Applied Mathematics E-Notes, 22(2022), 692-711 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

On Solutions To Open Problems And Volterra-Hammerstein Non-linear Integral Equation*

Said Beloul†, Anita Tomar‡, Meena Joshi§

Received 29 September 2021

Abstract

Taking into account the fact that the contractive conditions carry out the magnificent role in establishing coincidence and common fixed points, we introduce generalized condition (B) for self maps in $\mathcal{G}$-metric spaces and utilize it to establish a unique fixed point, a unique common fixed point, and a unique coincidence point. Conclusively, we deal with two questions about the survival of a fixed point in Abbas et al. [M. Abbas, G. V. R. Babu, and G. N. Alemayehu, On common fixed points of weakly compatible mappings satisfying generalized condition, Filomat 25 (2011), 9-19] and on the survival of contractive condition which assure the fixed point at the discontinuity of a map in Rhoades [B. E. Rhoades, Contractive definitions and continuity, Fixed Point Theory and its Applications (Berkeley 1986), Contemp. Math. (Amer. Math. Soc.), 72 (1988), 233–245]. Further, we introduce circle, fixed circle, common fixed circle, and $\mathfrak{u}_0$-generalized condition (B) via $\mathcal{G}$-metric to establish fixed circle and common fixed circle theorems. Also, we give examples and an application to solve Volterra-Hammerstein non-linear integral equation in order to demonstrate the significance of obtained results.

1 Introduction and Preliminaries

Mustafa and Sims [21] proposed an important extension of metric spaces, as $\mathcal{G}$-metric spaces or generalized metric spaces. Firstly, we recollect some basic definitions for these spaces.

Definition 1 ([21]) *Let $\mathcal{G} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ be a function on a non-empty set $\mathcal{U}$ satisfying*

1. $\mathcal{G}(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = 0$ *if* $\mathfrak{u} = \mathfrak{v} = \mathfrak{w}$,
2. $\mathcal{G}(\mathfrak{u}, \mathfrak{u}, \mathfrak{v}) > 0$, *whenever* $\mathfrak{u} \neq \mathfrak{v}$,
3. $\mathcal{G}(\mathfrak{u}, \mathfrak{u}, \mathfrak{v}) \leq \mathcal{G}(\mathfrak{u}, \mathfrak{v}, \mathfrak{w})$, *with* $\mathfrak{w} \neq \mathfrak{v}$,
4. $\mathcal{G}(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \mathcal{G}(\mathfrak{u}, \mathfrak{w}, \mathfrak{v}) = \mathcal{G}(\mathfrak{v}, \mathfrak{w}, \mathfrak{u}) = \dots$ *(symmetry in variables),*
5. $\mathcal{G}(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) \leq \mathcal{G}(\mathfrak{u}, a, a) + \mathcal{G}(a, \mathfrak{v}, \mathfrak{w})$ *(rectangle inequality),*

where $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, a \in \mathcal{U}$. Then the function $\mathcal{G}$ is $\mathcal{G}$-metric (generalized metric) on $\mathcal{U}$ and the pair $(\mathcal{U}, \mathcal{G})$ is a $\mathcal{G}$-metric space.

If we have $\mathcal{G}(\mathfrak{u}, \mathfrak{v}, \mathfrak{v}) = \mathcal{G}(\mathfrak{v}, \mathfrak{u}, \mathfrak{u})$, $\mathfrak{u}, \mathfrak{v} \in \mathcal{U}$, then $(\mathcal{U}, \mathcal{G})$ is a symmetric $\mathcal{G}$-metric space. Here it is magnificent to observe that there exist $\mathcal{G}$-metric spaces, which are not symmetric.

*Mathematics Subject Classifications: 47H10, 54H25.

†Department of Mathematics, Exact Sciences Faculty, University of El Oued, P.O.Box789, 39000, El Oued, Algeria
‡Department of Mathematics, Pt. L. M. S. Campus, Sridev Suman Uttarakhand University, Rishikesh, Uttarakhand, India
§Department of Mathematics, S. S. J. Campus, Soban Singh Jeena University Almora-263601, Uttarakhand, India

Example 1 ([21]) *Let $\mathcal{U} = \{2, 3\}$ and $\mathcal{G}$-metric $\mathcal{G} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to \mathbb{R}_+$ be defined as*

$$\mathcal{G}(2,2,3) = \mathcal{G}(3,2,2) = 1, \quad \mathcal{G}(2,2,2) = \mathcal{G}(3,3,3) = 0,$$

$$\mathcal{G}(2,3,3) = \mathcal{G}(3,2,3) = \mathcal{G}(3,3,2) = 2.$$

Clearly, $\mathcal{G}(3,2,2) \neq \mathcal{G}(3,3,2)$. Therefore G is not symmetric.

Proposition 1 ([21]) *The function $d_{\mathcal{G}} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ of a $\mathcal{G}$-metric space $(\mathcal{U}, \mathcal{G})$ defined by*

$$d_{\mathcal{G}}(\mathfrak{u}, \mathfrak{v}) = \mathcal{G}(\mathfrak{u}, \mathfrak{v}, \mathfrak{v}) + \mathcal{G}(\mathfrak{v}, \mathfrak{u}, \mathfrak{u}), \ \mathfrak{u}, \mathfrak{v} \in \mathcal{U}$$

is a metric on $\mathcal{U}$.

If $\mathcal{G}$ is symmetric, $d_{\mathcal{G}} = 2\mathcal{G}(\mathfrak{u}, \mathfrak{v}, \mathfrak{v})$, and if $\mathcal{G}$ is not symmetric $\frac{3}{2}\mathcal{G}(\mathfrak{u}, \mathfrak{v}, \mathfrak{v}) \leq d_{\mathcal{G}}(\mathfrak{u}, \mathfrak{v}) \leq 3\mathcal{G}(\mathfrak{u}, \mathfrak{v}, \mathfrak{v})$, $\mathfrak{u}, \mathfrak{v} \in \mathcal{U}$.

Definition 2 ([21]) *Let $(\mathcal{U}, \mathcal{G})$ be a $\mathcal{G}$-metric space.*

1. *A sequence $\{\mathfrak{u}_n\}$ in $\mathcal{U}$ is a $\mathcal{G}$-Cauchy sequence if there exists $n_0 \in \mathbb{N}$ so that $\lim\limits_{n\to\infty} \mathcal{G}(\mathfrak{u}_l, \mathfrak{u}_m, \mathfrak{u}_n)$, $l, m, n \geq n_0$ exists and is finite.*
2. *A sequence $\{\mathfrak{u}_n\}$ is $\mathcal{G}$-convergent if for $\varepsilon > 0$, there is a number $\mathfrak{u} \in \mathcal{U}$ so that $n, m \geq n_0$, $n_0 \in N$, $\mathcal{G}(\mathfrak{u}, \mathfrak{u}_n, \mathfrak{u}_m) < \varepsilon$.*
3. *$(\mathcal{U}, \mathcal{G})$ is $\mathcal{G}$-complete if every $\mathcal{G}$-Cauchy sequence $\{\mathfrak{u}_n\}$ in $\mathcal{U}$ is convergent to a point $\mathfrak{u} \in \mathcal{U}$ so that $\lim\limits_{n,m\to\infty} \mathcal{G}(\mathfrak{u}, \mathfrak{u}_n, \mathfrak{u}_m) = 0$, $n, m \in \mathbb{N}$.*

Proposition 2 ([21]) *The subsequent assumptions are equivalent in $\mathcal{G}$-metric space $(\mathcal{U}, \mathcal{G})$:*

1. *$\{\mathfrak{u}_n\}$ is $\mathcal{G}$-convergent to $\mathfrak{u}$ in $\mathcal{U}$.*
2. *$\mathcal{G}(\mathfrak{u}_n, \mathfrak{u}_m, \mathfrak{u}) \to 0$ as $n, m \to \infty$.*
3. *$\mathcal{G}(\mathfrak{u}_n, \mathfrak{u}_n, \mathfrak{u}) \to 0$ as $n \to \infty$.*
4. *$\mathcal{G}(\mathfrak{u}_n, \mathfrak{u}, \mathfrak{u}) \to 0$ as $n \to \infty$.*

Definition 3 ([11]) *Two self maps f and $\mathcal{A}$ are weakly compatible if they commute at their coincidence point, that is, $\mathfrak{f}\mathcal{A}u^* = \mathcal{A}\mathfrak{f}u^*$ if $\mathfrak{f}u^* = \mathcal{A}u^*$, $u^* \in \mathcal{U}$.*

For details on the weaker forms of commutativity allude to Singh and Tomar [28].

Berinde [7] launched weak contraction (later, he renamed it almost contraction [8]) to establish a fixed point that need not be unique.

Definition 4 ([7]) *A self map $\mathfrak{h}$ of a metric space $(\mathcal{U}, d)$ is (δ, L)-contraction or weak contraction if there exists $L \geq 0$ and $\delta \in (0, 1)$ so that*

$$d(\mathfrak{h}\mathfrak{u}, \mathfrak{h}\mathfrak{v}) \leq \delta d(\mathfrak{u}, \mathfrak{v}) + L d(\mathfrak{v}, \mathfrak{h}\mathfrak{u}), \ \mathfrak{u}, \mathfrak{v} \in \mathcal{U}.$$

Later Babu et al. [5] extended it to condition (B) to prove a unique fixed point.

Definition 5 ([5]) *A self map $\mathfrak{h}$ of metric space $(\mathcal{U}, d)$ is the condition (B) if there exists $L \geq 0$ and $\delta \in [0, 1)$ so that*

$$d(\mathfrak{h}\mathfrak{u}, \mathfrak{h}\mathfrak{v}) \leq \delta d(\mathfrak{u}, \mathfrak{v}) + L \min(d(\mathfrak{u}, \mathfrak{h}\mathfrak{u}), d(\mathfrak{v}, \mathfrak{h}\mathfrak{v}), d(\mathfrak{u}, \mathfrak{h}\mathfrak{v}), d(\mathfrak{v}, \mathfrak{h}\mathfrak{u})), \ \mathfrak{u}, \mathfrak{v} \in \mathcal{U}.$$

The condition (B) implies almost contraction / weak contraction / $(\delta, L)-$contraction. However, its reverse implication may not hold true. It is fascinating to see that condition (B) is not essentially continuous.

Abbas et al. [1] extended the condition (B) to a pair of maps as a generalized condition (B) and on the other hand, Abbas and Ilic [2] extended it as a generalized almost $\mathfrak{h}-$contraction.

Definition 6 ([1] (see, [2] also)) *A self map $\mathfrak{h}$ of a metric space $(\mathcal{U}, d)$ satisfies generalized condition (B) associated with the self map $\mathfrak{f}$ of $\mathcal{U}$ if there exists $L \geq 0$ and $\delta \in (0,1)$ so that*

$$d(\mathfrak{h}\mathfrak{u}, \mathfrak{h}\mathfrak{v}) \leq \delta M(\mathfrak{u}, \mathfrak{v}) + L \min\{d(f\mathfrak{u}, \mathfrak{h}\mathfrak{u}), d(f\mathfrak{v}, \mathfrak{h}\mathfrak{v}), d(f\mathfrak{u}, \mathfrak{h}\mathfrak{v}), d(f\mathfrak{v}, \mathfrak{h}\mathfrak{u})\},$$

where

$$M(\mathfrak{u}, \mathfrak{v}) = \max\left\{d(f\mathfrak{u}, f\mathfrak{v}), d(f\mathfrak{u}, \mathfrak{h}\mathfrak{u}), d(f\mathfrak{v}, \mathfrak{h}\mathfrak{v}), \frac{d(f\mathfrak{u}, \mathfrak{h}\mathfrak{v}) + d(f\mathfrak{v}, \mathfrak{h}\mathfrak{u})}{2}\right\}.$$

Noticeably, for $\mathfrak{f} = I$, the generalized condition (B) is similar to condition (B). It is interesting to notice that the Banach contraction [6], the almost contraction [8], the Kannan maps [18], the Chatterjea maps [9], and the Zamfirescu maps[32], including quasi-contractions $0 \leq \delta < 1$ [10] are incorporated in generalized condition (B) [1] and perform a remarkable role in establishing a coincidence point and common fixed points. In this work, we introduce a generalized condition (B) for two pairs of self maps in $\mathcal{G}$-metric spaces. As a by-product, we answer in affirmative to two problems proposed by Abbas et al. [1] and an open problem proposed by Rhoades [27]. Our conclusions generalize, improve, and extend numerous conclusions available in the literature [1]–[3], [5], [8]–[10], [17]–[19], [26], [32], and so on elucidating the significance of a generalized condition (B) in $\mathcal{G}$-metric space. Further, motivated by the fact that a self map may not always have a unique fixed point, the exploration of the geometry of the non-unique fixed points is quite natural, we investigate the geometry of a set of fixed points via $\mathcal{G}$-metric. Examples and an application of solving Volterra-Hammerstein non-linear integral equations via generalized condition (B) are provided to demonstrate the significance of established conclusions in $\mathcal{G}$-metric space.

