

A New Model For Pricing The Options In Islamic Finance*

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Abstract

Islamic banking and finance are increasingly attracting attention among investors and researchers worldwide. However, there is a paucity of the studies which have focused on Islamic derivatives. These Shariah compliant derivatives were created in order to enable Muslims to invest in international markets and hedge risks according to their beliefs. In this paper, we bridge the gap in the financial literature by proposing a new mathematical model for pricing the "urbun" in Islamic finance (used as a call option), taking into consideration its similarities and differences with conventional options. Also we provide many numerical examples in order to illustrate the model's solutions.

1 Introduction

Options are one of the most important financial derivatives in the financial market. They are widely used and their pricing and volatility forecasting are considered as an issue for both theoreticians and practitioners. For this reason, they have been widely studied in various fields such as finance, mathematics, etc.

Due to their use in hedging risks (see [34]), options have been priced using many models. All of these models assume a constant and positive risk-free interest rate (see [8, 19]). Taking into consideration the fact that the nominal interest rate dropped below zero percent for many countries during the last decade, some researches focused on models which allow for negative interest rates (see [27]).

Islamic Finance is based on a set of principles referred to as Shariah (see [12]), the most important among these principles is the prohibition of *riba*. The latter is commonly assimilated to interest-rate, even if some differences exist at the conceptual level, and some researchers argue that it is suitable to consider Islamic finance as *riba*-free instead of interest-free (see [33]). Hence, many products and financial tools were created, such as funds and indices (see [32]), using many Shariah screening criteria (see [14]).

From this point of view, derivatives are not compliant with Shariah principles as many Shariah scholars provide key objections and challenge the permissibility of derivatives under Islamic law (see [21]). Hence, all financial transactions must be directly linked to a real underlying economic transaction or asset, thus options and the other derivatives are banned (see [22]). Taking into consideration this restriction, Shariah scholars, jurists and financial authorities tried to suggest alternatives and structured products in order to enable Muslims to invest and according to their religious beliefs. Also Islamic finance has developed certain tools to hedge against some inevitable excessive market risks (see [2]).

In Islamic finance, down-payment can be given by a potential buyer to a potential seller in order to purchase a specific property. This down-payment sale is known also as "urbun". Hence, two cases could be distinguished (see [12]): If the buyer decides to complete the sale, the urbun is counted toward the total price. Otherwise (if the buyer does not execute the sale) he should forfeit the down-payment. In other words, the buyer has the right, but not the obligation, to buy a property, at a specified price within a specific time

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period. These characteristics meet to some extent those of call options in conventional finance (see [16]). However, *urbun* should be used for hedging real transactions (see [18]), not for speculation or intangible underlying assets such as currency, stock indices or interest rates (see [3, 1]).

Innovation is a necessary condition for the survival of any industry. Islamic finance is no exception (see [21]). Given the fact that financial markets are becoming more and more sophisticated, the question here is whether Islamic structured products are a real innovation or a mere replication of conventional ones (see [10]).

Unlike the widely studied conventional options (see [30, 31, 13]), "*urbun*" has not received high level of scrutiny. Most papers provided a preliminary framework for structuring "*urbun*" in Muslim stock markets and analyze these products from a conceptual point of view (see [10, 18, 15, 17]).

The contribution of our article is twofold. First, to bridge the gap in the financial literature studies by completing the scarce literature focused on Shariah-compliant options. Second, to provide a comprehensive modeling of "*urbun*", taking into consideration its similarities and differences with call-options.

The rest of the paper is organized as follows: Section 2 presents the Islamic option and securitization. The mathematical model and governing equations are presented in section 3. Section 4 displays residual power series method. We present some numerical illustrations in MATLAB in section 5. Section 6 gives the results and discussion. Finally, we present the proof of result in section 7.

