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Moments Of Power Lomax Distribution Through Generalized Order Statistics*

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Abstract

In this paper, we derive the exact expressions for single and product moments of generalized order statistics from the Power Lomax distribution. The obtained expressions involve the hypergeometric function.

1 Introduction

The Power Lomax (PoLo) distribution was proposed by [19], as a new extension of the Lomax distribution. Thanks to the addition of a new power/shape parameter, it gives a better fit as a model as compared to the Lomax distribution. For more distributional properties of the Power Lomax distribution, we may refer to [19]. The importance of the PoLo distribution was highlighted in various studies, including [2], [17] and [18]. However, some aspects of the related generalized order statistics (*gos*) remain unexplored, and are the purpose of this study. The motivations and details on these aspects are described below.

First and foremost, a random variable X is said to follow the PoLo distribution if its probability distribution function (*pdf*) is of the form

$$f(x) = \alpha\beta\lambda^{\alpha}x^{\beta-1}\left(\lambda + x^{\beta}\right)^{-(\alpha+1)}, \quad x \geq 0, \tag{1}$$

where $\alpha, \beta, \lambda > 0$. It is understood that $f(x) = 0$ for $x < 0$. The corresponding distribution function (*df*) is derived as

$$F(x) = 1 - \lambda^{\alpha}\left(\lambda + x^{\beta}\right)^{-\alpha} \quad x \geq 0, \tag{2}$$

and $F(x) = 0$ for $x < 0$. In view of Equations (1) and (2), the *pdf* and *df* of the PoLo distribution are defined by the following relation:

$$f(x) = \frac{\alpha\beta}{\lambda}\frac{x^{\beta-1}}{\left(1+\frac{x^{\beta}}{\lambda}\right)}\bar{F}(x), \tag{3}$$

where $\bar{F}(x) = 1 - F(x)$. Equation (3) is called the characterizing equation.

Now, the basics on the concept of *gos* are recalled. This concept was introduced by [7] as follows:

Let $n \in \mathbb{N}$, $k \geq 1$, and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \prod_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$ be the parameters such that, for $1 \leq i \leq n-1$,

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j > 0.$$

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The random variables $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)$ are said to be *gos* from a continuous random variable with the *df* $F(x)$ and the *pdf* $f(x)$ if their joint *pdf* has the following form:

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[\bar{F}(x_i)\right]^{m_i}f(x_i)\right)\left[\bar{F}(x_n)\right]^{k-1}f(x_n),$$

where it is supposed that $F^{-1}(0+) < x_1 \le x_2 \le \ldots \le x_n < F^{-1}(1)$.

The model of *gos* contains special cases such as ordinary order statistics

$$(\gamma_i = n-i+1;\ i=1,2,\ldots,n \ \text{ i.e. } m_1=m_2=\cdots=m_{n-1}=0,\ k=1),$$

k-th record values

$$(\gamma_i = k \ \text{ i.e. } m_1=m_2=\cdots=m_{n-1}=-1,\ k\in\mathbb{N}),$$

sequential order statistics ($\gamma_i=(n-i+1)\alpha_i;\ \alpha_1,\alpha_2,\ldots,\alpha_n>0$), order statistics with non-integral sample size ($\gamma_i=\alpha-i+1;\alpha>0$), Pfeifer's record values ($\gamma_i=\beta_i;\ \beta_1,\beta_2,\ldots,\beta_n>0$) and progressive type II censored order statistics ($m_i\in\mathbb{N},k\in\mathbb{N}$). We may refer the reader to [7] and [8].

With these notions in mind, we may consider two complementary cases as described below

Case I: $m_1=m_2=\cdots=m_{n-1}=m$.

Case II: $\gamma_i\neq\gamma_j$, $i\neq j$, $i,j=1,2,\ldots,n-1$.

The underlying theory behind these two cases is detailed below.

Case I: The *pdf* of the r-th *gos* $X(r,n,m,k)$ is given by [7] and can be expressed as

$$f_{X(r,n,m,k)}(x)=\frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r-1}f(x)[g_m(F(x))]^{r-1}.$$