2 Main Results

First, we put forward generalized condition (B) in $\mathcal{G}$-metric space.

Definition 7 *Let $\mathfrak{f}, \mathfrak{h}, \mathcal{A}$, and $\mathcal{T}$ be self maps of $\mathcal{G}$-metric space $(\mathcal{U}, \mathcal{G})$. Suppose there exists $L \geq 0$ and $\delta \in [0,1)$ so that*

$$\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathcal{T}\mathfrak{v}, \mathcal{T}\mathfrak{v}) \leq \delta M(\mathfrak{u}, \mathfrak{v}, \mathfrak{v}) + L \min\{\mathcal{G}(f\mathfrak{u}, \mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{u}), \mathcal{G}(\mathfrak{h}\mathfrak{v}, \mathcal{T}\mathfrak{v}, \mathcal{T}\mathfrak{v}), \mathcal{G}(f\mathfrak{u}, \mathcal{T}\mathfrak{v}, \mathcal{T}\mathfrak{v}), \mathcal{G}(\mathfrak{h}\mathfrak{v}, \mathfrak{h}\mathfrak{v}, \mathcal{A}\mathfrak{u})\} \quad (1)$$

where

$$M(\mathfrak{u}, \mathfrak{v}, \mathfrak{v}) = \max\left\{\mathcal{G}(f\mathfrak{u}, \mathfrak{h}\mathfrak{v}, \mathfrak{h}\mathfrak{v}), \mathcal{G}(\mathcal{A}\mathfrak{u}, f\mathfrak{u}, f\mathfrak{u}), \mathcal{G}(\mathfrak{h}\mathfrak{v}, \mathcal{T}\mathfrak{v}, \mathcal{T}\mathfrak{v}), \frac{1}{2}[\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{h}\mathfrak{v}, \mathfrak{h}\mathfrak{v}) + \mathcal{G}(f\mathfrak{u}, \mathcal{T}\mathfrak{v}, \mathcal{T}\mathfrak{v})]\right\}, \ \mathfrak{u}, \mathfrak{v} \in \mathcal{U}.$$

Inequality (1) is called a generalized condition (B) for two pairs of self maps in $\mathcal{G}$-metric space.

Theorem 1 *Let $\mathfrak{f}, \mathfrak{h}, \mathcal{A}$, and $\mathcal{T}$ be self-maps of a $\mathcal{G}$-metric space $(\mathcal{U}, \mathcal{G})$ so that $\mathcal{T}\mathcal{U} \subset \mathfrak{f}\mathcal{U}$, $\mathcal{A}\mathcal{U} \subset \mathfrak{h}\mathcal{U}$ and verify the generalized condition (B) (1). If $\mathfrak{f}\mathcal{U}$ or $\mathfrak{h}\mathcal{U}$ is closed, then pairs $\{\mathfrak{f}, \mathcal{A}\}$ and $\{\mathfrak{h}, \mathcal{T}\}$ have a coincidence point. Besides $\mathfrak{f}, \mathfrak{h}, \mathcal{A}$, and $\mathcal{T}$ have a unique common fixed point, if pairs $\{\mathfrak{f}, \mathcal{A}\}$ and $\{\mathfrak{h}, \mathcal{T}\}$ are weakly compatible.*

Proof. Let $\mathfrak{u}_0 \in \mathcal{U}$. Since $\mathcal{A}\mathcal{U} \subset \mathfrak{h}\mathcal{U}$ and $\mathcal{T}\mathcal{U} \subset \mathfrak{f}\mathcal{U}$, there exists $\mathfrak{u}_1 \in \mathcal{U}$, so that $\mathfrak{h}\mathfrak{u}_1 = \mathcal{A}\mathfrak{u}_0$, and there exists $\mathfrak{u}_2 \in \mathcal{U}$ so that $\mathfrak{f}\mathfrak{u}_2 = \mathcal{T}\mathfrak{u}_1$, so by continuing, we can construct two sequences $\{\mathfrak{u}_n\}$ and $\{\mathfrak{v}_n\}$ in $\mathcal{U}$ as follows

$$\begin{cases} \mathfrak{v}_{2n+1} = \mathfrak{h}\mathfrak{u}_{2n+1} = \mathcal{A}\mathfrak{u}_{2n}, \\ \mathfrak{v}_{2n+2} = \mathfrak{f}\mathfrak{u}_{2n+2} = \mathcal{T}\mathfrak{u}_{2n+1}, \end{cases} \quad (2)$$

Firstly, we claim

$$\lim_{n\to\infty} \mathcal{G}(\mathfrak{v}_{2n+1}, \mathfrak{v}_{2n+2}, \mathfrak{v}_{2n+2}) = 0.$$

We observe that

$$\begin{aligned}
& \mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2}) \\
= \; & \mathcal{G}(\mathcal{A}\mathfrak{u}_{2n},\mathcal{T}\mathfrak{u}_{2n+1},\mathcal{T}\mathfrak{u}_{2n+1}) \\
\leq \; & \delta \max\Big\{\{\mathcal{G}(\mathfrak{f}\mathfrak{u}_{2n},\mathfrak{h}\mathfrak{u}_{2n+1},\mathfrak{h}\mathfrak{u}_{2n+1}),\mathcal{G}(\mathcal{A}\mathfrak{u}_{2n},\mathfrak{f}\mathfrak{u}_{2n},\mathfrak{f}\mathfrak{u}_{2n}),\mathcal{G}(\mathfrak{h}\mathfrak{u}_{2n+1},\mathcal{T}\mathfrak{u}_{2n+1},\mathcal{T}\mathfrak{u}_{2n+1}), \\
& \frac{1}{2}(\mathcal{G}(\mathcal{A}\mathfrak{u}_{2n},\mathfrak{h}\mathfrak{u}_{2n+1},\mathfrak{h}\mathfrak{u}_{2n+1})+(\mathcal{G}(\mathfrak{f}\mathfrak{u}_{2n},\mathcal{T}\mathfrak{u}_{2n+1},\mathcal{T}\mathfrak{u}_{2n+1}))\Big\}+L\min\Big\{\mathcal{G}(\mathfrak{f}\mathfrak{u}_{2n},\mathcal{A}\mathfrak{u}_{2n},\mathcal{A}\mathfrak{u}_{2n}), \\
& \mathcal{G}(\mathfrak{h}\mathfrak{u}_{2n+1},\mathcal{T}\mathfrak{u}_{2n+1},\mathcal{T}\mathfrak{u}_{2n+1}),\mathcal{G}(\mathfrak{f}\mathfrak{u}_{2n},\mathcal{T}\mathfrak{u}_{2n+1},\mathcal{T}\mathfrak{u}_{2n+1}),\mathcal{G}(\mathfrak{h}\mathfrak{u}_{2n+1},\mathfrak{h}\mathfrak{u}_{2n+1},\mathcal{A}\mathfrak{u}_{2n})\Big\} \\
= \; & \delta \max\Big\{\mathcal{G}(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+1}),\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n},\mathfrak{v}_{2n}),\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2}), \\
& \frac{1}{2}(\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+1})+\mathcal{G}(\mathfrak{v}_{2n},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2}))\Big\}+L\min\Big\{\mathcal{G}(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+1}) \\
& \mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2}),\mathcal{G}(\mathfrak{v}_{2n},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2}),\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+1})\Big\} \\
\leq \; & \delta \max\Big\{\mathcal{G}(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+1}),\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n},\mathfrak{v}_{2n}),\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2}), \\
& \frac{1}{2}[\mathcal{G}(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+1})+\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2})]\Big\} \\
& +L\min\Big\{\mathcal{G}(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+1}),\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2},0)\Big\} \\
\leq \; & \delta \max\{\mathcal{G}(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+1}),\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n},\mathfrak{v}_{2n}),\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2})\}.
\end{aligned}$$

If $\mathcal{G}(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+1}) \leq \mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2})$, then

$$\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2}) \leq \delta\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2}) < \mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2}),$$

which is a contradiction. So

$$\begin{aligned}
\mathcal{G}(\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+2},\mathfrak{v}_{2n+2}) &\leq \delta\mathcal{G}(\mathfrak{v}_{2n},\mathfrak{v}_{2n+1},\mathfrak{v}_{2n+1}) \\
&\leq \delta^2\mathcal{G}(\mathfrak{v}_{2n-1},\mathfrak{v}_{2n},\mathfrak{v}_{2n}) \\
&\leq \delta^3\mathcal{G}(\mathfrak{v}_{2n-2},\mathfrak{v}_{2n-1},\mathfrak{v}_{2n-1}) \\
&\vdots \qquad \dots \\
&\leq \delta^{2n+1}\mathcal{G}(\mathfrak{v}_0,\mathfrak{v}_1,\mathfrak{v}_1),
\end{aligned}$$

that is, $\mathcal{G}(\mathfrak{v}_{n+1},\mathfrak{v}_{n+2},\mathfrak{v}_{n+2}) \leq \delta^{n+1}\mathcal{G}(\mathfrak{v}_0,\mathfrak{v}_1,\mathfrak{v}_1)$. For all $m > n$, $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
\mathcal{G}(\mathfrak{v}_n,\mathfrak{v}_m,\mathfrak{v}_m) &\leq \mathcal{G}(\mathfrak{v}_n,\mathfrak{v}_{n+1},\mathfrak{v}_{n+1})+\mathcal{G}(\mathfrak{v}_{n+1},\mathfrak{v}_{n+2},\mathfrak{v}_{n+2})+...+\mathcal{G}(\mathfrak{v}_{m-1},\mathfrak{v}_m,\mathfrak{v}_m) \\
&\leq \delta^n(1+\delta+...+\delta^{m-n})\mathcal{G}(\mathfrak{v}_0,\mathfrak{v}_1,\mathfrak{v}_1) \\
&\leq \delta^n\frac{1-\delta^{m-n+1}}{1-\delta}\mathcal{G}(\mathfrak{v}_0,\mathfrak{v}_1,\mathfrak{v}_1) \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

that is, $\{\mathfrak{v}_n\}$ is a convergent sequence in $(\mathcal{U},\mathcal{G})$. So its subsequence $\{\mathfrak{v}_{2n+2}\} = \{\mathfrak{f}\mathfrak{u}_{2n+2}\}$ is also convergent. Suppose $\mathfrak{f}\mathcal{U}$ is closed. Hence, there exists $z \in \mathfrak{f}\mathcal{U}$ so that $\{\mathfrak{v}_{2n+2}\} \to z$, consequently $\{\mathfrak{v}_n\} \to z$.