2 Islamic Option and Securitization

2.1 Islamic Securitization

The securitization could be defined as: The process of packaging financial promises and transforming them into a form whereby they can be freely transferred among a multitude of investors (see [9]). It consists of pooling together a various types of debt instruments and/or illiquid assets, and transforming them into liquid and negotiable bonds. These bonds could be either Asset Backed securities (ABS) or Collateralized Debt Obligations (CDO). This ability to package assets is deemed to add values for the involved parties and enhance the market liquidity.

In Islamic finance, many religious constraints limit the number of Shariah-compliant instruments (see [21]). This is why all the debt based securities are not authorized from a Shariah point of view. The asset securitization is also constrained by the fact that not all Islamic scholars are in agreement about the permissibility of some instruments, thus some differences of interpretation exist from a country to another (see [22]). Under a set of Shariah principles, a new form of Islamic securities called *sukuk* arises. These contracts are, in general, based on an entrepreneur and a financier taking a business risk together and sharing the profits and losses (see [28]). The profit-and-loss sharing principle of Islamic finance where banks share the profits and bear losses (*Mudarabah*) or share both profits and losses (*Musharaka*) with the firm, could lead to some less risky investments (see [6]).

Despite the new terminologies related to the securitization of assets, the Islamic instruments could be traceable to a number of old practices in very rudimentary forms (see [11]). The modern forms of Islamic debt securitization in contemporary finance started in the early 1990's. Hence, Malaysia was among the first countries to issue Islamic securitized instruments, and the financial markets saw the issuance of the first corporate and sovereign *sukuk*. Since their inception, the issuance of *Sukuk* continued its growth trend, and its issuer base is broadening, and has started to gain popularity with new issuances in Africa, America, East Asia and Europe.

Structuring *sukuk* is, to some extent, similar to structuring conventional ABS. However, the underlying asset should be tangible as well as Shariah compliant (not prohibited sectors). There are many ways to structure *sukuk* (*mudaraba*, *mucharaka*, *salam*, *istisnaa*, etc.), but the lease-based securitization, known as *ijarah*, is the most popular vehicle in Islamic finance (see [12]). The design of the security consists of a securitization process in which an unaffiliated SPV (Special Purpose Vehicle) is set up to acquire the assets of the borrower and to issue financial claims by investors on the assets (see [28]).

The securitization process of an *ijarah sukuk* could be described as follows (see [11]): it starts with

the identification of suitable underlying asset, then the originator sell the identified asset to the SPV, and this SPV will enter into a lease contract with investors, which creates a stream of income in the form of rental payment in favor of the SPV. From an Islamic legal perspective, the amount of sukuk issuance is limited to the value of assets held by the SPV, and the buyers of sukuk (sukuk holders) are co-owners of the leased asset because they effectively bought a portion of this tangible asset. Thus, the sukuk could be described as undivided proportionate ownership of the underlying asset. At the maturity of the sukuk which is also the end of the lease period, the originator will redeem the sukuk from the holders, buying back the underlying asset from them, at whatever price that the parties agree to. In addition to the Islamic asset-based securities, Islamic equity-based securities have emerged since the early 2000's, as a new form of Shariah-compliant securitization. This reflects a new milestone of Islamic securities market.

Despite a series of economic challenges, such as low energy prices and geopolitical conflicts, the sukuk market has been growing rapidly, with a double-digit asset growth rate over the past decade, from approximately US\$ 97.3 billion in 2009 to an overall total value of US\$ 775.7 billion by the end of 2021. Thus, the sukuk market represents 25.4% of the global Islamic financial Industry (see [20]).

2.2 Islamic Options

Derivatives in Islamic finance are not as sophisticated as in mainstream finance. This is also due to many restrictions and religious constraints, mainly because Shariah bans interest-based activities and excessive speculations (known as *gharar*). Hence, in Islamic equity-based contracts, lenders and borrowers agree to share any gains of profitable projects based on the degree of funding or ownership by each party. In Islamic asset-based contracts, the borrower leases from the lender one or more assets which have previously been acquired from either the borrower or a third party (see [21]).