Furthermore, with reference to [7], the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1\le r<s\le n$ is given by

$$\begin{aligned}f_{X(r,n,m,k),X(s,n,m,k)}(x,y)&=\frac{C_{s-1}}{(r-1)!\ (s-r-1)!}[\bar{F}(x)]^m[g_m(F(x))]^{r-1}\\&\times[h_m(F(y))-h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(x)f(y),\quad x<y,\end{aligned}\tag{4}$$

where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i,\quad \gamma_i=k+(n-i)(m+1),\quad h_m(x)=\begin{cases}-\frac{1}{m+1}(1-x)^{m+1}, & m\neq-1\\ -\ln(1-x), & m=-1\end{cases}$$

and

$$g_m(x)=h_m(x)-h_m(0)=\int_0^x(1-t)^m dt,\quad x\in[0,1).$$

In this setting, let $E[X^j(r,n,m,k)]=\mu^j_{r,n,m,k}$ denotes the j-th moment of the r-th *gos* $X(r,n,m,k)$. Similarly $E[X^i(r,n,m,k)X^j(s,n,m,k)]=\mu^{i,j}_{r,s,n,m,k}$ denotes the (i,j)-th product moment of the r-th and s-th *gos*.

Case II: The *pdf* of the r-th *gos* $X(r,n,\tilde{m},k)$ is specified by [8]. It can be expressed as

$$f_{X(r,n,\tilde{m},k)}(x)=C_{r-1}\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i-1}f(x).$$

Also, the joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1\le r<s\le n$, is given by

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y)=C_{s-1}\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(x)}\right\}^{\gamma_i}\right]\left[\sum_{i=1}^{r}a_i(r)\left\{\bar{F}(x)\right\}^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)},$$

where

$$a_i(r)=\prod_{j=1}^{r}\frac{1}{\gamma_j-\gamma_i},\quad j\neq i,\quad \gamma_j\neq\gamma_i,\quad 1\leq i\leq r\leq n$$

and

$$a_i^{(r)}(s)=\prod_{j=r+1}^{n}\frac{1}{\gamma_j-\gamma_i},\quad j\neq i,\quad \gamma_j\neq\gamma_i,\quad r+1\leq i\leq s\leq n.$$

For $m_1=m_2=\cdots=m_{n-1}=m\neq-1$, based on a result of [11], it can be shown that

$$a_i(r)=\frac{(-1)^{r-i}}{(m+1)^{r-1}(r-1)!}\binom{r-1}{r-i} \tag{5}$$

and

$$a_i^{(r)}(s)=\frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!}\binom{s-r-1}{s-i}. \tag{6}$$

In this setting, let $E[X^j(r,n,\tilde{m},k)]=\mu^j_{r,n,\tilde{m},k}$ denotes the j-th moment of the r-th *gos* $X(r,n,\tilde{m},k)$. Similarly $E[X^i(r,n,\tilde{m},k)X^j(s,n,\tilde{m},k)]=\mu^{i,j}_{r,s,n,\tilde{m},k}$ denotes the (i,j)-th product moment of the r-th and s-th *gos*.

In the context of the PoLo distribution, based on the cases above, the author in [1] derived the recurrence relations of order statistics. References [14] and [20] have derived the recurrence relations for the *gos* with $m_1=m_2=\cdots=m_{n-1}=m$ and $\gamma_i\neq\gamma_j$, $i\neq j$, $i,j=1,2,\ldots,n-1$, respectively, for the same distribution. The authors in [15] also found the moments of the PoLo distribution based on order statistics. For some additional related topics, one may refer to [3], [4], [9], [10], [11], [12], [13], among others. However, to our knowledge the exact expressions for single and product moments of *gos* from the PoLo distribution for Cases I and II remain unexplored and deserve a complete study. This article offers these expressions.

Section 2 provides the single and product moments of *gos* from the PoLo distribution for Case I. Similarly, Section 3 provides the single and product moments of *gos* from the PoLo distribution for Case II. The paper ends with a numerical study in Section 4.

2 Moments of *gos* for Case I

Single moments

In this paragraph, we derive the exact expressions for single moments of *gos*. Before presenting the main result, the following lemma is proved.

Lemma 1 *Let us consider the following integral function:*

$$\Phi_j(a)=\int_0^{+\infty}\frac{x^{j+\beta-1}}{\left(1+\frac{x^\beta}{\lambda}\right)^{a\alpha+1}}dx. \tag{7}$$

Then, provided that it exists, we have

$$\Phi_j(a)=\frac{\lambda^{1+\frac{j}{\beta}}}{\beta}B\Big(a\alpha-\frac{j}{\beta},1+\frac{j}{\beta}\Big), \tag{8}$$

where $B(a,b)$ is the beta function defined by $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt$, and

$$\Phi_0(a)=\frac{\lambda}{\beta a\alpha}. \tag{9}$$

Proof. Substituting $u = \frac{1}{1+\frac{x^\beta}{\lambda}}$ in Equation (7), after some algebra, we get

$$\Phi_j(a) = \frac{\lambda^{1+\frac{j}{\beta}}}{\beta} \int_0^1 u^{a\alpha - \frac{j}{\beta} - 1}(1-u)^{\frac{j}{\beta}} du,$$

which gives the result given in Equation (8). To prove Equation (9), it is enough to put $j = 0$ in Equation (8). ■