Also, there exists $u^* \in \mathcal{U}$ so that $z = \mathfrak{f}u^*$. We submit that $z = \mathcal{A}u^*$, otherwise utilizing rectangular inequality and inequality (1) for $\mathfrak{u} = u^*$ and $\mathfrak{v} = \mathfrak{u}_{2n+1}$,

$$\begin{aligned}
\mathcal{G}(\mathcal{A}u^*, \mathfrak{f}u^*, \mathfrak{f}u^*) &\leq \mathcal{G}(\mathcal{A}u^*, \mathcal{T}\mathfrak{u}_{2n+1}, \mathcal{T}\mathfrak{u}_{2n+1}) + \mathcal{G}(\mathcal{T}\mathfrak{u}_{2n+1}, \mathfrak{f}u^*, \mathfrak{f}u^*) \\
&\leq \delta \max\Big\{\mathcal{G}(\mathfrak{f}u^*, \mathfrak{h}\mathfrak{u}_{2n+1}, \mathfrak{h}\mathfrak{u}_{2n+1}), \mathcal{G}(\mathcal{A}u^*, \mathfrak{f}u^*, \mathfrak{f}u^*), \\
&\quad \mathcal{G}(\mathfrak{h}\mathfrak{u}_{2n+1}, \mathcal{T}\mathfrak{u}_{2n+1}, \mathcal{T}\mathfrak{u}_{2n+1}), \frac{1}{2}(\mathcal{G}(\mathcal{A}u^*, \mathfrak{h}\mathfrak{u}_{2n+1}, \mathfrak{h}\mathfrak{u}_{2n+1}) + \mathcal{G}(fu^*, \mathcal{T}\mathfrak{u}_{2n+1}, \mathcal{T}\mathfrak{u}_{2n+1}))\Big\} \\
&\quad + L \min\Big\{\mathcal{G}(\mathfrak{f}u^*, \mathcal{A}u^*, \mathcal{A}u^*), \mathcal{G}(\mathfrak{h}\mathfrak{u}_{2n+1}, \mathcal{T}\mathfrak{u}_{2n+1}, \mathcal{T}\mathfrak{u}_{2n+1}), \mathcal{G}(\mathfrak{f}u^*, \mathcal{T}\mathfrak{u}_{2n+1}, \mathcal{T}\mathfrak{u}_{2n+1}), \\
&\quad \mathcal{G}(\mathfrak{h}\mathfrak{u}_{2n+1}, \mathfrak{h}\mathfrak{u}_{2n+1}, \mathcal{A}u^*)\} + \mathcal{G}(\mathfrak{v}_{2n+1}, \mathfrak{f}u^*, \mathfrak{f}u^*)\Big\} \\
&= \delta \max\Big\{\mathcal{G}(z, \mathfrak{v}_{2n+1}, \mathfrak{v}_{2n+1}), \mathcal{G}(\mathcal{A}u^*, z, z), \mathcal{G}(\mathfrak{v}_{2n+1}, \mathfrak{v}_{2n+2}, \mathfrak{v}_{2n+2}), \\
&\quad \frac{1}{2}(\mathcal{G}(\mathcal{A}u^*, \mathfrak{v}_{2n+1}, \mathfrak{v}_{2n+1}) + \mathcal{G}(z, \mathfrak{v}_{2n+2}, \mathfrak{v}_{2n+2}))\Big\} + L \min\Big\{\mathcal{G}(z, \mathcal{A}u^*, \mathcal{A}u^*), \\
&\quad \mathcal{G}(\mathfrak{v}_{2n+1}, \mathfrak{v}_{2n+2}, \mathfrak{v}_{2n+2}), \mathcal{G}(z, \mathfrak{v}_{2n+2}, \mathfrak{v}_{2n+2}) + \mathcal{G}(\mathfrak{v}_{2n+1}, \mathfrak{v}_{2n+1}, \mathcal{A}u^*)\Big\} \\
&\quad + \mathcal{G}(\mathfrak{v}_{2n+1}, \mathfrak{f}u^*, \mathfrak{f}u^*). \qquad (3)
\end{aligned}$$

Taking $\mathcal{G}$-metric as continuous and letting $n \to \infty$, we get

$$\mathcal{G}(\mathcal{A}u^*, \mathfrak{f}u^*, \mathfrak{f}u^*) \leq \delta\mathcal{G}(\mathcal{A}u^*, \mathfrak{f}u^*, \mathfrak{f}u^*) < \mathcal{G}(\mathcal{A}u^*, \mathfrak{f}u^*, \mathfrak{f}u^*),$$

which is a contradiction. Then $\mathcal{A}u^* = \mathfrak{f}u^* = z$, that is, u^* is a coincidence point for $\mathcal{A}$ and $\mathfrak{f}$.

On the other hand, $\mathcal{A}\mathcal{U} \subseteq \mathfrak{h}\mathcal{U}$ implies that $z \in \mathfrak{h}\mathcal{U}$ and there is a point $v^* \in \mathcal{U}$ so that $\mathfrak{h}v^* = z$. Next, we submit that $\mathcal{T}v^* = z$, if not by using inequality (1) we get

$$\begin{aligned}
\mathcal{G}(z, \mathcal{T}v^*, \mathcal{T}v^*) &= \mathcal{G}(\mathcal{A}u^*, \mathcal{T}v^*, \mathcal{T}v^*) \\
&\leq \delta \max\Big\{\mathcal{G}(\mathfrak{f}u^*, \mathfrak{h}v^*, \mathfrak{h}v^*), \mathcal{G}(\mathcal{A}u^*, \mathfrak{f}v^*, \mathfrak{f}v^*), \mathcal{G}(\langle v^*, \mathcal{T}v^*, \mathcal{T}v^*), \\
&\quad \frac{1}{2}\left[\mathcal{G}(\mathcal{A}u^*, \mathfrak{h}v^*, \mathfrak{h}v^*) + \mathcal{G}(\mathfrak{f}u^*, \mathcal{T}v^*, \mathcal{T}v^*)\right]\Big\} + L \min\Big\{\mathcal{G}(\mathfrak{f}u^*, \mathcal{A}u^*, \mathcal{A}u^*), \\
&\quad \mathcal{G}(\mathfrak{h}v^*, \mathcal{T}v^*, \mathcal{T}v^*), \mathcal{G}(\mathfrak{f}u^*, \mathcal{T}u^*, \mathcal{T}u^*), \mathcal{G}(\mathfrak{h}u^*, \mathfrak{h}u^*, \mathcal{A}u^*)\Big\} \\
&\leq \delta\mathcal{G}(\mathfrak{f}u^*, \mathcal{T}v^*, \mathcal{T}v^*) + L.0 \\
&< \mathcal{G}(z, \mathcal{T}v^*, \mathcal{T}v^*),
\end{aligned}$$

which is a contradiction. Then $\mathcal{T}v^* = z$ and v^* is a coincidence point for $\mathfrak{h}$ and $\mathcal{T}$. Utilizing the weak compatibility of the pair $\{\mathfrak{f}, \mathcal{A}\}$, $\mathcal{A}z = \mathcal{A}\mathfrak{f}u^* = \mathfrak{f}\mathcal{A}u^* = \mathfrak{f}z$, that is, $\mathcal{A}z = \mathfrak{f}z$. Similarly, utilizing the weak compatibility of the pair $\{\mathfrak{h}, \mathcal{T}\}$ we get $\mathfrak{h}z = \mathcal{T}z$. Now, we assert $z = \mathfrak{f}z$, if not by utilizing inequality (1), we obtain

$$\begin{aligned}\mathcal{G}(\mathfrak{f}z, z, z) &= \mathcal{G}(\mathcal{A}z, \mathcal{T}v^*, \mathcal{T}v^*)\\ &\leq \delta \max\Big\{\mathcal{G}(\mathfrak{f}z, \mathfrak{h}v^*, \mathfrak{h}v^*), \mathcal{G}(\mathcal{A}z, \mathfrak{f}z, \mathfrak{f}z), \mathcal{G}(\mathfrak{h}v^*, \mathcal{T}v^*, \mathcal{T}v^*),\\ &\quad \frac{1}{2}(\mathcal{G}(\mathcal{A}z, \mathfrak{h}v^*, \mathfrak{h}v^*) + \mathcal{G}(\mathfrak{f}z, \mathcal{T}v^*, \mathcal{T}v^*))\Big\}\\ &\quad + L \min\Big\{\mathcal{G}(\mathfrak{f}z, \mathcal{A}u^*, \mathcal{A}u^*),\ \mathcal{G}(\mathfrak{h}v^*, \mathcal{T}v^*, \mathcal{T}v^*), \mathcal{G}(\mathfrak{f}z, \mathcal{T}z, \mathcal{T}z), \mathcal{G}(\mathfrak{h}z, \mathfrak{h}z, \mathcal{A}z)\Big\}\\ &\leq \delta\mathcal{G}(\mathfrak{f}z, z, z) + L.0\\ &< \mathcal{G}(\mathfrak{f}z, z, z),\end{aligned}$$

which is a contradiction. Hence, $z = \mathfrak{f}z = \mathcal{A}z$.

On the same lines, we conclude that $z = \mathfrak{h}z = Tz$, that is, z is a common fixed point for $\mathfrak{f}, \mathfrak{h}, \mathcal{A}$, and $\mathcal{T}$. If we assume w to be one more common fixed point and utilize inequality (1) we obtain

$$\begin{aligned}\mathcal{G}(z, w^*, w^*) &= \mathcal{G}(\mathcal{A}z, \mathcal{T}w^*, \mathcal{T}w^*)\\ &\leq \delta \max\Big\{\mathcal{G}(\mathfrak{f}z, \mathfrak{h}w^*, \mathfrak{h}w^*), \mathcal{G}(\mathcal{A}z, \mathfrak{f}z, \mathfrak{f}z), \mathcal{G}(\mathfrak{h}w^*, \mathcal{T}w^*, \mathcal{T}w^*),\\ &\quad \frac{1}{2}(\mathcal{G}(\mathcal{A}z, \mathfrak{h}w^*, \mathfrak{h}w^*) + \mathcal{G}(\mathfrak{f}z, \mathcal{T}w^*, \mathcal{T}w^*))\Big\} + L \min\Big\{\mathcal{G}(\mathfrak{f}z, \mathcal{A}z, \mathcal{A}z),\\ &\quad \mathcal{G}(\mathfrak{h}w^*, \mathcal{T}w^*, \mathcal{T}w^*), \mathcal{G}(\mathfrak{f}z, \mathcal{T}w^*, \mathcal{T}w^*), \mathcal{G}(\mathfrak{h}w^*, \mathfrak{h}w^*, \mathcal{A}z)\Big\}\\ &= \delta \max\Big\{\mathcal{G}(z, w^*, w^*), \mathcal{G}(z, z, z), \mathcal{G}(w^*, w^*, w^*),\\ &\quad \frac{1}{2}(\mathcal{G}(z, w^*, w^*) + \mathcal{G}(z, w^*, w^*))\Big\} + L \min\Big\{\mathcal{G}(z, z, z),\\ &\quad \mathcal{G}(\mathfrak{h}w^*, \mathcal{T}w^*, \mathcal{T}w^*), \mathcal{G}(\mathfrak{f}z, \mathcal{T}w^*, \mathcal{T}w^*), \mathcal{G}(\mathfrak{h}w^*, \mathfrak{h}w^*, \mathcal{A}z)\Big\}\\ &\leq \delta\mathcal{G}(z, w^*, w^*) + L.0\\ &< \mathcal{G}(z, w^*, w^*),\end{aligned}$$

which is a contradiction. Hence, z is unique. ■

Theorem 1 is an enhancement of Theorem 2.1 [17] to four maps utilizing a closedness of the range space to $\mathcal{G}$-metric spaces, which is more natural than the completeness of the space. Also, it extends and generalizes Theorem 3.2 [1], Theorem 2.3 [5], Theorem 1 and 2 [7], and so on existing in the literature.

Now, if $\mathcal{A} = \mathcal{T}$ and $\mathfrak{f} = \mathfrak{h}$, we get the subsequent definition in $\mathcal{G}$-metric space.

Definition 8 *Let $\mathfrak{f}$ and $\mathcal{A}$ be self maps of a $\mathcal{G}$-metric space $(\mathcal{U}, \mathcal{G})$. If there exist $L \geq 0$ and $\delta \in [0, 1)$ such that*

$$\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}, \mathcal{A}\mathfrak{v})) \leq \delta M(\mathfrak{u}, \mathfrak{v}, \mathfrak{v}) + L \min\{\mathcal{G}(\mathfrak{f}\mathfrak{u}, \mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{u}), \mathcal{G}(\mathfrak{f}\mathfrak{v}, \mathcal{A}\mathfrak{v}, \mathcal{A}\mathfrak{v}), \mathcal{G}(\mathfrak{f}\mathfrak{u}, \mathcal{A}\mathfrak{v}, \mathcal{A}\mathfrak{v}), \mathcal{G}(\mathfrak{f}\mathfrak{v}, \mathfrak{f}\mathfrak{v}, \mathcal{A}\mathfrak{u})\}, \quad (4)$$

where

$$M(\mathfrak{u}, \mathfrak{v}, \mathfrak{v}) = \max\Big\{\mathcal{G}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{v}, \mathfrak{f}\mathfrak{v}), \mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{u}), \mathcal{G}(\mathfrak{f}\mathfrak{v}, \mathcal{A}\mathfrak{v}, \mathcal{A}\mathfrak{v}), \frac{1}{2}[\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{f}\mathfrak{v}, \mathfrak{f}\mathfrak{v}) + \mathcal{G}(\mathfrak{f}\mathfrak{u}, \mathcal{A}\mathfrak{v}, \mathcal{A}\mathfrak{v})]\Big\},$$

for $\mathfrak{u}, \mathfrak{v} \in \mathcal{U}$, then this inequality is called a generalized condition (B) for a pair of self maps in $\mathcal{G}$-metric space.