In order to enable Muslims to invest in financial markets and hedge risks according to their beliefs, the shariah requirements should be met, some ethics should be respected (see [26]) and therefore some alternative Shariah-compliant products were created. Hence, according to ([2]), Islamic finance has developed certain tools to hedge against some inevitable excessive market risks. These tools include unilateral binding promises, which are known as "Islamic derivatives" (see [7]). Obviously, Islamic derivatives could not function in the same way as conventional ones, but there are actually some similar features (see [16, 7]). *Urbun* is considered as one of these tools, which consists of a down-payment sale in which the buyer deposits earnest money with the seller as part payment of the price in advance, and accept to forfeit the deposit money in the case he does not buy as agreed (see [24]). In other words, *urbun* is a contract which entitles the buyer to the right to buy an asset but does not obligate the buyer to do so. If the buyer does in fact purchase the asset, the *urbun* becomes part of the purchase price, and if the buyer does not purchase the asset, the seller retains the *urbun* (see [23]).

For the above-mentioned reasons, many researchers suggest using *urbun* as a call option (see [4, 25, 12, 23]), even if others consider that options are more flexible than *urbun* (see [24]), hence we should mention that *urbun* is not tradable according to Standard 53 of AAOIFI (see [1]). The basis for the impermissibility of taking consideration for sale or transfer of put or call options is resolution No (63) 7/1 of the International Islamic Fiqh Academy (IIFA), which is based on recognised evidences. Unlike the widely studied conventional options, *urbun* has not received high level of scrutiny. Our literature review shows that the subject is only documented in terms of legal aspects, and the previous researches focused on conceptual analysis of these products. In order to provide an in depth analysis, we suggest providing a mathematical modelling of *urbun*, as a call option.

Hence, the contribution of our article is twofold. First, we aim to complete the literature focused on Islamic derivatives. Second, we also provide a comprehensive modeling of *urbun*, taking into consideration its similarities and differences with call options.

3 Mathematical Model of Islamic Option Pricing

Definition 1 In Equation (1), V_τ is the price of the sukuk, M is the maturity payment (i.e., the face value), T is the maturity date, τ is time, and r is the discount rate, which should not be based on any interest-bearing benchmark (see [28]):

$$V_\tau = \frac{M}{(1+r)^{T-\tau}}. \quad (1)$$

We assume the risky asset to be a stock. Since, the stock Islamic option value C_t is a random variable, the seller of the Islamic option is faced with a risk in selling it. However, the seller can manage the risk by buying certain shares (denoted as Δ) of the stocks to hedge the risk in the Islamic option. This is the idea of Δ -hedging.

Definition 2 Δ -hedging: For a given Islamic option value C , trade Δ shares of the underlying asset S in the opposite direction, so that the Islamic portfolio

$$V = -C + \Delta S = -C + \frac{\partial C}{\partial S} S \quad (2)$$

is risk-free.

The idea of hedging: It is possible to construct an Islamic investment portfolio with S and C such that it is risk-free.

Basic assumptions

- (a) The underlying asset price follows the geometrical Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad (3)$$

where

- μ expected return rate (constant parameter),
- σ volatility (constant parameter),
- dW_t standard Brownian motion (i.e $E(dW_t) = 0$, $Var(dW_t) = dt$).

- (b) The discount rate r is a constant, (see Definition 2).
- (c) Underlying asset pays no dividend.
- (c) No transaction cost and no tax.
- (d) The market is arbitrage-free.

Problem: Let $C = C(S, t)$ denote the option price, the boundary condition at option expiration is at maturity ($t = T$),

$$C(S, t = T) = |S - K|,$$

where K is the strike price. What is the "Urbun"'s value during its lifetime ($0 \leq t \leq T$)?