Theorem 1 *The single moment for the PoLo distribution is*

$$\mu^j_{r,n,m,k} = \frac{C_{r-1}}{(r-1)!} \frac{1}{(m+1)^{r-1}} \left(\frac{\alpha\beta}{\lambda}\right) \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \Phi_j(\gamma_{r-i}). \tag{10}$$

Proof. From the definition of $\mu^j_{r,n,m,k}$, we have

$$\mu^j_{r,n,m,k} = \frac{C_{r-1}}{(r-1)!} \int_0^{+\infty} x^j [\bar{F}(x)]^{k+(n-r)(m+1)-1} [g_m(F(x))]^{r-1} f(x) dx.$$

Now, using Equation (3), we get

$$\begin{aligned}
\mu^j_{r,n,m,k} &= \frac{C_{r-1}}{(r-1)!} \frac{1}{(m+1)^{r-1}} \left(\frac{\alpha\beta}{\lambda}\right) \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \int_0^{+\infty} \frac{x^{j+\beta-1}}{\left(1+\frac{x^\beta}{\lambda}\right)^{\alpha(\gamma_{r-i})+1}} dx \\
&= \frac{C_{r-1}}{(r-1)!} \frac{1}{(m+1)^{r-1}} \left(\frac{\alpha\beta}{\lambda}\right) \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \Phi_j(\gamma_{r-i}).
\end{aligned}$$

■

Remark 1 *Setting $m = -1$ in Equation (10) gives an indeterminate form. The single moments of the PoLo distribution based on upper record values can be calculated using L'Hospital Rule, by differentiating the numerator and denominator of Equation (10), $(r-1)$ times with respect to m and taking the limit $m \to -1$. By assuming that j/β is a positive integer, an expression for the moments of generalized (k-th) record values from the PoLo distribution is*

$$\mu^j_{r,n,-1,k} = E\big(Y_r^{(k)}\big)^j = (\alpha k)^r \lambda^{\frac{j}{\beta}-1} \sum_{p=0}^{j/\beta} (-1)^p \binom{j/\beta}{p} \frac{1}{[\alpha k + p - j/\beta]^r},$$

where $Y_r^{(k)}$ denotes the k-th upper record values. Putting $k = 1$ in, we deduce the explicit expression for the moments of ordinary upper record values from the PoLo distribution.

Remark 2 *Setting $m = 0$ and $k = 1$ in Equation (10), we get the result for the order statistics from the PoLo distribution as*

$$\mu^j_{r,n,0,1} = \mu^j_{r:n} = C_{r:n} \left(\frac{\alpha\beta}{\lambda}\right) \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \Phi_j(n-r+i+1),$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}, \quad \gamma_{r-i} = n-r+i+1, \quad C_{r-1} = \frac{n!}{(n-r)!}.$$

Remark 3 *Setting $j = 0$ in Equation (10), we get*

$$\sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{\gamma_{r-i}} = \frac{(r-1)!(m+1)^{r-1}}{C_{r-1}}, \qquad m \neq 1. \tag{11}$$

Product moments

In this paragraph, we derive the exact expressions for product moments of *gos*. Before coming to the main result, the following lemma is proved.

Lemma 2 *Let us consider the following double integral function:*

$$\Phi_{j,l}(a,b) = \int_0^{+\infty}\int_0^{y} \frac{x^{j+\beta-1}}{\left(1+\frac{x^\beta}{\lambda}\right)^{a\alpha+1}} \frac{y^{l+\beta-1}}{\left(1+\frac{y^\beta}{\lambda}\right)^{\alpha b+1}} dxdy. \tag{12}$$

Then, provided that it exists, we have

$$\begin{aligned}\Phi_{j,l}(a,b) &= \frac{\lambda^{\frac{j+l}{\beta}+2}}{\beta(j+\beta)} B\Big(\frac{j+l}{\beta}+2, \alpha b - \frac{l}{\beta}\Big) \\ &\quad \times {}_3F_2\Big[\frac{j}{\beta}+1, 1-a\alpha+\frac{j}{\beta}, \frac{j+l}{\beta}+2; \frac{j}{\beta}+2, \frac{j}{\beta}+\alpha b+2; 1\Big],\end{aligned} \tag{13}$$

where ${}_pF_q[a_1,\ldots,a_p;b_1,\ldots,b_q;x]$ *denotes the hypergeometric function defined by*

$${}_pF_q[a_1,\ldots,a_p;b_1,\ldots,b_q;x] = \sum_{r=0}^{+\infty}\Big[\prod_{j=1}^{p}\frac{\Gamma(a_j+r)}{\Gamma(a_j)}\Big]\Big[\prod_{j=1}^{q}\frac{\Gamma(b_j)}{\Gamma(b_j+r)}\Big]\frac{x^r}{r!},$$

