Then for any given $\epsilon > 0$, there exists a set $E \subset (1,+\infty)$ with $lmE < +\infty$ such that for all z with $|z| = r \notin ([0,1] \cup E)$,*

$$|g(z)| \le \exp\left\{ r^{\sigma(g)+\epsilon} \right\} \text{ as } r \to \infty.$$

Applying Lemma 3 to $\frac{1}{g(z)}$, it is clear that for any given $\varepsilon > 0$, there exists a set $E \subset (1,+\infty)$ with $lmE < +\infty$ such that for all z with $|z| = r \notin ([0,1] \cup E)$,

$$\exp\left\{ -r^{\sigma(g)+\epsilon} \right\} \le |g(z)| \le \exp\left\{ r^{\sigma(g)+\epsilon} \right\} \text{ as } r \to \infty. \tag{4}$$

Lemma 4 **([6])** *Let $f(z)$ be a transcendental entire function and let $E \subset [1,+\infty)$ be a set of finite logarithmic measure. Then there exists $\left\{ z_k = r_k e^{i\theta_k} \right\}$ such that $|f(z_k)| = M(r_k, f)$, $\theta_k \in [0, 2\pi)$, $\lim_{k\to+\infty} \theta_k = \theta_0 \in [0, 2\pi]$, $|z_k| = r_k \notin E$ and if $0 < \sigma(f) < +\infty$, then for any given $\varepsilon > 0$ and sufficiently large r_k,*

$$r_k^{\sigma(f)-\varepsilon} < \nu(r_k, f) < r_k^{\sigma(f)+\varepsilon}.$$

If $\sigma(f) = +\infty$, then for any given large $C > 0$ and sufficiently large r_k, $\nu(r_k, f) > r_k^C$.

3 The Proof of Theorem 4

We consider the following two cases. Case 1: $\sigma(f) < +\infty$ and Case 2: $\sigma(f) = +\infty$.

Case 1. Suppose $\sigma(f) < +\infty$. Then $\sigma_1(f) = 0$.

Case 2. Suppose $\sigma(f) = +\infty$. Now from (3), it follows that

$$\frac{\frac{P[f]}{f^d} - \frac{\beta}{f}\frac{1}{f^{d-1}}}{1 - \frac{\alpha}{f}\frac{1}{f^{d-1}}} = Q(z). \tag{5}$$

For each $j = 1, 2, ..., t$,

$$M_j[f] = f^{\left(\sum_{i=0}^{k} n_{ij}\right)} \prod_{i=1}^{k} \left(\frac{f^{(i)}}{f} \right)^{n_{ij}} = f^{\gamma_{M_j}} \prod_{i=1}^{k} \left(\frac{f^{(i)}}{f} \right)^{n_{ij}}.$$

By Lemma 2, there exists a subset $E_0 \subset (1,+\infty)$ with finite logarithmic measure such that for all z satisfying $r = |z| \notin E_1 = [0,1] \cup E_0$ and $|f(z)| = M(r,f)$, we get

$$\frac{f^{(i)}(z)}{f(z)} = \left\{ \frac{\nu(r,f)}{z} \right\}^i (1 + o(1)), \quad i = 1, 2, ..., k.$$

Thus it follows that

$$\frac{M_j[f]}{f^{\gamma_{M_j}}} = \prod_{i=1}^{k} \left\{ \frac{\nu(r,f)}{z} \right\}^{i.n_{ij}} (1 + o(1)) = \left\{ \frac{\nu(r,f)}{z} \right\}^{\left(\sum_{i=1}^{k} i.n_{ij}\right)} (1 + o(1))$$

and

$$\left|\frac{P[f]}{f^d}\right| \leq \sum_{j=1}^{t}|Q_j|\left|\frac{M_j[f]}{f^d}\right| \leq \sum_{j=1}^{t}|Q_j|\left|\frac{M_j[f]}{f^{\gamma_{M_j}}}\right| = \sum_{j=1}^{t}|Q_j|\left|\frac{\nu(r,f)}{z}\right|^{\left(\sum_{i=1}^{k} i.n_{ij}\right)}(1+o(1)). \tag{6}$$