3.1 Ito's Lemma and the Model of Islamic Option Pricing

Since

$$C_t = C(S_t, t),$$

where the stochastic process S_t satisfies the stochastic differential equation (3), we see that by the Ito formula

$$dC = \left(\frac{\partial C}{\partial S} \mu S + \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 \right) + \frac{\partial C}{\partial S} \sigma S dW. \quad (4)$$

By using Δ -hedging strategy to eliminate all risk with an combination of $\frac{\partial C}{\partial S}$ in the asset and short one option. We will create a portfolio of *sukūk salam* to cancel the random component. The value of the Islamic portfolio:

$$V = -C + \frac{\partial C}{\partial S}S.$$

The change in the portfolio value becomes:

$$dV = -dC + \frac{\partial C}{\partial S}dS. \quad (5)$$

Substituting (3) and (4) into (5):

$$\begin{aligned} dV &= -\left(\frac{\partial C}{\partial \tau} + \frac{\partial C}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)d\tau - \frac{\partial C}{\partial S}\sigma SdW + \frac{\partial C}{\partial S}(\mu Sd\tau + \sigma SdW) \\ &= -\frac{\partial C}{\partial \tau}d\tau - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2d\tau. \end{aligned}$$

This equation does not include the stochastic Winer process dW . In time $d\tau$, and using equation (1), the portfolio earns capital gains dV is:

$$\begin{aligned} dV &= \ln(1+r)Vd\tau, \\ -\frac{\partial C}{\partial \tau}d\tau - \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2d\tau &= \ln(1+r)(-C + \frac{\partial C}{\partial S}S)d\tau, \\ \frac{\partial C}{\partial \tau}d\tau + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2d\tau + \ln(1+r)\frac{\partial C}{\partial S}Sd\tau &= \ln(1+r)Cd\tau, \\ \left(\frac{\partial C}{\partial \tau} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 + \ln(1+r)\frac{\partial C}{\partial S}S\right)d\tau &= \ln(1+r)Cd\tau. \end{aligned}$$

We get the following partial differential equation for "Urbun":

$$\frac{\partial C}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \ln(1+r)S \frac{\partial C}{\partial S} - \ln(1+r)C = 0.$$

This equation describes the "Urbun" option price movement. Therefore, in order to determine the "Urbun" option value at any time in $[0, T]$, we need to solve the following PDE problem in the domain $\Sigma = \{0 \leq S < +\infty, 0 \leq t \leq T\}$:

$$\begin{cases} \frac{\partial C}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \ln(1+r)S \frac{\partial C}{\partial S} - \ln(1+r)C = 0, & \Sigma, \\ C(S, t = T) = |S - K|, & \text{(Urbun)}, & S \in \mathbb{R}^+, \\ C(0, \tau) = 0, \quad \lim_{S \rightarrow +\infty} C(S, \tau) = S, & & \tau \in [0, T], \end{cases} \quad (6)$$

where $C(S, \tau)$ is the value of the Urbun (or the deposit of "Urbun") at the asset price S and at time τ , K is the exercise price, T is the maturity date, $r > 0$ is the discount rate, (which should not be based on any interest bearing benchmark), and $\sigma > 0$ represents volatility function of underlying asset.

Remark 1 1. The line segment $\{S = 0, 0 \leq t \leq T\}$ is also a boundary of the domain Σ . However, since the equation of the problem (6) is degenerated at $S = 0$, according to the PDE theory, there is no need to specify the boundary value at $S = 0$.

2. The linear differential operator given by:

$$\mathcal{L}_{BS} = \frac{\partial}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \ln(1+r)S \frac{\partial}{\partial S} - \ln(1+r),$$

has a financial interpretation as a measure of the difference between the return on a hedged option portfolio (the first two terms) and the return on a bank deposit (the last two terms).

3. The equation of the problem (6) does not contain the drift parameter μ of the underlying asset. Hence the price of the options will be independent of how rapidly or slowly an asset grows. The price will depend on the volatility σ however. A consequence of this is that two people may have quite differ views on μ , yet still agree on the value of an option.