$\Gamma(a)$ *denotes the gamma function defined by* $\Gamma(a) = \int_0^{+\infty} t^{a-1}e^{-t}dt$, *for* $p = q+1$ *and* $\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j > 0$.

Proof. First, we notice that

$$\Phi_{j,l}(a,b) = \int_0^{+\infty} \frac{y^{l+\beta-1}}{\left(1+\frac{y^\beta}{\lambda}\right)^{\alpha b+1}} \left[\int_0^{y} \frac{x^{j+\beta-1}}{\left(1+\frac{x^\beta}{\lambda}\right)^{a\alpha+1}} dx\right] dy. \tag{14}$$

Now, let us set

$$B(y) = \int_0^{y} \frac{x^{j+\beta-1}}{\left(1+\frac{x^\beta}{\lambda}\right)^{a\alpha+1}} dx. \tag{15}$$

Substituting $1-u = \frac{1}{1+\frac{x^\beta}{\lambda}}$ in Equation (15), we get

$$B(y) = \frac{\lambda^{1+\frac{j}{\beta}}}{\beta}\int_0^{\frac{\frac{y^\beta}{\lambda}}{1+\frac{y^\beta}{\lambda}}} u^{\frac{j}{\beta}}(1-u)^{a\alpha-\frac{j}{\beta}-1}du = \frac{\lambda^{1+\frac{j}{\beta}}}{\beta} B_{\frac{\frac{y^\beta}{\lambda}}{1+\frac{y^\beta}{\lambda}}}\Big(\frac{j}{\beta}+1, a\alpha-\frac{j}{\beta}\Big).$$

From Equation (14), we have

$$\Phi_{j,l}(a,b) = \frac{\lambda^{1+\frac{j}{\beta}}}{\beta}\int_0^{+\infty} \frac{y^{l+\beta-1}}{\left(1+\frac{y^\beta}{\lambda}\right)^{\alpha b+1}} B_{\frac{\frac{y^\beta}{\lambda}}{1+\frac{y^\beta}{\lambda}}}\Big(\frac{j}{\beta}+1, a\alpha-\frac{j}{\beta}\Big)dy, \tag{16}$$

where $B_x(p,q) = \int_0^x u^{p-1}(1-u)^{q-1}du$. Owing to [16], we know that

$$B_x(p,q) = p^{-1}x^p{}_2F_1[p, 1-q; p+1; x]$$

and

$$\int_0^1 u^{a-1}(1-u)^{b-1}{}_2F_1[c,d;e;x]du = B(a,b){}_3F_2[c,d,a;e,a+b;1].$$

Substituting these results in Equation (16), we get

$$\Phi_{j,l}(a,b) = \frac{\lambda^{1+\frac{j}{\beta}}}{\beta}\int_0^{+\infty}\frac{y^{l+\beta-1}}{(1+\frac{y^\beta}{\lambda})^{\alpha b+1}}\left(\frac{\frac{y^\beta}{\lambda}}{1+\frac{y^\beta}{\lambda}}\right)^{\frac{j}{\beta}+1}\left(\frac{j}{\beta}+1\right)^{-1} \times {}_2F_1\Big[\frac{j}{\beta}+1, 1-a\alpha+\frac{j}{\beta}; \frac{j}{\beta}+2; \Big(\frac{\frac{y^\beta}{\lambda}}{1+\frac{y^\beta}{\lambda}}\Big)\Big]dy. \quad (17)$$

Setting $t=\frac{\frac{y^\beta}{\lambda}}{1+\frac{y^\beta}{\lambda}}$ in Equation (17), we have

$$\begin{aligned}\Phi_{j,l}(a,b) &= \frac{\lambda^{\frac{j+l}{\beta}+2}}{\beta(j+\beta)}\int_0^1 t^{\frac{j+l}{\beta}+1}(1-t)^{\alpha b-\frac{l}{\beta}-1}_2 F_1\Big[\frac{j}{\beta}+1, 1-a\alpha+\frac{j}{\beta}; \frac{j}{\beta}+2; t\Big]dt \\ &= \frac{\lambda^{\frac{j+l}{\beta}+2}}{\beta(j+\beta)}B\Big(\frac{j+l}{\beta}+2, \alpha b-\frac{l}{\beta}\Big) \\ &\quad \times {}_3F_2\Big[\frac{j}{\beta}+1, 1-a\alpha+\frac{j}{\beta}, \frac{j+l}{\beta}+2; \frac{j}{\beta}+2, \frac{j}{\beta}+\alpha b+2; 1\Big].\end{aligned}$$