Theorem 2 *Theorem 1 is true even if we substitute*

$$M(\mathfrak{u},\mathfrak{v},\mathfrak{v})=\max\{\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathfrak{h}\mathfrak{v},\mathfrak{h}\mathfrak{v}),\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u})+\mathcal{G}(\mathfrak{h}\mathfrak{v},\mathcal{T}\mathfrak{v},\mathcal{T}\mathfrak{v}),\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{v},\mathcal{T}\mathfrak{v})+\mathcal{G}(\mathfrak{h}\mathfrak{v},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u})\}$$

in inequality (1) by

$$M'(\mathfrak{u},\mathfrak{v},\mathfrak{v})=\max\left\{\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathfrak{h}\mathfrak{v},\mathfrak{h}\mathfrak{v}),\frac{1}{2}(\mathcal{G}(f\mathfrak{u},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u})+\mathcal{G}(\mathfrak{h}\mathfrak{v},\mathcal{T}\mathfrak{v},\mathcal{T}\mathfrak{v})),\frac{1}{2}(\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{v},\mathcal{T}\mathfrak{v})+\mathcal{G}(\mathfrak{h}\mathfrak{v},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u}))\right\}.$$

If $\mathcal{A}=\mathcal{T}$ and $\mathfrak{f}=\mathfrak{h}$, we have the subsequent corollary.

Corollary 1 *Let $\mathfrak{f}$ and $\mathcal{A}$ be self- maps of a $\mathcal{G}$-metric space $(\mathcal{U},\mathcal{G})$ so that $\mathcal{A}\mathcal{U}\subseteq\mathfrak{f}\mathcal{U}$ and verify the generalized condition (B) (4). If $\mathfrak{f}\mathcal{U}$ is closed, then the pair $\{\mathfrak{f},\mathcal{A}\}$ has a coincidence point. Besides $\mathfrak{f}$ and $\mathcal{A}$ have a unique common fixed point, if a pair $\{\mathfrak{f},\mathcal{A}\}$ is weakly compatible.*

Corollary 1 extends Corollary 2.2 [17] to $\mathcal{G}$-metric spaces.

Corollary 2 *Corollary 1 is true even if we substitute*

$$M(\mathfrak{u},\mathfrak{v},\mathfrak{v})=\max\{\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{v},\mathfrak{f}\mathfrak{v}),\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u}),\mathcal{G}(f\mathfrak{v},\mathcal{A}\mathfrak{v},\mathcal{A}\mathfrak{v}),\mathcal{G}(f\mathfrak{u},\mathcal{A}\mathfrak{v},\mathcal{A}\mathfrak{v}),\mathcal{G}(\mathfrak{f}\mathfrak{v},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u})\}$$

in inequality (1) by

$$M'(\mathfrak{u},\mathfrak{v},\mathfrak{v},)=\max\left\{\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{v},\mathfrak{f}\mathfrak{v}),\frac{1}{2}[\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u})+\mathcal{G}(\mathfrak{f}\mathfrak{v},\mathcal{A}\mathfrak{v},\mathcal{A}\mathfrak{v})],\frac{1}{2}[(\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathcal{A}\mathfrak{v},\mathcal{A}\mathfrak{v})+\mathcal{G}(\mathfrak{f}\mathfrak{v},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u})]\right\}.$$

Now, we support our conclusions with two subsequent examples.

Example 2 *Let $\mathcal{U}=[0,2]$ be endowed with a G-metric*

$$\mathcal{G}(\mathfrak{u},\mathfrak{v},\mathfrak{w})=\max\{|\mathfrak{u}-\mathfrak{v}|,|\mathfrak{u}-\mathfrak{w}|,|\mathfrak{v}-\mathfrak{w}|\}.$$

Consider the maps as

$$\mathfrak{f}\mathfrak{u}=\begin{cases}2-\mathfrak{u}, & 0\le\mathfrak{u}\le1,\\ \frac{3}{2}, & 1<\mathfrak{u}\le2,\end{cases}\quad,\quad\mathfrak{h}\mathfrak{u}=\begin{cases}2\mathfrak{u}, & 0\le\mathfrak{u}<1,\\ 1, & \mathfrak{u}=1,\\ 2, & 1<\mathfrak{u}\le2,\end{cases}$$

$$\mathcal{A}\mathfrak{u}=\begin{cases}\mathfrak{u}, & 0\le\mathfrak{u}\le1,\\ \frac{\mathfrak{u}}{2}, & 1<\mathfrak{u}\le2,\end{cases}\quad,\quad and\quad\mathcal{T}\mathfrak{u}=\begin{cases}1, & 0\le\mathfrak{u}\le1,\\ \frac{3}{4}, & 1<\mathfrak{u}\le2.\end{cases}$$

Noticeably, $(\mathcal{U},\mathcal{G})$ is a $\mathcal{G}$-metric space, $\mathfrak{h}\mathcal{U}=[0,2]$ is closed, $\mathcal{T}\mathcal{U}=\{\frac{3}{4},1\}\subset\mathfrak{f}\mathcal{U}=[1,2]$ and $\mathcal{A}\mathcal{U}=[0,1]\subset\mathfrak{h}\mathcal{U}$.

Choosing $\delta=\frac{4}{5}$, in Theorem 1, we get

1. *For $\mathfrak{u},\mathfrak{v}\in[0,1]$, we obtain*

$$\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{v},\mathcal{T}\mathfrak{v})=|1-\mathfrak{u}|\le\frac{8}{5}|1-\mathfrak{u}|=\frac{4}{5}\mathcal{G}(\mathfrak{f}\mathfrak{u},f\mathfrak{u},\mathcal{A}\mathfrak{u}).$$

2. *For $\mathfrak{u}\in[0,1)$ and $\mathfrak{v}\in(1,2]$, we obtain*

$$\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{v},\mathcal{T}\mathfrak{v})=\frac{1}{2}|2\mathfrak{u}-1|\le\frac{6}{5}=\frac{4}{5}\mathcal{G}(\mathfrak{h}\mathfrak{u},\mathcal{T}\mathfrak{v},\mathcal{T}\mathfrak{v}).$$

3. *For $1<\mathfrak{u}\le2$ and $\mathfrak{v}\in[0,1]$, we obtain*

$$\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{v},\mathcal{T}\mathfrak{v})=\frac{1}{2}|2-\mathfrak{u}|\le\frac{12}{5}|3-\mathfrak{u}|=\frac{4}{5}\mathcal{G}(f\mathfrak{u},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u}).$$

4. *For* $\mathfrak{u}, \mathfrak{v} \in (1, 2]$, *we obtain*

$$\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathcal{T}\mathfrak{v}, \mathcal{T}\mathfrak{v}) = \frac{1}{2}|\mathfrak{u} - 1| \leq 1 = \frac{4}{5}\mathcal{G}(\mathfrak{h}\mathfrak{v}, \mathcal{T}\mathfrak{v}, \mathcal{T}\mathfrak{v}).$$

That is, maps $\mathfrak{f}, \mathfrak{h}, \mathcal{A}$ *and* $\mathcal{T}$ *satisfy generalized condition (1). Also,* 1 *is a coincidence point satisfying* $\mathfrak{f}\mathcal{A}1 = \mathcal{A}\mathfrak{f}1 = 1$, *and* $\mathfrak{h}\mathcal{T}1 = \mathcal{T}\mathfrak{h}1 = 1$, *so the pairs* $\{\mathfrak{f}, \mathcal{A}\}$ *and* $\{\mathfrak{h}, \mathcal{T}\}$ *are weakly compatible. Also,* $\mathfrak{f}, \mathfrak{h}, \mathcal{A}$, *and* $\mathcal{T}$ *are discontinuous at* 1. *Consequently, suppositions of Theorem 1 are validated. Hence,* 1 *is a unique common fixed point for the four maps.*

Example 3 *Let* $\mathcal{U} = \mathbb{R}^+$ *be endowed with a* $\mathcal{G}$*-metric*

$$\mathcal{G}(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \max\{|\mathfrak{u} - \mathfrak{v}|, |\mathfrak{u} - \mathfrak{w}|, |\mathfrak{v} - \mathfrak{w}|\}.$$

Define $\mathfrak{f}$ *and* $\mathcal{A}$ *by*

$$\mathfrak{f}\mathfrak{u} = \begin{cases} 2\mathfrak{u} + 1, & 0 \leq \mathfrak{u} < 2, \\ 2, & \mathfrak{u} = 2, \\ 5, & \mathfrak{u} > 2, \end{cases} \quad \text{and} \quad \mathcal{A}\mathfrak{u} = \begin{cases} \frac{\mathfrak{u}+2}{2}, & 0 \leq \mathfrak{u} \leq 2, \\ \frac{4}{3}, & \mathfrak{u} > 2. \end{cases}$$

Noticeably, $\mathfrak{f}\mathcal{U} = [1, 5]$ *is closed and* $\mathcal{A}\mathcal{U} = [1, 2] \subset f\mathcal{U}$. *Taking* $\delta = \frac{2}{3}$, *in Corollary 1, we get*

1. *For* $\mathfrak{u}, \mathfrak{v} \in [0, 2]$, *we have*

$$\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}, \mathcal{A}\mathfrak{v}) = \frac{1}{2}|\mathfrak{u} - \mathfrak{v}| \leq \frac{4}{3}|\mathfrak{u} - \mathfrak{v}| = \frac{2}{3}\mathcal{G}(\mathfrak{f}\mathfrak{u}, \mathfrak{f}\mathfrak{v}, \mathfrak{f}\mathfrak{v}).$$

2. *For* $\mathfrak{u} \in [0, 2)$ *and* $\mathfrak{v} > 2$, *we have*

$$\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}, \mathcal{A}\mathfrak{v}) = \frac{1}{6}|3\mathfrak{u} - 2| \leq \frac{22}{9} = \frac{2}{3}\mathcal{G}(\mathfrak{h}\mathfrak{v}, \mathcal{T}\mathfrak{v}, \mathcal{T}\mathfrak{v}).$$

3. *For* $\mathfrak{u} > 2$ *and* $\mathfrak{v} \in [0, 2]$, *we have*

$$\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}, \mathcal{A}\mathfrak{v}) = \frac{1}{6}|3\mathfrak{v} - 2| \leq \frac{22}{9} = \frac{2}{3}\mathcal{G}(f\mathfrak{u}, \mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{u}).$$

4. *For* $\mathfrak{u}, \mathfrak{v} \in (2, \infty)$, *we have* $\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{v}, \mathcal{A}\mathfrak{v}) = 0$.

That is, maps $\mathfrak{f}$ *and* $\mathcal{A}$ *satisfy the generalized condition (1). Also,* 2 *is a coincidence point satisfying* $\mathfrak{f}\mathcal{A}2 = \mathcal{A}\mathfrak{f}2 = 2$, *so pair* $\{\mathfrak{f}, \mathcal{A}\}$ *is weakly compatible. Also,* $\mathfrak{f}$ *and* $\mathcal{A}$ *are discontinuous at* 2.
Consequently, all the suppositions of Corollary 1 are verified and 2 *is a unique common fixed point for the maps* $\mathfrak{f}$ *and* $\mathcal{A}$.