By the transformation

$$S = Ke^x, \quad C(S, \tau) = Kw(x, t), \quad t = (T - \tau)\sigma^2/2, \quad \lambda = 2 \ln(1 + r)/\sigma^2. \quad (7)$$

Hence, problem (6) is reduced to a Cauchy problem of a parabolic equation with constant coefficients:

$$\begin{cases} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + (\lambda - 1) \frac{\partial w}{\partial x} - \lambda w, & (x, t) \in \mathbb{R}^+ \times (0, T), \\ w(x, 0) = |e^x - 1|, & x \in \mathbb{R}^+. \end{cases} \quad (8)$$

By the PDE theory, the Cauchy problem (8) is well-posed. Thus the original problem (6) is also well-posed.

Notice that this system of equations contains just two dimensionless parameters $\lambda = 2 \ln(1 + r)/\sigma^2$ which λ represents the balance between the discount rate and the variability of stock returns and the dimensionless time to expiry $\sigma^2 T/2$, even though there are four dimensional parameters, k , A , T , σ^2 and r , in the original statement of the problem.

Solving the PDE (8) with the appropriate boundary condition. An alternative is to solve the PDE (8) analytically.

4 Procedure of Solution: Residual Power Series Method (RPSM)

Step 1: Let us assume that problem (8) about the point $t = t_0$ is written as

$$w(x, t) = \sum_{k=0}^{\infty} b_k(x) \frac{t^k}{\Gamma(k+1)}, \quad 0 \leq t. \quad (9)$$

In order to evaluate the value of $w(x, t)$, let $w_m(x, t)$ signifies the m th truncated series of $w(x, t)$ as

$$w_m(x, t) = \sum_{k=0}^m b_k(x) \frac{t^k}{\Gamma(k+1)}, \quad 0 \leq t. \quad (10)$$

For $m = 0$, the 0th RPS solution of $w(x, t)$ may be written as

$$w_0(x, t) = b_0(x) = |e^x - 1|. \quad (11)$$

Using Eqs. (11) and (10) can be modified as

$$w_m(x, t) = b_0(x) + \sum_{k=1}^m b_k(x) \frac{t^k}{\Gamma(k+1)}, \quad 0 \leq t, \quad m = 1, 2, 3, \dots$$

So m th RPS solution can be evaluated after obtaining all $b_k(x)$, $k = 1, 2, \dots, m$.

Step 2: Let us consider the residual function (RF) of the problem (8) as

$$\text{res}(x, t) = \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} - (\lambda - 1) \frac{\partial w}{\partial x} + \lambda w,$$

and m th RF may be written as

$$\text{res}_m(x, t) = \frac{\partial w_m}{\partial t} - \frac{\partial^2 w_m}{\partial x^2} - (\lambda - 1) \frac{\partial w_m}{\partial x} + \lambda w_m, \quad m = 1, 2, 3, \dots \quad (12)$$

Some useful results about $res_m(x, t)$ have been included in [5] which are given below

$$\begin{aligned}
 & i. \text{res}(x, t) = 0, \\
 & ii. \lim_{m \rightarrow \infty} \text{res}_m(x, t) = \text{res}(x, t), \\
 & iii. \frac{\partial \text{res}}{\partial t}(x, 0) = \frac{\partial \text{res}_m}{\partial t}(x, 0) = 0, \quad i = 0, 1, 2, \dots, m.
 \end{aligned} \tag{13}$$

Theorem 1 Let $C(S, \tau)$ denote the price of "Urbun" option (or the deposit of Urbun), with the same strike price K and expiration date T . Then the value of "Urbun" option is given by the formula

$$C(S, \tau) = |S - K|e^{-\lambda(T-\tau)\sigma^2/2} + S \left(1 - e^{-\lambda(T-\tau)\sigma^2/2}\right), \tag{14}$$

with $\lambda = 2 \ln(1 + r)/\sigma^2$ and the profit and loss of Urbun is

$$P\&L = \max[S - (K - C), 0] - C.$$

Proof. See Appendix. ■

5 Numerical Examples

To investigate the properties of our freshly derived urbun option pricing formula as well as the corresponding attitude in practice, we now envisage some numerical examples with explicit parameter settings. According to Theorem 1, we can likewise calculate the appropriate price of the urbun option call option by using residual power series method RPS approach, which is proved to be an efficacious method in option valuation (see for example, [5]).