■

Lemma 3 *Let $\Phi_{j,l}$ be defined in (12). We have*

$$\Phi_{0,l}(a,b) = \frac{\lambda}{a\alpha\beta}[\Phi_l(b) - \Phi_l(a+b)] \quad (18)$$

and

$$\Phi_{j,0}(a,b) = \frac{\lambda}{b\alpha\beta}[\Phi_j(a+b)], \quad (19)$$

where $\Phi_j(a)$ is defined in Equation (7), and

$$\Phi_{0,0}(a,b) = \Big(\frac{\lambda}{\alpha\beta}\Big)^2\frac{1}{b(a+b)}. \quad (20)$$

Proof. Putting $j=0$ in Equation (12), we get

$$\begin{aligned}\Phi_{0,l}(a,b) &= \int_0^{+\infty}\frac{y^{l+\beta-1}}{(1+\frac{y^\beta}{\lambda})^{\alpha b+1}}\left[\int_0^y\frac{x^{\beta-1}}{(1+\frac{x^\beta}{\lambda})^{a\alpha+1}}dx\right]dy \\ &= \frac{\lambda}{a\alpha\beta}\int_0^{+\infty}\frac{y^{l+\beta-1}}{(1+\frac{y^\beta}{\lambda})^{\alpha b+1}}\left[1-\frac{1}{(1+\frac{y^\beta}{\lambda})^{a\alpha}}\right]dy \\ &= \frac{\lambda}{a\alpha\beta}[\Phi_l(b)-\Phi_l(a+b)].\end{aligned}$$

Equations (19) and (20) can be proved by noting that

$${}_3F_2(a,b,c;c,d;1) = {}_2F_1(a,b;d;1) = \frac{\Gamma(d)\Gamma(d-a-b)}{\Gamma(d-a)\Gamma(d-b)}.$$

■

Theorem 2 *The generalized product moment for the PoLo distribution is given as*

$$\mu^{j,l}_{r,s,n,m,k} = \frac{C_{r-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\Big(\frac{\alpha\beta}{\lambda}\Big)^2 \times\sum_{i=0}^{r-1}\sum_{t=0}^{s-r-1}(-1)^{i+t}\binom{r-1}{i}\binom{s-r-1}{t}\Phi_{j,l}(\gamma_{r-i}-\gamma_{s-t}, \gamma_{s-t}). \quad (21)$$

Proof. Based on the definition of $\mu^{j,l}_{r,s,n,m,k}$, we can write

$$\begin{aligned}\mu^{j,l}_{r,s,n,m,k} &= \frac{C_{r-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\int_0^{+\infty}\int_0^y x^j y^l[\bar{F}(x)]^m[1-\bar{F}(x)^{m+1}]^{r-1}\\ &\quad\times[\bar{F}(x)^{m+1}-\bar{F}(y)^{m+1}]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1}f(x)f(y)dxdy.\end{aligned}$$

Using Equation (3), for $m \neq 0$ and $1 \leq r < s \leq n$, after some operations, we get