Remark 1 (*i*) *We have evidenced common fixed point and coincidence point theorems for two pairs of self maps in a* $\mathcal{G}$*-metric space* $(\mathcal{U}, \mathcal{G})$ *via generalized condition (B) without utilizing continuity or its variant like reciprocal continuity, weak reciprocal continuity, conditional reciprocal continuity, sub-sequential continuity, sequential continuity of type* (A_f) *or* $(A_{\mathfrak{h}})$, *and so on (see, Tomar and Karapinar [29]). Further, the more natural notion, closedness of range space is taken in place of completeness of space. Our conclusions extend, generalize, and refine the conclusions of Abbas et al. [1]–[4], Babu et al. [5], Banach [6], Berinde [7]–[8], Chatterjea [9], Ćirić [10], Kannan [18], Kikina et al. [19], Zamfirescu [32], and references therein to the* $\mathcal{G}$*-metric spaces.*

(*ii*) *Abbas et al. [1] presented an open problem, if Theorem 3.1 [1] is true for* $\frac{1}{2} < \delta < 1$*? Here, we answer in the affirmative, in a non-complete* $\mathcal{G}$*-metric space. It is clear from Examples 2 and 3 that our Theorem 1 and Corollary 1 are valid for* $\delta = \frac{4}{5}$ *and* $\delta = \frac{2}{3}$ *respectively, taking the closedness of any one of the range spaces* $\mathfrak{f}\mathcal{U}$ *(or* $\mathfrak{h}\mathcal{U}$*) and the pairs* $(\mathfrak{f}, \mathcal{A})$ *and* $(\mathfrak{h}, \mathcal{T})$ *to be weakly compatible.*

(*iii*) *Abbas et al. [1] posed one more problem: Under what additional assumptions, either on a pair of self maps or on the domain of pair of self maps, do the maps satisfying condition (B) has a common fixed point? We also answer this problem in the affirmative in a non-complete $\mathcal{G}$-metric space. By assuming containment of an involved pair of self maps and one of the range spaces to be closed, the given discontinuous and a weakly compatible pair of self maps have a unique common fixed point (refer to Example 3).*

(*iv*) *If $\mathcal{G}$-metric space $(\mathcal{U},\mathcal{G})$ is not symmetric, a generalized condition (B) does not turn down to any metric condition. Consequently, our conclusions do not turn down to the coincidence and common fixed point theorems already present in metric spaces $(\mathcal{U}, d_{\mathcal{G}})$.*

(*v*) *Theorems 1, 6 and Corollaries 1, 2 (see, Examples 2 and 3) support a significant fact that there exist contractive conditions in $\mathcal{G}$-metric spaces assuring the common fixed point at the point of discontinuity of a self map (answer to Rhoades problem in [27]).*

Now, motivated by the exploration of the geometry of the set of fixed points (see, [12]–[16], [20], [22]–[25], [30]–[31], and so on), of a self-map, we define the circle, the fixed circle, and the common fixed circle via $\mathcal{G}$-metric and introduce $\mathfrak{u}_0$-generalized conditions to obtain the fixed circle and common fixed circle theorems.

Definition 9 *A circle $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r}$ with radius $r>0$ having centre at $\mathfrak{u}_0\in\mathcal{U}$ in $\mathcal{G}$-metric space $(\mathcal{U},\mathcal{G})$ is defined as*

$$\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r}=\{\mathfrak{u}\in\mathcal{U}:\mathcal{G}(\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0)=r\}. \tag{5}$$

Define

$$\begin{aligned}
r_{\mathcal{A}} &= \inf\{\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{u},\mathfrak{u}):\mathfrak{u}\neq\mathcal{A}\mathfrak{u},\mathfrak{u}\in\mathcal{U}\},\\
r_{\mathcal{T}} &= \inf\{\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{u},\mathfrak{u}):\mathfrak{u}\neq\mathcal{T}\mathfrak{u},\mathfrak{u}\in\mathcal{U}\},\\
r_{\mathfrak{f}} &= \inf\{\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathfrak{u},\mathfrak{u}):\mathfrak{u}\neq\mathfrak{f}\mathfrak{u},\mathfrak{u}\in\mathcal{U}\},\\
r_{\mathfrak{h}} &= \inf\{\mathcal{G}(\mathfrak{h}\mathfrak{u},\mathfrak{u},\mathfrak{u}):\mathfrak{u}\neq\mathfrak{h}\mathfrak{u},\mathfrak{u}\in\mathcal{U}\},\\
r_{\mathcal{A}\mathcal{T}} &= \inf\{\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u}):\mathcal{A}\mathfrak{u}\neq\mathcal{T}\mathfrak{u},\mathfrak{u}\in\mathcal{U}\},\\
r_{\mathfrak{f}\mathfrak{h}} &= \inf\{\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathfrak{h}\mathfrak{u},\mathfrak{h}\mathfrak{u}):\mathfrak{f}\mathfrak{u}\neq\mathfrak{h}\mathfrak{u},\mathfrak{u}\in\mathcal{U}\},
\end{aligned}$$

and $r^*=\inf\{r_{\mathcal{A}},r_{\mathcal{T}},r_{\mathfrak{f}},r_{\mathfrak{h}},r_{\mathcal{A}\mathcal{T}},r_{\mathfrak{f}\mathfrak{h}}\}$ for four self maps $\mathfrak{f},\mathfrak{h},\mathcal{A},\mathcal{T}$ and $\mathbf{r}^*=\inf\{r_{\mathcal{A}},r_{\mathcal{T}},r_{\mathcal{A}\mathcal{T}}\}$ for two self maps $\mathcal{A}$ and $\mathcal{T}$.

Definition 10 *Let $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r}$ be a circle having radius r and centre at $\mathfrak{u}_0$ in $\mathcal{G}$-metric space $(\mathcal{U},\mathcal{G})$. Then $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r}$ is known as a fixed circle of map $\mathcal{A}:\mathcal{U}\to\mathcal{U}$ if $\mathcal{A}\mathfrak{u}=\mathfrak{u}$, $\mathfrak{u}\in\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r}$.*

Definition 11 *Let $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,\mathbf{r}^*}$ be a circle having radius $\mathbf{r}^*$ and centre at $\mathfrak{u}_0$ in $\mathcal{G}$-metric space $(\mathcal{U},\mathcal{G})$. Then $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,\mathbf{r}^*}$ is known as a common fixed circle of two maps $\mathcal{A},\mathcal{T}:\mathcal{U}\to\mathcal{U}$ if $\mathcal{A}\mathfrak{u}=\mathcal{T}\mathfrak{u}=\mathfrak{u}$, $\mathfrak{u}\in\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,\mathbf{r}^*}$.*

Definition 12 *Let $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r^*}$ be a circle having radius r^* and centre at $\mathfrak{u}_0$ in $\mathcal{G}$-metric space $(\mathcal{U},\mathcal{G})$. Then $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r^*}$ is known as a common fixed circle of four maps $\mathfrak{f},\mathfrak{h},\mathcal{A},\mathcal{T}:\mathcal{U}\to\mathcal{U}$ if $\mathcal{A}\mathfrak{u}=\mathcal{T}\mathfrak{u}=\mathfrak{f}\mathfrak{u}=\mathfrak{h}\mathfrak{u}=\mathfrak{u}$, $\mathfrak{u}\in\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r^*}$.*

Definition 13 *Let $\mathcal{A}:\mathcal{U}\to\mathcal{U}$ be a self map in $\mathcal{G}$-metric space $(\mathcal{U},\mathcal{G})$. Suppose there exists $\mathfrak{u}_0\in\mathcal{U}$, $L\geq 0$ and $\delta\in[0,1)$ so that for $\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{u},\mathfrak{u})>0$, $\mathfrak{u}\in\mathcal{U}$, we have*

$$\begin{aligned}
\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{u},\mathfrak{u}) \leq\ & \delta\max\left\{\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{u},\mathfrak{u}),\frac{1}{2}(\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0)+\mathcal{G}(\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0))\right\}\\
&+L\min\left\{\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u},\mathfrak{u}),\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u})\right\}.
\end{aligned} \tag{6}$$

Then, inequality (6) is called a $\mathfrak{u}_0$-generalized condition (B) for a self map $\mathcal{A}$ in G-metric space.

Definition 14 *Let $\mathcal{A}$ and $\mathcal{T}$ be self maps of a $\mathcal{G}$-metric space $(\mathcal{U},\mathcal{G})$. Suppose there exists $\mathfrak{u}_0 \in \mathcal{U}$, $L \geq 0$ and $\delta \in [0,1)$ so that for $\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u}) > 0$, $\mathfrak{u} \in \mathcal{U}$, we have*

$$\begin{aligned}\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u})) \quad &\leq \quad \delta\mathbf{M}(\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0) + L\min\Big\{\mathcal{G}(\mathcal{T}\mathfrak{u},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u}),\mathcal{G}(\mathcal{T}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0),\\ &\qquad \mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}),\mathcal{G}(\mathcal{T}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0,\mathcal{A}\mathfrak{u})\Big\}, \qquad (7)\end{aligned}$$

where

$$\begin{aligned}\mathbf{M}(\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0) \quad &= \quad \max\Big\{\mathcal{G}(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u}),\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0),\\ &\qquad \frac{1}{2}[\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0)+\mathcal{G}(\mathcal{T}\mathfrak{u},\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0)]\Big\}.\end{aligned}$$

Then, inequality (7) is called a $\mathfrak{u}_0$-generalized condition (B) for a single pair of self maps in $\mathcal{G}$-metric space.

Definition 15 *Let $\mathfrak{f},\mathfrak{h},S$, and $\mathcal{T}$ be self maps of a $\mathcal{G}$-metric space $(\mathcal{U},\mathcal{G})$. Suppose there exists $\mathfrak{u}_0 \in \mathcal{U}$, $L \geq 0$, and $\delta \in [0,1)$ so that for $\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{h}\mathfrak{u},\mathfrak{h}\mathfrak{u}) > 0$, $\mathfrak{u} \in \mathcal{U}$, we have*

$$\begin{aligned}\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{h}\mathfrak{u},\mathfrak{h}\mathfrak{u}) \quad &\leq \quad \delta\mathbf{M}'(\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0) + L\min\Big\{\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u}),\mathcal{G}(\mathcal{T}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{f}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0),\\ &\qquad \mathcal{G}(\mathfrak{h}\mathfrak{u},\mathcal{T}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0)\Big\},\end{aligned}$$

where

$$\begin{aligned}\mathbf{M}'(\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0) = \max\Big\{&\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0),\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{u}),\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0),\\ &\frac{1}{2}[\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{h}\mathfrak{u}_0,\mathfrak{h}\mathfrak{u}_0)+\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0)]\Big\}. \qquad (8)\end{aligned}$$

Then, inequality (8) is called a $\mathfrak{u}_0$-generalized condition (B) for two pairs of self maps in $\mathcal{G}$-metric space.

Next, we explore new fixed-circle results via $\mathfrak{u}_0$-generalized condition (B) in $\mathcal{G}$-metric spaces for a single map, a pair, and two pairs of self maps .

Theorem 3 *Suppose there exists a self-map $\mathcal{A}:\mathcal{U}\to\mathcal{U}$ in $\mathcal{G}$-metric space $(\mathcal{U},\mathcal{G})$ so that for $\mathfrak{u},\mathfrak{u}_0 \in \mathcal{U}$,*

1. *a map $\mathcal{A}$ satisfies a $\mathfrak{u}_0$-generalized condition (B) (6),*
2. *$r_{\mathcal{A}} \leq \mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0) \leq \mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{u},\mathfrak{u})$.*

Then $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r_{\mathcal{A}}}$, a circle having radius $r_{\mathcal{A}}$ and centre at $\mathfrak{u}_0$ in $(\mathcal{U},\mathcal{G})$ is a fixed circle of $\mathcal{A}$.