In this section, we construe numerically the proposed model of valuing the urbun option by executing the RSP method in MATLAB to solve the PDE (3.8) numerically.

Example 1 Party A is a grain merchant who wishes to purchase 10 tons of grain from Party B in the future. The spot price of grain is \$10,000 per ton. Party B offers to sell Party A 10 tons of grain at the market price, but Party A is hesitant to close the deal as he has yet to find a buyer for the commodity willing to offer a reasonable price. Party A offers to pay a deposit to Party B, requesting that Party B holds 10 tons of grain for Party A for 30 days, but does not guarantee that Party A will buy the grain in 30 days. The deposit amount is \$2,000 or 20 percent of the total purchase price. If Party A buys the grain, this amount will be part of the total price of \$10,000, and if Party A does not purchase he grain, Party B can hold onto the \$2,000 as compensation for holding 10 tons of stock for Party A.

We consider, $r = 0.05$, $\sigma = 0.25$, $T = 1$, $K = 1$, $S \in [0, 2]$ and $\tau \in [0, 1]$. For $S = 0.3$, value of Urbun is $C = 0.681533$ and

$$P\&L = \max[S - (K - C), 0] - C = -0.681533.$$

For $S = 1.4$, value of Urbun is $C = 0.446165$ and

$$P\&L = \max[S - (K - C), 0] - C = 0.4.$$

Example 2 (Urbun on Real Estate) Consider a Urbun with three months to expiry. The real estate price is 60, the strike price is 65, the risk-free interest rate is 8% per year, and the volatility is 30% per annum. Thus, $S = 60$, $K = 65$, $T = 0.25$, $r = 0.08$, and $\sigma = 0.3$, we obtain

$$\lambda = 2 \ln(1 + r)/\sigma^2 = 2 \ln(1 + 0.08)/0.3^2 = 0.7427,$$

$$C = |S - K|e^{-\lambda T \sigma^2/2} + S \left(1 - e^{-\lambda T \sigma^2/2}\right) = 5.4576,$$

$$P\&L = \max[S - (K - C), 0] - C = -5.$$

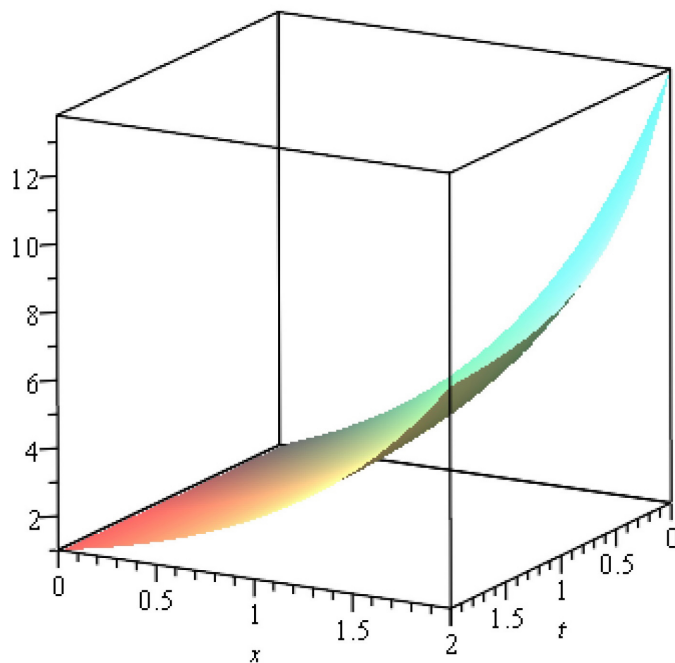


Figure 1: The plot of Eq. (14) represents the surface $w(x, t)$ for $r = 0.04$, $\sigma = 0.2$ and $t, x \in [0, 2]$.