$$\begin{aligned}\mu^{j,l}_{r,s,n,m,k} &= \frac{C_{r-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\left(\frac{\alpha\beta}{\lambda}\right)^2\int_0^{+\infty}\int_0^y \frac{x^{j+\beta-1}}{\left(1+\frac{x^\beta}{\lambda}\right)}\frac{y^{l+\beta-1}}{\left(1+\frac{y^\beta}{\lambda}\right)}\\ &\quad\times[\bar{F}(x)]^{m+1}[1-\bar{F}(x)^{m+1}]^{r-1}[\bar{F}(x)^{m+1}-\bar{F}(y)^{m+1}]^{s-r-1}[\bar{F}(y)^{m+1}]^{\frac{\gamma_s}{m+1}}dxdy\\ &= \frac{C_{r-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\left(\frac{\alpha\beta}{\lambda}\right)^2\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\\ &\quad\times\int_0^{+\infty}\int_0^y \frac{x^{j+\beta-1}}{\left(1+\frac{x^\beta}{\lambda}\right)}\frac{y^{l+\beta-1}}{\left(1+\frac{y^\beta}{\lambda}\right)}[\bar{F}(x)^{m+1}]^{i+1}[\bar{F}(x)^{m+1}-\bar{F}(y)^{m+1}]^{s-r-1}[\bar{F}(y)^{m+1}]^{\frac{\gamma_s}{m+1}}dxdy\\ &= \frac{C_{r-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\left(\frac{\alpha\beta}{\lambda}\right)^2\sum_{i=0}^{r-1}\sum_{t=0}^{s-r-1}(-1)^{i+t}\binom{r-1}{i}\binom{s-r-1}{t}\\ &\quad\times\int_0^{+\infty}\int_0^y \frac{x^{j+\beta-1}}{\left(1+\frac{x^\beta}{\lambda}\right)^{\alpha(\gamma_{r-i}-\gamma_{s-t})+1}}\frac{y^{l+\beta-1}}{\left(1+\frac{y^\beta}{\lambda}\right)^{\alpha(\gamma_{s-t})+1}}dxdy\\ &= \frac{C_{r-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\left(\frac{\alpha\beta}{\lambda}\right)^2\\ &\quad\times\sum_{i=0}^{r-1}\sum_{t=0}^{s-r-1}(-1)^{i+t}\binom{r-1}{i}\binom{s-r-1}{t}\Phi_{j,l}(\gamma_{r-i}-\gamma_{s-t},\gamma_{s-t}).\end{aligned}$$

The result given in Equation (21) is proved. ∎

In case of record values, an exact expression for the product moment from the PoLo distribution cannot be obtained.

Remark 4 *Setting $j = 0$, $l = 0$ and using Equation (11), Equation (21) reduces to an identity and it is*

$$\sum_{t=0}^{s-r-1}(-1)^t\binom{s-r-1}{t}\frac{1}{\gamma_{s-t}} = \frac{C_{r-1}(s-r-1)!(m+1)^{r-1}}{C_{s-1}}, \qquad m \neq 1. \tag{22}$$

Remark 5 *Setting $m = 0$, $k = 1$ in Equation (21), we get the result for order statistics, that is,*

$$\begin{aligned}\mu^{j,l}_{r,s,n,0,1} &= \mu^{j,l}_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}\left(\frac{\alpha\beta}{\lambda}\right)^2\\ &\quad\times\sum_{i=0}^{r-1}\sum_{t=0}^{s-r-1}(-1)^{i+t}\binom{r-1}{i}\binom{s-r-1}{t}\Phi_{j,l}(s-r-t+i,n-s+t+1),\end{aligned} \tag{23}$$

where $\Phi_{j,l}(a,b)$ is defined in Equation (13).

3 Moments of *gos* for Case II

Single moments

This paragraph focuses on the single moments of the *gos*, beginning with the following theorem.

Theorem 3 *The single moment for the PoLo distribution is*

$$\mu^{j}_{r,n,\tilde{m},k} = C_{r-1}\Big(\frac{\alpha\beta}{\lambda}\Big)\sum_{i=0}^{r} a_i(r)\Phi_j(\gamma_i). \tag{24}$$

Proof. By the definition of $\mu^{j}_{r,n,\tilde{m},k}$, we have

$$\begin{aligned}\mu^{j}_{r,n,\tilde{m},k} &= C_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^{+\infty} x^j[\bar{F}(x)]^{\gamma_i-1}f(x)dx\\ &= C_{r-1}\Big(\frac{\alpha\beta}{\lambda}\Big)\sum_{i=1}^{r} a_i(r)\int_0^{+\infty}\frac{x^{j+\beta-1}}{(1+\frac{x^\beta}{\lambda})^{\alpha\gamma_i+1}}dx.\end{aligned}$$

In view of Equation (7), we get

$$\mu^{j}_{r,n,\tilde{m},k} = C_{r-1}\Big(\frac{\alpha\beta}{\lambda}\Big)\sum_{i=0}^{r} a_i(r)\Phi_j(\gamma_i).$$

■

Remark 6 *Setting $j = 0$ in Equation (24), we obtain*

$$\sum_{i=0}^{r}\frac{a_i(r)}{\gamma_i} = \frac{1}{C_{r-1}}. \tag{25}$$

Remark 7 *Setting $m_1 = m_2 = \cdots = m_{n-1} = m$ and using Equation (5) in Equation (24), we get*

$$\mu^{j}_{r,n,m,k} = \frac{C_{r-1}}{(r-1)!}\frac{1}{(m+1)^{r-1}}\Big(\frac{\alpha\beta}{\lambda}\Big)\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}\Phi_j(\gamma_{r-i}),$$

as specified in Equation (10).