Since $\sigma(f) = +\infty$, it follows from Lemma 4 that there exists an infinite sequence $\left\{z_k = r_k e^{i\theta_k}\right\}_{k=1}^{+\infty}$ with $|f(z_k)| = M(r_k, f)$, $\theta_k \in [0, 2\pi)$, $\lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi]$, $|z_k| = r_k \notin E_2 \subset [1, +\infty)$ such that for any large constant $C > 0$ and for sufficiently large r_k, we have

$$\nu(r_k, f) > r_k^C$$

and so from (6) we have

$$\left|\frac{P[f]}{f^d}\right| \leq \left|\frac{\nu(r_k,f)}{z_k}\right|^{\chi}\left(\sum_{j=1}^{t}|Q_j|\right)(1+o(1)). \tag{7}$$

Since $\max\{\sigma(\alpha), \sigma(\beta)\} < \lambda(f)$, from the definitions of order and lower order, it follows for sufficiently large values of r_k that

$$\frac{|\alpha(z_k)|}{|f(z_k)|} \to 0 \text{ and } \frac{|\beta(z_k)|}{|f(z_k)|} \to 0. \tag{8}$$

Thus from (5), (7) and (8) we obtain that

$$\left\{\frac{\nu(r_k,f)}{r_k}\right\}^{\chi} \cdot \left(\sum_{j=1}^{t}|Q_j|\right)(1+o(1)) = |Q(z_k)| \tag{9}$$

for sufficiently large values of r_k. From (9) it follows that

$$M(r_k, Q(z_k)) \geq \left\{\frac{\nu(r_k,f)}{r_k}\right\}^{\chi}\left(\sum_{j=1}^{t}|Q_j|\right)$$

$$\Rightarrow \{\nu(r_k,f)\}^{\chi} \leq \left(\sum_{j=1}^{t}|Q_j|\right)^{-1}.(r_k)^{\chi}.M(r_k, Q(z_k)). \tag{10}$$

Thus from (10) and Lemma 1 we get

$$\begin{aligned}\sigma_1(f) &= \limsup_{r_k\to+\infty}\frac{\log\log\nu(r_k,f)}{\log r_k}\\ &\leq \limsup_{r_k\to+\infty}\frac{\log\log\left\{\left(\sum_{j=1}^{t}|Q_j|\right)^{-1}(r_k)^{\chi}M(r_k,Q(z_k)).\right\}}{\log r_k}\end{aligned}$$

$$\Rightarrow \sigma_1(f) \leq \sigma(Q). \tag{11}$$

Now if $\sigma(Q) < +\infty$, then it follows from (4) that for any given $\varepsilon > 0$, there exists a set $E_3 \subset (1, +\infty)$ with $lmE_3 < +\infty$ such that for all z with $|z| = r \notin [0,1] \cup E_3$

$$\exp\left\{-r^{\sigma(Q)+\epsilon}\right\} \leq |Q(z)| \leq \exp\left\{r^{\sigma(Q)+\epsilon}\right\} \text{ as } r \to \infty. \tag{12}$$

Now from (9) and (12) we get

$$\begin{aligned}(\sigma(Q)+\epsilon)\log\log r_k &\leq \log\left|\log\left|Q(z_k)\right|\right| \\ &\leq \log\log\nu(r_k,f)+\log\log\left(\sum_{j=1}^{t}|Q_j|\right)+\log\log r_k+O(1) \\ &\leq \log\log\nu(r_k,f)+O(\log\log r_k). \end{aligned} \tag{13}$$

Since $\epsilon > 0$ is arbitrary, from (12) and Lemma 1 we obtain

$$\sigma(Q) \leq \sigma_1(f). \tag{14}$$

Combining (11) and (14) we get

$$\sigma(Q) = \sigma_1(f).$$

This completes the proof of the theorem.

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