Proof. Suppose $\mathcal{A}\mathfrak{u}_0 \neq \mathfrak{u}_0$, that is, $\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0) > 0$. Now, using hypotheses 1 and 2,

$$\begin{aligned}\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0)) \quad &\leq \quad \delta\max\Big\{\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0),\frac{1}{2}[\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0)+\mathcal{G}(\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0)]\Big\}\\ &\qquad +L\min\Big\{\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0)\Big\}\\ &= \quad \delta\max\Big\{\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0),\frac{1}{2}\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0)\Big\} + L.0\\ &\leq \quad \delta\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0),\end{aligned}$$

which is a contradiction. Hence, $\mathcal{A}\mathfrak{u}_0 = \mathfrak{u}_0$. If $r_{\mathcal{A}} = 0$, then $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0, r_{\mathcal{A}}} = \{\mathfrak{u}_0\}$ and $\mathcal{A}\mathfrak{u}_0 = \mathfrak{u}_0$, that is, $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0, r_{\mathcal{A}}}$ is a fixed circle of $\mathcal{A}$, and the result is concluded. So, let $r_{\mathcal{A}} > 0$ and $\mathfrak{u} \in \mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0, r_{\mathcal{A}}}$ be any point such that $\mathcal{A}\mathfrak{u} \neq \mathfrak{u}$. From definition of $r_{\mathcal{A}}$, $\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{u}, \mathfrak{u}) > r_{\mathcal{A}}$. Again, since $\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{u}, \mathfrak{u}) > 0$,

$$\begin{aligned}
\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{u}, \mathfrak{u})) &\leq \delta \max\left\{\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{u}_0, \mathfrak{u}_0), \mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{u}, \mathfrak{u}), \frac{1}{2}[\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{u}_0, \mathfrak{u}_0) + \mathcal{G}(\mathfrak{u}, \mathfrak{u}_0, \mathfrak{u}_0)]\right\} \\
&\quad + L \min\left\{\mathcal{G}(\mathcal{A}\mathfrak{u}_0, \mathfrak{u}, \mathfrak{u}), \mathcal{G}(\mathcal{A}\mathfrak{u}_0, \mathfrak{u}_0, \mathfrak{u}_0), \mathcal{G}(\mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{u}_0, \mathcal{A}\mathfrak{u}_0), \mathcal{G}(\mathcal{A}\mathfrak{u}_0, \mathcal{A}\mathfrak{u}, \mathcal{A}\mathfrak{u})\right\} \\
&= \delta \max\left\{\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{u}_0, \mathfrak{u}_0), \mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{u}, \mathfrak{u}), \frac{1}{2}[\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{u}_0, \mathfrak{u}_0) + r]\right\} + L.0 \\
&\leq \delta \mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{u}, \mathfrak{u}),
\end{aligned}$$

which is a contradiction. Hence, $\mathcal{A}\mathfrak{u} = \mathfrak{u}$, for all $\mathfrak{u} \in \mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0, r_{\mathcal{A}}}$, that is, $\mathcal{A}$ fixes the circle $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0, r_{\mathcal{A}}}$. ■

Example 4 *Let $U = \mathbb{R}$ and a $\mathcal{G}$-metric $\mathcal{G} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ be*

$$\mathcal{G}(\mathfrak{u}, \mathfrak{v}, \mathfrak{w}) = \max\{|\mathfrak{u} - \mathfrak{v}|, |\mathfrak{v} - \mathfrak{w}|, |\mathfrak{w} - \mathfrak{u}|\}.$$

Define $\mathcal{A} : \mathcal{U} \to \mathcal{U}$ as

$$\mathcal{A}\mathfrak{u} = \begin{cases} \mathfrak{u}, & \mathfrak{u} \in (-5, 5), \\ \frac{\mathfrak{u}}{5}, & \textit{otherwise}. \end{cases}$$

Now,

$$r_{\mathcal{A}} = \inf\left\{\frac{4\mathfrak{u}}{5} : \mathfrak{u} \geq 5\right\} = 4.$$

If $\mathfrak{u}_0 = 0$, then for $\mathfrak{u} \in \mathcal{U}$, a map $\mathcal{A}$ satisfies all the suppositions of Theorem (3) with $\delta \in [\frac{2}{9}, 1)$ and $L \geq 0$. Hence, $\mathcal{C}^{\mathcal{G}}_{0,3} = \{-4, 4\}$ is a fixed circle of $\mathcal{A}$.

Theorem 4 *Suppose there exist self-maps $\mathcal{A}, \mathcal{T} : \mathcal{U} \to \mathcal{U}$ in $\mathcal{G}$-metric space $(\mathcal{U}, \mathcal{G})$ so that for $\mathfrak{u}, \mathfrak{u}_0 \in \mathcal{U}$,*

1. *a pair of maps $\mathcal{A}, \mathcal{T}$ satisfies the $\mathfrak{u}_0$-generalized condition (B) (7),*
2. *$\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{u}_0, \mathfrak{u}_0) \leq \mathcal{G}(\mathcal{A}\mathfrak{u}, \mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{u})$,*
3. *$\mathcal{G}(\mathcal{T}\mathfrak{u}, \mathfrak{u}_0, \mathfrak{u}_0) \leq \mathcal{G}(\mathcal{A}\mathfrak{u}, \mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{u})$,*
4. *$\mathcal{A}\mathfrak{u}_0 = \mathcal{T}\mathfrak{u}_0 = \mathfrak{u}_0$,*
5. *$\mathcal{A}$ (or $\mathcal{T}$) satisfies the $\mathfrak{u}_0$-generalized condition (B) (6).*

Then $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0, \mathbf{r}^*}$ is a common fixed circle of $\mathcal{A}$ and $\mathcal{T}$.

Proof. Let $\mathbf{r}^* = 0$. Then $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0, \mathbf{r}^*} = \{\mathfrak{u}_0\}$ and $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0, \mathbf{r}^*}$ is a common fixed circle of $\mathcal{A}$ and $\mathcal{T}$, and this concludes the result. So presume that $\mathbf{r}^* > 0$ and $\mathfrak{u} \in \mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0, \mathbf{r}^*}$ be any point so that $\mathcal{A}\mathfrak{u} \neq \mathcal{T}\mathfrak{u}$, that is, $\mathcal{G}(\mathcal{A}\mathfrak{u}, \mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{u}) > 0$.

Now, using hypotheses 1–3,

$$\begin{aligned}\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u})) &\leq \delta\max\Big\{\mathcal{G}(\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u}),\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0),\\ &\frac{1}{2}[\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0)+\mathcal{G}(\mathcal{T}\mathfrak{u},\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0)]\Big\}+\mathbf{L}\min\Big\{\mathcal{G}(\mathcal{T}\mathfrak{u},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u}),\mathcal{G}(\mathcal{T}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0),\\ &\mathcal{G}(\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}),\mathcal{G}(\mathcal{T}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0,\mathcal{A}\mathfrak{u})\Big\}\\ &= \delta\max\Big\{\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u}),\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0),\frac{1}{2}[\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0)+\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0)]\Big\}\\ &+\mathbf{L}\min\Big\{\mathcal{G}(\mathcal{T}\mathfrak{u},\mathcal{A}\mathfrak{u},\mathcal{A}\mathfrak{u}),\mathcal{G}(\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathfrak{u}_0,\mathfrak{u}_0,\mathcal{T}\mathfrak{u}),\mathcal{G}(\mathfrak{u}_0,\mathfrak{u}_0,\mathcal{A}\mathfrak{u})\Big\}\\ &= \delta\mathcal{G}(\mathcal{A}\mathfrak{u},\mathcal{T}\mathfrak{u},\mathcal{T}\mathfrak{u}),\end{aligned}$$

which is a contradiction. So, $\mathcal{A}\mathfrak{u}=\mathcal{T}\mathfrak{u}$. Now, suppose $\mathcal{A}$ satisfies $\mathfrak{u}_0$-generalized condition (B) (6), then following the pattern of Theorem 3, $\mathcal{A}\mathfrak{u}=\mathfrak{u}=\mathcal{T}\mathfrak{u}$. If we assume, $\mathcal{T}$ satisfies $\mathfrak{u}_0$-generalized condition (B) (6), then again following the similar pattern $\mathcal{T}\mathfrak{u}=\mathfrak{u}=\mathcal{A}\mathfrak{u}$. Consequently, $\mathcal{A}\mathfrak{u}=\mathcal{T}\mathfrak{u}=\mathfrak{u},\mathfrak{u}\in\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r^*}$, that is, $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r^*}$ is a common fixed circle of $\mathcal{A}$ and $\mathcal{T}$. ■

Example 5 *Let $\mathcal{U}=\mathbb{R}$ and a $\mathcal{G}$-metric $\mathcal{G}:\mathcal{U}\times\mathcal{U}\times\mathcal{U}\to\mathbb{R}^+$ be*

$$\mathcal{G}(\mathfrak{u},\mathfrak{v},\mathfrak{w})=\max\{|\mathfrak{u}-\mathfrak{v}|,|\mathfrak{v}-\mathfrak{w}|,|\mathfrak{w}-\mathfrak{u}|\}.$$

Define $\mathcal{A},\mathcal{T}:\mathcal{U}\to\mathcal{U}$ as

$$\mathcal{A}\mathfrak{u}=\begin{cases}\mathfrak{u}, & \mathfrak{u}\in[0,12),\\ \frac{\mathfrak{u}}{4}, & \textit{otherwise},\end{cases}\quad \textit{and}\quad \mathcal{T}\mathfrak{u}=\begin{cases}\mathfrak{u}, & \mathfrak{u}\in[0,12),\\ \frac{\mathfrak{u}}{2}, & \textit{otherwise}.\end{cases}$$

Now,

$$r_{\mathcal{A}}=\inf\{\frac{3\mathfrak{u}}{4}:\mathfrak{u}\geq 12\}=9,$$

$$r_{\mathcal{T}}=\inf\{\frac{\mathfrak{u}}{2}:\mathfrak{u}\geq 12\}=6,$$

$$r_{\mathcal{AT}}=\inf\{\frac{\mathfrak{u}}{4}:\mathfrak{u}\geq 12\}=3,$$

that is, $\mathbf{r}^*=\inf\{r_{\mathcal{A}},r_{\mathcal{T}},r_{\mathcal{AT}}\}=3$. If $\mathfrak{u}_0=0$, then for all $\mathfrak{u}\in\mathcal{U}$, maps $\mathcal{A}$ and $\mathcal{T}$ validate the suppositions of Theorem 4 for $\delta\in[\frac{1}{2},1)$ and $L\geq 0$. Hence, $\mathcal{C}^{\mathcal{G}}_{0,3}=\{-3,3\}$ is a common fixed circle of $\mathcal{A}$ and $\mathcal{T}$.

Theorem 5 *Suppose there exist self-maps $\mathfrak{f},\mathfrak{h},\mathcal{A},\mathcal{T}:\mathcal{U}\to\mathcal{U}$ in $\mathcal{G}$-metric space $(\mathcal{U},\mathcal{G})$ so that $\mathcal{AU}\subseteq\mathfrak{f}\mathcal{U}$ or $\mathcal{TU}\subseteq\mathfrak{h}\mathcal{U}$ and pairs $\{\mathcal{A},\mathfrak{h}\}$ and $\{\mathcal{T},\mathfrak{f}\}$ are weakly compatible so that*

1. *two pairs of maps $(\mathfrak{f},\mathfrak{h})$ and $(\mathcal{A},\mathcal{T})$ satisfy the $\mathfrak{u}_0$-generalized condition (B) (8),*
2. *$\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0)\leq\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{u})$,*
3. *$\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0)\leq\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{u})$,*
4. *$\mathcal{G}(\prime\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{u})\leq\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{h}\mathfrak{u},\mathfrak{h}\mathfrak{u})$,*
5. *$\mathcal{A}\mathfrak{u}_0=\mathcal{T}\mathfrak{u}_0=\mathfrak{f}\mathfrak{u}_0=\mathfrak{h}\mathfrak{u}_0=\mathfrak{u}_0$,*
6. *map $\mathcal{A}$ (or $\mathfrak{h}$) and $\mathcal{T}$ (or $\mathfrak{f}$) satisfy a $\mathfrak{u}_0$-generalized condition (B) (6).*

Then $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r^*}$ *is a common fixed circle of* $\mathfrak{f},\mathfrak{h},\mathcal{A}$, *and* $\mathcal{T}$.