6 Summary and Conclusions

In order to respond to the objections to the derivatives in Islamic jurisprudence and to meet the Shariah principles, we proposed the model urbun (used as a call option) based on the principles of asset backing, profit and loss sharing. This approach is considered as an alternative to the conventional options. Indeed, the buyer has the right, but not the obligation, to buy a property, at a specified price within a specific time period. Since the buyer could exercise the option at any time prior to maturity, urbun is deemed to be closer to American options rather than European options.

The main contribution of our article was to complete the literature focused on Shariah compliant derivatives by providing a mathematical modeling of urbun, taking into consideration its similarities and differences with call options. Hence, the suggested model enables to calculate the value of the urbun at any time, using the asset price, the exercise price, the maturity date, the volatility function of the underlying asset, and the discount rate, which should not be based on any interest bearing benchmark. Two numerical examples were performed in this article based on two underlying assets, namely: commodities and real estate.

The main policy implication of this modeling is that it could be used by the investors as a tool to overcome the Islamic financial restrictions on options. However, our research has two limitations. The first one lies in the nature of Islamic derivatives, which are less sophisticated than their mainstream counterparts. Urbun could be used as a call option, but there is no put option in Islamic finance. The second limitation is related to the provided numerical examples, limited to options on commodities and real estate. This is due to the lack of data recording other underlying. Future researches could go further enhance the modeling in Islamic derivatives and provide more examples to test their efficiency and relevance.

7 Appendix

Consider the problem (8) and Eq. (9), according to the RPSM, $w_0(x, t) = |e^x - 1|$ and the infinite series solution of the problem (8) can be written as

$$w(x, t) = |e^x - 1| + \sum_{k=1}^{\infty} b_k(x) \frac{t^k}{\Gamma(k+1)},$$

m th truncated series solution of $w(x, t)$ becomes

$$w_m(x, t) = |e^x - 1| + \sum_{k=1}^m b_k(x) \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)}, \quad m = 1, 2, 3, \dots$$

For $m = 1$, 1st RPS solution for Eq. (8) may be written as

$$w_1(x, t) = |e^x - 1| + b_1(x) \frac{t}{\Gamma(1+1)}. \quad (15)$$

To determine the value of $b_1(x)$, we substitute Eq. (15) in the 1st residual function of Eq. (12) this gives

$$\text{res}_1(x, t) = \frac{\partial w_1}{\partial t} - \frac{\partial^2 w_1}{\partial x^2} - (\lambda - 1) \frac{\partial w_1}{\partial x} + \lambda w_1,$$

this gives

$$\text{res}_1(x, t) = b_1(x) - e^x - b_1''(x) \frac{t}{\Gamma(2)} - (k-1) \left(e^x + b_1'(x) \frac{t}{\Gamma(2)} \right) + k \left(|e^x - 1| + b_1(x) \frac{t}{\Gamma(2)} \right).$$

Using (iii) of Eq. (13), for $i = 0$, that is $\text{res}(x, 0) = \text{res}_1(x, 0) = 0$, we get

$$\text{res}_1(x, 0) = b_1(x) - \lambda e^x + \lambda |e^x - 1| = 0.$$

So,

$$b_1(x) = k e^x - k |e^x - 1|.$$

For $m = 2$, 2nd RPS solution for Eq. (8) can be written as

$$w_2(x, t) = |e^x - 1| + (k e^x - k |e^x - 1|) \times \frac{t}{\Gamma(2)} + b_2(x) \frac{t^2}{\Gamma(3)}.$$

To determine the value of $b_2(x)$, we substitute Eq. (15) in the 1st residual function of Eq. (12) this gives

$$\text{res}_2(x, t) = \frac{\partial w_2}{\partial t} - \frac{\partial^2 w_2}{\partial x^2} - (\lambda - 1) \frac{\partial w_2}{\partial x} + \lambda w_2,$$

this gives

$$\begin{aligned} \text{res}_2(x, t) = & \lambda e^x - \lambda |e^x - 1| + b_2(x) \frac{t}{\Gamma(2)} \\ & - e^x - b_2''(x) \frac{t^2}{\Gamma(3)} - (k-1) \left[e^x + b_2'(x) \frac{t^2}{\Gamma(3)} \right] \\ & + \lambda \left[|e^x - 1| + (\lambda e^x - \lambda |e^x - 1|) \frac{t}{\Gamma(2)} + b_2(x) \frac{t^2}{\Gamma(3)} \right]. \end{aligned}$$