Product moments

In this paragraph, the product moments of the *gos* are investigated.

Theorem 4 *The generalized product moment for the PoLo distribution is given as*

$$\mu^{j,l}_{r,s,n,\tilde{m},k} = C_{s-1}\Big(\frac{\alpha\beta}{\lambda}\Big)^2\left[\sum_{t=r+1}^{s} a_t^{(r)}(s)\left(\sum_{i=1}^{r} a_i(r)\Phi_{j,l}(\gamma_i-\gamma_t,\gamma_t)\right)\right]. \tag{26}$$

Proof. We have

$$\begin{aligned}\mu^{j,l}_{r,s,n,\tilde{m},k} &= C_{s-1}\int_0^{+\infty}\int_0^{y} x^j y^l\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\Big\{\frac{\bar{F}(y)}{\bar{F}(x)}\Big\}^{\gamma_i}\right]\Big(\sum_{i=1}^{r} a_i(r)\{\bar{F}(x)\}^{\gamma_i}\Big)\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dxdy\\ &= C_{s-1}\sum_{t=r+1}^{s} a_t^{(r)}(s)\left[\sum_{i=1}^{r} a_i(r)\int_0^{+\infty} y^l\{\bar{F}(y)\}^{\gamma_t}\Big(\int_0^{y} x^j\{\bar{F}(x)\}^{\gamma_i-\gamma_t}\frac{f(x)}{\bar{F}(x)}dx\Big)\frac{f(y)}{\bar{F}(y)}dy\right]\\ &= C_{s-1}\Big(\frac{\alpha\beta}{\lambda}\Big)^2\sum_{t=r+1}^{s} a_t^{(r)}(s)\left[\sum_{i=1}^{r} a_i(r)\int_0^{+\infty}\frac{y^{l+\beta-1}}{(1+\frac{y^\beta}{\lambda})^{\alpha(\gamma_t)+1}}\Big(\int_0^{y}\frac{x^{j+\beta-1}}{(1+\frac{x^\beta}{\lambda})^{\alpha(\gamma_i-\gamma_t)+1}}dx\Big)dy\right].\end{aligned}$$

Hence the result follows by an application of Equation (12). ■

Corollary 1 *For the single moment of the PoLo distribution, we have*

$$\mu^{l}_{r,n,\tilde{m},k} = C_{s-1}\Big(\frac{\alpha\beta}{\lambda}\Big)\sum_{i=0}^{s} a_i(s)\Phi_l(\gamma_i) \tag{27}$$

as given in Theorem 3.

Proof. Putting $j=0$ in Equation (26) and using Equation (18), we get

$$\begin{aligned}
\mu^{l}_{r,s,n,\tilde{m},k} &= C_{s-1}\Big(\frac{\alpha\beta}{\lambda}\Big)\Bigg[\sum_{t=r+1}^{s}\frac{a_t^{(r)}(s)}{(\gamma_i-\gamma_t)}\Bigg(\sum_{i=1}^{r} a_i(r)\big\{\Phi_l(\gamma_t)-\Phi_l(\gamma_i)\big\}\Bigg)\Bigg]\\
&= C_{s-1}\Big(\frac{\alpha\beta}{\lambda}\Big)\Bigg[\sum_{t=r+1}^{s} a_t^{(r)}(s)\Phi_l(\gamma_t)\Bigg(\sum_{i=1}^{r}\frac{a_i(r)}{(\gamma_i-\gamma_t)}\Bigg)\Bigg]\\
&\quad + C_{s-1}\Big(\frac{\alpha\beta}{\lambda}\Big)\Bigg[\sum_{i=1}^{r} a_i(r)\Phi_l(\gamma_i)\Bigg(\sum_{t=r+1}^{s}\frac{a_t^{(r)}(s)}{(\gamma_i-\gamma_t)}\Bigg)\Bigg].
\end{aligned}$$

Now, by virtue of the results of [5], we have

$$\sum_{i=1}^{r}\frac{a_i(r)}{\gamma_i-\gamma_j} = \prod_{j=1}^{r}\frac{1}{\gamma_i-\gamma_j}, \quad j\neq i, \quad \gamma_j\neq\gamma_i, \quad 1\leq i\leq r\leq n,$$

and

$$\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{\gamma_i-\gamma_j} = \prod_{j=r+1}^{s}\frac{1}{\gamma_i-\gamma_j}, \quad j\neq i, \quad \gamma_j\neq\gamma_i, \quad r+1\leq i\leq s\leq n.$$