Proof. Let $r^*=0$, then $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r^*}=\{\mathfrak{u}_0\}$ and $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r^*}$ is a common fixed circle of $\mathfrak{f},\mathfrak{h},\mathcal{A}$, and $\mathcal{T}$, and the validation of conclusion is complete. So presume that $r^*>0$ and $\mathfrak{u}^*\in\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r^*}$ be any point so that $\mathfrak{u}^*=\mathcal{A}\mathfrak{u}\neq\mathcal{T}\mathfrak{u}$. Now, using hypotheses 1–5,

$$\begin{aligned}
\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{h}\mathfrak{u},\mathfrak{h}\mathfrak{u})) &\leq \delta\max\Big\{\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0),\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{u}),\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathcal{T}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0),\\
&\quad \frac{1}{2}[\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{h}\mathfrak{u}_0,\mathfrak{h}\mathfrak{u}_0)+\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0)]\Big\}+\mathbf{L}\min\Big\{\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0),\\
&\quad \mathcal{G}(\mathcal{T}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{f}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0),\mathcal{G}(\mathfrak{h}\mathfrak{u},\mathcal{T}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0)\Big\}\\
&= \delta\max\Big\{\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{u}),\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0),\\
&\quad \frac{1}{2}[\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0)+\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0)]\Big\}+\mathbf{L}\min\Big\{\mathcal{G}(\mathfrak{f}\mathfrak{u},\mathfrak{u}_)),\mathfrak{u}_0),\\
&\quad \mathcal{G}(\mathcal{T}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathfrak{h}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0)\Big\}\\
&\leq \delta\max\Big\{\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{u}),\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{u}),\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{u}),\\
&\quad \frac{1}{2}[\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{u})+\mathcal{G}(\mathcal{T}\mathfrak{u},\mathfrak{f}\mathfrak{u},\mathfrak{f}\mathfrak{u})]\Big\}\\
&\leq \delta\mathcal{G}(\mathcal{A}\mathfrak{u},\mathfrak{h}\mathfrak{u},\mathfrak{h}\mathfrak{u}),
\end{aligned}$$

which is a contradiction. Hence, $\mathcal{A}\mathfrak{u}=\mathfrak{h}\mathfrak{u}=\mathfrak{u}^*$. On the other hand, $\mathcal{A}\mathcal{U}\subseteq\mathfrak{f}\mathcal{U}$ implies that $\mathfrak{u}^*\in\mathfrak{f}\mathcal{U}$ and there exists a point $\mathfrak{v}\in\mathcal{U}$ so that $\mathfrak{f}\mathfrak{v}=\mathfrak{u}^*$. Next, we submit that $\mathfrak{f}\mathfrak{v}=\mathcal{T}\mathfrak{v}$, if not by using inequality (8) and hypothesis 4, we get

$$\begin{aligned}
\mathcal{G}(\mathcal{T}\mathfrak{v},\mathfrak{f}\mathfrak{v},\mathfrak{f}\mathfrak{v}) &\leq \mathcal{G}(\mathcal{A}\mathfrak{v},\mathfrak{h}\mathfrak{v},\mathfrak{h}\mathfrak{v})\\
&\leq \delta\max\Big\{\mathcal{G}(\mathcal{T}\mathfrak{v},\mathfrak{f}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0),\mathcal{G}(\mathcal{T}\mathfrak{v},\mathfrak{f}\mathfrak{v},\mathfrak{f}\mathfrak{v}),\mathcal{G}(\mathfrak{f}\mathfrak{v},\mathcal{T}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0),\\
&\quad \frac{1}{2}[\mathcal{G}(\mathcal{T}\mathfrak{v},\mathfrak{h}\mathfrak{u}_0,\mathfrak{h}\mathfrak{u}_0)+\mathcal{G}(\mathfrak{f}\mathfrak{v},\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0)]\Big\}+\mathbf{L}\min\Big\{\mathcal{G}(\mathfrak{f}\mathfrak{v},\mathcal{A}\mathfrak{u}_0,\mathcal{A}\mathfrak{u}_0),\\
&\quad \mathcal{G}(\mathcal{T}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{v},\mathfrak{f}\mathfrak{u}_0,\mathfrak{f}\mathfrak{u}_0),\mathcal{G}(\mathfrak{h}\mathfrak{u},\mathcal{T}\mathfrak{u}_0,\mathcal{T}\mathfrak{u}_0)\Big\}\\
&= \delta\max\Big\{\mathcal{G}(\mathcal{T}\mathfrak{v},\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathcal{T}\mathfrak{v},\mathfrak{f}\mathfrak{v},\mathfrak{f}\mathfrak{v}),\mathcal{G}(\mathfrak{f}\mathfrak{v},\mathfrak{u}_0,\mathfrak{u}_0),\\
&\quad \frac{1}{2}[\mathcal{G}(\mathcal{T}\mathfrak{v},\mathfrak{u}_0,\mathfrak{u}_0)+\mathcal{G}(\mathfrak{f}\mathfrak{v},\mathfrak{u}_0,\mathfrak{u}_0)]\Big\}+\mathbf{L}\min\Big\{\mathcal{G}(\mathfrak{f}\mathfrak{v},\mathfrak{u}_0,\mathfrak{u}_0),\\
&\quad \mathcal{G}(\mathcal{T}\mathfrak{u}_0,\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathcal{A}\mathfrak{v},\mathfrak{u}_0,\mathfrak{u}_0),\mathcal{G}(\mathfrak{h}\mathfrak{u},\mathfrak{u}_0,\mathfrak{u}_0)\Big\}\\
&\leq \delta\max\Big\{\mathcal{G}(\mathcal{T}(\mathfrak{v},\mathfrak{f}\mathfrak{v},\mathfrak{f}\mathfrak{v}),\mathcal{G}(\mathcal{T}\mathfrak{v},\mathfrak{f}\mathfrak{v},\mathfrak{f}\mathfrak{v}),\mathcal{G}(\mathcal{T}\mathfrak{v},\mathfrak{f}\mathfrak{v},\mathfrak{f}\mathfrak{v}),\\
&\quad \frac{1}{2}[\mathcal{G}(\mathcal{T}\mathfrak{v},\mathfrak{f}\mathfrak{v},\mathfrak{f}\mathfrak{v})+\mathcal{G}(\mathcal{T}\mathfrak{v},\mathfrak{f}\mathfrak{v},\mathfrak{f}\mathfrak{v})]\Big\}\\
&= \delta\mathcal{G}(\mathcal{T}\mathfrak{v},\mathfrak{f}\mathfrak{v},\mathfrak{f}\mathfrak{v}),
\end{aligned}$$

which is a contradiction. Then $\mathcal{T}\mathfrak{v} = \mathfrak{f}\mathfrak{v} = \mathfrak{u}^*$ and $\mathfrak{v}$ is a coincidence point for $\mathcal{T}$ and $\mathfrak{f}$. Utilizing weak compatibility of the pair $\{\mathcal{A},\mathfrak{h}\}$, $\mathcal{A}\mathfrak{u}^* = \mathcal{A}\mathfrak{h}\mathfrak{u} = \mathfrak{h}\mathcal{A}\mathfrak{u} = \mathfrak{h}\mathfrak{u}^*$, that is, $\mathcal{A}\mathfrak{u}^* = \mathfrak{h}\mathfrak{u}^*$. Similarly, utilizing the weak compatibility of the pair $\{\mathcal{T},\mathfrak{f}\}$ we get $\mathcal{T}\mathfrak{u}^* = \mathfrak{f}\mathfrak{u}^*$.

Suppose $\mathcal{A}$ satisfies $\mathfrak{u}_0$-generalized condition (B) (6), then following the pattern of Theorem 3, $\mathcal{A}\mathfrak{u}^* = \mathfrak{u}^*$, consequently, $\mathcal{A}\mathfrak{u}^* = \mathfrak{h}\mathfrak{u}^* = \mathfrak{u}^*,\ \mathfrak{u}^* \in \mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r^*}$. If we suppose that $\mathcal{T}$ satisfies $\mathfrak{u}_0$-generalized condition (B) (6), then again following the similar pattern $\mathcal{T}\mathfrak{u}^* = \mathfrak{f}\mathfrak{u}^* = \mathfrak{u}^*,\ \mathfrak{u}^* \in \mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r^*}$. A similar result holds if we assume that $\mathfrak{h}$ as well $\mathfrak{f}$ satisfies the $\mathfrak{u}_0$-generalized condition (B) (6), that is, $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r^*}$ is a common fixed circle of $\mathcal{A},\mathcal{T},\mathfrak{f}$, and $\mathfrak{h}$. ■

Example 6 *Let $\mathcal{U} = \mathbb{R}$ and a $\mathcal{G}$-metric $\mathcal{G} : \mathcal{U}\times\mathcal{U}\times\mathcal{U} \to \mathbb{R}^+$ be*

$$\mathcal{G}(\mathfrak{u},\mathfrak{v},\mathfrak{w}) = \max\{|\mathfrak{u}-\mathfrak{v}|, |\mathfrak{v}-\mathfrak{w}|, |\mathfrak{w}-\mathfrak{u}|\}.$$

Define $\mathcal{A},\mathcal{T} : \mathcal{U} \to \mathcal{U}$ as

$$\mathcal{A}\mathfrak{u} = \begin{cases} \mathfrak{u}, & \mathfrak{u} \in [-10,10], \\ 2, & \textit{otherwise}, \end{cases} \quad \mathcal{T}\mathfrak{u} = \begin{cases} \mathfrak{u}, & \mathfrak{u} \in (-10,10), \\ 0, & \textit{otherwise}, \end{cases}$$

$$\mathfrak{f}\mathfrak{u} = \begin{cases} \mathfrak{u}, & \mathfrak{u} \in (-10,10), \\ \frac{\mathfrak{u}}{5}, & \textit{otherwise}, \end{cases} \quad \textit{and} \quad \mathfrak{h}\mathfrak{u} = \begin{cases} \mathfrak{u}, & \mathfrak{u} \in [-10,10], \\ 5, & \textit{otherwise}. \end{cases}$$

Now, $\mathcal{A}\mathcal{U} \subseteq \mathfrak{f}\mathcal{U}$, $\mathcal{T}\mathcal{U} \subseteq \mathfrak{h}\mathcal{U}$, pairs $\{\mathcal{A},\mathfrak{h}\}$, $\{\mathcal{T},\mathfrak{f}\}$ are weakly compatible, and

$$\begin{aligned} r_{\mathcal{A}} &= \inf\{|\mathfrak{u}-2| : \mathfrak{u} \in (-\infty,10) \cup (10,\infty)\} = 8, \\ r_{\mathcal{T}} &= \inf\{|\mathfrak{u}| : \mathfrak{u} \in (-\infty,10] \cup (10,\infty)\} = 10, \\ r_{\mathcal{A}\mathcal{T}} &= \inf\{|2-0| : \mathfrak{u} \in (-\infty,10) \cup (10,\infty)\} = 2, \\ r_{\mathcal{A}} &= \inf\{\frac{4\mathfrak{u}}{5} : \mathfrak{u} \in (-\infty,10] \cup [10,\infty)\} = 8, \\ r_{\mathfrak{h}} &= \inf\{|\mathfrak{u}-5| : \mathfrak{u} \in (-\infty,10) \cup [10,\infty)\} = 5, \\ r_{\mathfrak{f}\mathfrak{h}} &= \inf\{\frac{|25-\mathfrak{u}|}{5} : \mathfrak{u} \in (-\infty,10) \cup [10,\infty)\} = 3, \end{aligned}$$

that is, $r^* = \inf\{r_{\mathcal{A}}, r_{\mathcal{T}}, r_{\mathfrak{f}}, r_{\mathfrak{h}}, r_{\mathcal{A}\mathcal{T}}, r_{\mathfrak{f}\mathfrak{h}}\} = 2$. If $\mathfrak{u}_0 = 5$, then for all $\mathfrak{u} \in \mathcal{U}$ maps $\mathcal{A}$, $\mathcal{T}$, $\mathfrak{f}$, and $\mathfrak{h}$ validate the suppositions of Theorem 5 with $\delta \in [\frac{2}{3},1)$ and $L \geq 0$. Hence, $\mathcal{C}^{\mathcal{G}}_{5,2} = \{3,7\}$ is a common fixed circle of maps $\mathcal{A},\mathcal{T},\mathfrak{f}$, and $\mathfrak{h}$.

Remark 2 *(i) Noticeably, the radius of the fixed circle, as well as the common fixed circle is independent of the selection of center in Theorems 3, 4, and 5 (see, Examples 4, 5, and 6).*

(ii) Theorems 4 and 5 have answered, in the setting of $\mathcal{G}$-metric space, regarding the existence of the conditions to make any circle $\mathcal{C}^{\mathcal{G}}_{\mathfrak{u}_0,r}$ as the common fixed circle for two as well as four self-maps respectively (see, Examples 5, and 6).

(iii) The investigation of new conditions which ensure a disc to be fixed by self-map is also very significant. If we replace the equality in (5) (Definition 9) by less than or equal to sign, we get the definition of a disc in a $\mathcal{G}$-metric space. On the lines of Definition 10-12, we may define the fixed discs and common fixed discs (for a pair and two pairs of self-maps) and by slightly modifying the postulates of Theorems 3, 4, and 5, we may establish the fixed disc and common fixed disc conclusions. For more fixed disc conclusions, we refer to [15], [25], [31], and so on.