Using (iii) of Eq. (13), for $i = 0$, that is $\frac{\partial}{\partial t} \text{res}(x, 0) = \frac{\partial}{\partial t} \text{res}_2(x, 0) = 0$, we get

$$b_2(x) = \lambda^2 |e^x - 1| - \lambda^2 e^x.$$

For $m = 3$, 3rd RPS solution for the Eq. (8) can be written as

$$w_3(x, t) = |e^x - 1| + (\lambda e^x - \lambda |e^x - 1|) \frac{t}{\Gamma(2)} + (\lambda^2 |e^x - 1| - \lambda^2 e^x) \frac{t^2}{\Gamma(3)} + b_3(x) \frac{t^3}{\Gamma(4)}. \quad (16)$$

Putting Eq. (16) in the 3rd residua function of Eq. (12), we obtain

$$\text{res}_3(x, t) = \frac{\partial w_3}{\partial t} - \frac{\partial^2 w_3}{\partial x^2} - (\lambda - 1) \frac{\partial w_3}{\partial x} + \lambda w_3.$$

This gives

$$\begin{aligned} \text{res}_3(x, t) &= \lambda e^x - \lambda |e^x - 1| + (\lambda^2 |e^x - 1| - \lambda^2 e^x) \frac{t}{\Gamma(2)} \\ &+ b_3(x) \frac{t^2}{\Gamma(3)} - e^x - b_3''(x) \frac{t^3}{\Gamma(4)} - (\lambda - 1) \left[e^x + b_3'(x) \frac{t^3}{\Gamma(4)} \right] \\ &+ \lambda \left[|e^x - 1| + (\lambda e^x - \lambda |e^x - 1|) \frac{t}{\Gamma(2)} \right. \\ &\left. + (\lambda^2 |e^x - 1| - \lambda^2 e^x) \frac{t^2}{\Gamma(3)} + b_3(x) \frac{t^3}{\Gamma(4)} \right]. \end{aligned}$$

Using (iii) of Eq. (13), for $i = 0$, that is $\frac{\partial}{\partial t} \text{res}(x, 0) = \frac{\partial}{\partial t} \text{res}_2(x, 0) = 0$, we get

$$b_3(x) = \lambda^3 e^x - \lambda^3 |e^x - 1|.$$

Continuing this way, one may find the values of $b_4(x)$, $b_5(x), \dots$. So the solution of Eq. (8) may be written as

$$\begin{aligned} w(x, t) &= |e^x - 1| + (\lambda e^x - \lambda |e^x - 1|) \frac{t}{\Gamma(2)} + (\lambda^2 |e^x - 1| - \lambda^2 e^x) \frac{t^2}{\Gamma(3)} \\ &+ (\lambda^3 e^x - \lambda^3 |e^x - 1|) \frac{t^3}{\Gamma(4)} + \dots \\ &= |e^x - 1| \left[1 - \frac{\lambda t}{\Gamma(2)} + \frac{(\lambda t)^2}{\Gamma(3)} - \frac{(\lambda t)^3}{\Gamma(4)} + \dots \right] \\ &+ e^x \left[\frac{\lambda t}{\Gamma(2)} - \frac{(\lambda t)^2}{\Gamma(3)} + \frac{(\lambda t)^3}{\Gamma(4)} - \dots \right] \\ &= |e^x - 1| e^{-\lambda t} + e^x (1 - e^{-\lambda t}) \\ &= \frac{|e^x - 1|}{(1+r)^{2t/\sigma^2}} + e^x \left(1 - \frac{1}{(1+r)^{2t/\sigma^2}} \right). \end{aligned}$$

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