Therefore,

$$\begin{aligned}
\mu^{l}_{r,s,n,\tilde{m},k} &= C_{s-1}\Big(\frac{\alpha\beta}{\lambda}\Big)\Bigg[\sum_{t=r+1}^{s} a_t^{(r)}(s)\Phi_l(\gamma_t)\Bigg(\prod_{j=1}^{r}\frac{1}{\gamma_i-\gamma_j}\Bigg)\Bigg]\\
&\quad + C_{s-1}\Big(\frac{\alpha\beta}{\lambda}\Big)\Bigg[\sum_{i=1}^{r} a_i(r)\Phi_l(\gamma_i)\Bigg(\prod_{j=r+1}^{s}\frac{1}{\gamma_i-\gamma_j}\Bigg)\Bigg].
\end{aligned}$$

The desired result follows. ■

Remark 8 *Setting $j=0$ and $l=0$ in Equation (26), we get*

$$\sum_{i=1}^{r}\sum_{t=r+1}^{s}\frac{a_i(r)a_t^{r}(s)}{\gamma_i\gamma_t} = \frac{1}{C_{s-1}}. \tag{28}$$

Combining Equations (25) and (28), we determine another identity, which is

$$\sum_{t=r+1}^{s}\frac{a_t^{r}(s)}{\gamma_t} = \frac{C_{r-1}}{C_{s-1}}.$$

Remark 9 *Setting $m_1 = m_2 = \cdots = m_{n-1} = m$ and using Equation (6), Theorem 4 reduces to Theorem 2.*

4 Numerical Computations

Here, we calculate the means and variances for order statistics (Tables 1 and 2) and *gos* (Tables 3 and 4) but the results are obtained for progressive type II censoring with a suitable adjustment of $m_1, m_2, \ldots, m_{n-1}$. All computations are obtained using Mathematica. Mathematica, like other algebraic manipulation packages, allows for arbitrary precisions, so the accuracy of the given values is not an issue. In case of order statistics, based on [6], the following identities are used to check the calculation of means and variances:

$$\sum_{r=1}^{n} \mu^{j}_{r,n,0,1} = n\mu^{j}_{1,1,0,1}, \qquad j = 1, 2.$$

For any fixed sample size n, increase in r results in increase in mean and variance. For any fixed r increase in sample size n leads to decrease in variance, which is universally true.

	$n\,(\beta{=}2,\ \alpha{=}2,\ \lambda{=}1)$						
r	1	2	3	4	5	6	7
1	0.785398	0.490874	0.386563	0.329039	0.291337	0.264189	0.243444
2		1.07992	0.699495	0.559136	0.479848	0.427073	0.388663
3			1.27014	0.839854	0.678067	0.585399	0.523097
4				1.41356	0.947713	0.770735	0.668469
5					1.53003	1.03620	0.847435
6						1.62879	1.111710
7							1.714970

Table 1: Means of order statistics from the PoLo distribution

	$n\,(\beta{=}2,\ \alpha{=}2,\ \lambda{=}1)$						
r	1	2	3	4	5	6	7
1	0.383150	0.092376	0.050569	0.034590	0.026234	0.021113	0.017658
2		0.500443	0.110707	0.058796	0.039587	0.029730	0.023766
3			0.586744	0.123216	0.064035	0.042589	0.031730
4				0.658988	0.133590	0.068305	0.044990
5					0.722498	0.142739	0.072064
6						0.779943	0.151051
7							0.832768

Table 2: Variances of order statistics from the PoLo distribution

	$n\,(\beta{=}2,\ \alpha{=}2,\ \lambda{=}1,\ m{=}1,\ k{=}2)$						
r	1	2	3	4	5	6	7
1	0.245437	0.164519	0.132095	0.113462	0.10099	0.091895	0.084883
2		0.163177	0.114685	0.093996	0.081674	0.073235	0.066982
3			0.093712	0.067686	0.056240	0.049276	0.044434
4				0.051193	0.037659	0.031602	0.027866
5					0.027288	0.020344	0.017202
6						0.014339	0.010800
7							0.007464

Table 3: Means of *gos* from the PoLo distribution

	n (β=2, α=2, λ=1, m=1, k=2)						
r	1	2	3	4	5	6	7
1	0.106428	0.044362	0.028005	0.020460	0.016116	0.013294	0.011313
2		0.104325	0.048536	0.032074	0.024031	0.019236	0.016045
3			0.074010	0.036652	0.024947	0.019025	0.015410
4				0.045702	0.023573	0.016385	0.012662
5					0.026333	0.013984	0.009875
6						0.014601	0.007927
7							0.007911

Table 4: Variances of *gos* from the PoLo distribution

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