3 Application

We utilize Theorem 1 to investigate the solution for a system of non-linear Volterra-Hammerstein integral equations in real-valued and measurable functions on $(0,\infty)$, that is, $\mathcal{U} = L^2((0,\infty),\mathbb{R})$. Define $d : \mathcal{U}\times\mathcal{U} \to \mathbb{R}^+$ so that

$$d(\mathfrak{f},\mathfrak{h}) = \int_0^\infty |\mathfrak{f}(\mathfrak{u}) - \mathfrak{h}(\mathfrak{u})| d\mathfrak{u},$$

for each $\mathfrak{f},\mathfrak{h} \in \mathcal{U}$. Endow $\mathcal{U}$ with a $\mathcal{G}$-metric as

$$\mathcal{G}(\mathfrak{u},\mathfrak{v},\mathfrak{w}) = \max\{d(\mathfrak{u},\mathfrak{v}), d(\mathfrak{v},\mathfrak{w}), d(\mathfrak{u},\mathfrak{w})\}, \quad \mathfrak{u},\mathfrak{v},\mathfrak{w} \in \mathcal{U}.$$

Clearly, $(\mathcal{U},\mathcal{G})$ is a $\mathcal{G}$-metric space. Assume the system of Volterra-Hammerstein non-linear integral equations to be

$$\mathfrak{u}(t) = p_1(t) - p_2(t) + \lambda \int_0^t m(t,s) q_i(s,\mathfrak{u}(s))ds + \mu \int_0^\infty k(t,s) h_j(s,\mathfrak{u}(s))ds, \tag{9}$$

$t \in [0,\infty)$, where p_1, p_2 are known, with $p_1(t) \geq p_2(t)$, λ,μ are real numbers, $m(t,s)$, $k(t,s)$, q_i, h_j, $i \neq j$, $i,j = 1,2$, and are real-valued functions measurable in t and s on $[0,\infty)$, so that the subsequent conditions are satisfied

(C_1) $\sup_{\mathfrak{u}\in[0,\infty)} \int_0^\infty |m(t,s)|dt = M_1 < \infty$;

(C_2) $\sup_{\mathfrak{u}\in[0,\infty)} \int_0^\infty |k(t,s)|dt = M_2 < \infty$;

(C_3) $q_i \in \mathcal{U}, i = 1,2$ and there exists $K_1 > 0$ so that for $s \in [0,\infty)$

$$|q_i(s,\mathfrak{u}(s)) - q_j(s,\mathfrak{v}(s))| \leq K_1 |\mathfrak{u}(s) - \mathfrak{v}(s)| \ ;$$

(C_4) $h_i \in \mathcal{U}$ and there exists $K_2 > 0$ so that for $s \in [0,\infty)$

$$|h_i(s,\mathfrak{u}(s)) - h_j(s,\mathfrak{v}(s))| \leq K_2 |\mathfrak{u}(s) - \mathfrak{v}(s)| \ \text{ for all } \mathfrak{u},\mathfrak{v} \in \mathcal{U}.$$

Theorem 6 *Under the conditions (C_1)–(C_4), assume that the subsequent postulates hold*

(a) $\mu \int_0^\infty k(t,s) h_i\Big(s, \lambda \int_0^s m(s,t) q_j(t,\mathfrak{u}(t))dt + p_1(s) - p_2(s)\Big)ds = 0,\ \ i \neq j$ *and* $i,j = 1,2.$

(b) *For* $\mathfrak{u} \in \mathcal{U}$

$$\lambda \int_0^t m(t,s) q_i(s,\mathfrak{u}(s))ds = -p_1 + p_2 + \mu \int_0^\infty k(t,s) h_i(s,\mathfrak{u}(s))ds = \Gamma_i(s) \in \mathcal{U}.$$

(c) *For some* $\Gamma_i(t) \in \mathcal{U}$. *there exists* $\Theta_i(t) \in \mathcal{U}$ *such that*

$$\begin{aligned} & -p_2(t) + \lambda \int_0^t m(t,s) q_i(s,\mathfrak{u}(s) - \Gamma_i)(s) + p_2(s)ds \\ = \ & -p_1 + \mu \int_0^\infty k(t,s)\Big(h_i(s,\Gamma_i(s) - p_2(s)) - h_i(s,\mathfrak{u}(s))\Big)ds = \Theta_i(t), \quad i = 1,2. \end{aligned}$$

Then the system of Volterra-Hammerstein non-linear integral equations (9) has a unique solution in $\mathcal{U}$ if $|\mu|K_2M_2 < 1$ and $\frac{|\lambda|K_1M_1}{1-|\mu|K_2M_2} < 1$.

Proof. Define $\mathfrak{f}, \mathfrak{h}, \mathcal{A}$, and $\mathcal{T}$ as

$$\mathfrak{f}\mathfrak{u}(t) = \mathfrak{u}(t) - p_1(t) + \mu \int_0^\infty k(t,s) h_2(s, \mathfrak{u}(s)) ds,$$

$$\mathfrak{h}\mathfrak{u}(t) = \mathfrak{u}(t) - p_1(t) + \mu \int_0^\infty k(t,s) h_1(s, \mathfrak{u}(s)) ds,$$

$$\mathcal{A}\mathfrak{u}(t) = -p_2(t) + \lambda \int_0^t m(t,s) q_1(s, \mathfrak{u}(s)) ds,$$

$$\mathcal{T}\mathfrak{u}(t) = -p_2(t) + \lambda \int_0^t m(t,s) q_2(s, \mathfrak{u}(s)) ds.$$

The point $\mathfrak{u}$ is a solution of (9) if and only if $\mathfrak{u}$ is a common fixed point for $\mathfrak{f}, \mathfrak{h}, \mathcal{A}$, and $\mathcal{T}$. We assert that the postulates of Theorem 1 are valid. Firstly, we show that the maps are self maps on $\mathcal{U}$.

$$\begin{aligned} |\mathcal{A}\mathfrak{u}(t)| &\leq |p_2(t)| + |\lambda| \int_0^t |m(t,s) q_1(s, \mathfrak{u}(s))| ds \\ &\leq |p_2(t)| + |\lambda| \sup_{t \in [0,\infty)} |m(t,s)| \int_0^t |q_1(s, \mathfrak{u}(s))| ds \end{aligned}$$

and

$$\int_0^\infty |\mathcal{A}\mathfrak{u}(t)| dt \leq \int_0^\infty |p_2(t)| dt + |\lambda| \sup_{t \in [0,\infty)} \int_0^\infty \Big(|m(t,s)| \int_0^\infty |q_1(s, \mathfrak{u}(s))| ds \Big) dt.$$

Since $p_2 \in \mathcal{U}$ and using conditions (C_1), we get

$$\int_0^\infty |\mathcal{A}\mathfrak{u}(t)| dt \leq \int_0^\infty |p_2(t)| dt + |\lambda| M_1 \int_0^\infty |q_1(s, \mathfrak{u}(s))| ds < +\infty.$$

Hence, $\mathcal{A} \in \mathcal{U}$. Similarly, we find $\mathcal{T} \in \mathcal{U}$.

For map $\mathfrak{f}$, since $\mathfrak{u}, p_1, h_2 \in \mathcal{U}$ and using (C_2), we get

$$\int_0^\infty |\mathfrak{f}\mathfrak{u}(t)| dt \leq \int_0^\infty |p_1(t)| + \int_0^\infty |\mathfrak{u}(t)| dt + M_2 |\mu| \int_0^\infty |h_2(s, \mathfrak{u}(s))| ds < +\infty.$$

Hence, $\mathfrak{f} \in \mathcal{U}$, similarly $\mathfrak{h} \in \mathcal{U}$. Now, we assert that $\mathcal{T}\mathcal{U} \subseteq \mathfrak{f}\mathcal{U}$, by using (a) we get

$$\begin{aligned} \mathfrak{f}(\mathcal{T}\mathfrak{u}(t) + p_1(t)) &= \mathcal{T}\mathfrak{u}(t) + \mu \int_0^\infty k(t,s) h_2(s, p_1 - p_2 + \lambda \int_0^t m(t,s) q_1(t, \mathfrak{u}(t)) dt) ds \\ &= \mathcal{T}\mathfrak{u}(t). \end{aligned}$$

Hence, $\mathcal{T}\mathcal{U} \subseteq \mathfrak{f}\mathcal{U}$. Similarly, we find $\mathcal{A}\mathcal{U} \subseteq \mathfrak{h}\mathcal{U}$. Now, we prove that $\mathfrak{f}\mathcal{U}$ is $\mathcal{G}$-closed. Take a sequence $\{\mathfrak{u}_n\} \subseteq \mathcal{U}$ converging to $\mathfrak{u} \in \mathcal{U}$ and the sequence $\{\mathfrak{f}\mathfrak{u}_n\}$ converges to $\mathfrak{v}$. We claim that $\mathfrak{v} = \mathfrak{f}\mathfrak{u} \in \mathfrak{f}\mathcal{U}$, by using condition (C_4), we get

$$\begin{aligned} d(\mathfrak{f}\mathfrak{u}_n, \mathfrak{f}\mathfrak{u}) = \int_0^\infty |\mathfrak{f}\mathfrak{u}_n - \mathfrak{f}\mathfrak{u}| dt &\leq d(\mathfrak{u}_n, \mathfrak{u}) + \int_0^\infty |\mu| M_2 \int_0^\infty |h_1(s, \mathfrak{u}_n(s)) - h_2(s, \mathfrak{u}(s))| ds \\ &\leq d(\mathfrak{u}_n, \mathfrak{u}) + |\mu| M_2 K_2 \int_0^\infty |\mathfrak{u}_n(s) - \mathfrak{u}(s)| ds \\ &= (1 + |\mu| M_2 K_2) d(\mathfrak{u}_n, \mathfrak{u}) \to 0 \text{ as } n \to \infty, \end{aligned}$$

that is, $\lim_{n\to\infty} \mathcal{G}(\mathfrak{u}_n, \mathfrak{u}, \mathfrak{u}) = 0$. Hence, $\mathfrak{f}\mathcal{U}$ is $\mathcal{G}$-closed.

Next, we establish that pair $\{\mathfrak{f}, \mathcal{A}\}$ is weakly compatible. Suppose that $\mathfrak{u}$ is a coincidence point for $\mathfrak{f}$ and $\mathcal{A}$, we have

$$\begin{aligned}
\mathfrak{f}\mathcal{A}\mathfrak{u} &= \mathcal{A}\mathfrak{u} - p_1(t) + \mu \int_0^\infty \Big(k(t,s)h_2(s, -p_2 + \lambda \int_0^s m(s,t)q_1(t,\mathfrak{u}(t)))dt\Big)ds \\
&= -p_2(t) + \lambda \int_0^t m(t,s)q_1(s,\mathfrak{u}(s))ds - p_1(t) \\
&\quad + \mu \int_0^\infty \Big(k(t,s)h_2(s, -p_2 + \lambda \int_0^s m(s,t)q_1(t,\mathfrak{u}(t)))dt\Big)ds,
\end{aligned}$$

$$\mathcal{A}\mathfrak{f}\mathfrak{u} = -p_2(t) + \lambda \int_0^t \Big(m(t,s)q_1(s,\mathfrak{u}(s) - p_1 + \mu \int_0^\infty k(t,s)h_2(t,\mathfrak{u}(t)))dt\Big)ds,$$

$$\begin{aligned}
|\mathfrak{f}\mathcal{A}\mathfrak{u} - \mathcal{A}\mathfrak{f}\mathfrak{u}| &= \Big| -p_1(t) - p_2(t) + \lambda \int_0^t m(t,s)q_1(s,\mathfrak{u}(s))ds \\
&\quad + \mu \int_0^\infty \Big(k(t,s)h_2(s, -p_2 + \lambda \int_0^s m(s,t)q_1(t,\mathfrak{u}(t))dt\Big)ds + p_2(t) - \\
&\quad \lambda \int_0^t \Big(m(t,s)q_1(s,\mathfrak{u}(s) - p_1 + \mu \int_0^\infty k(t,s)h_2(t,\mathfrak{u}(t))dt)\Big)ds \Big| \\
&= \Big| \lambda \int_0^t m(t,s)q_1(s,\mathfrak{u}(s))ds + \mu \int_0^\infty \Big(k(t,s)h_2(s, \Gamma_2(s) - p_2(s))ds\Big) \\
&\quad + \mu \int_0^\infty k(t,s)h_2(s,\mathfrak{u}(s))ds - \lambda \int_0^t m(t,s)q_1(s,\mathfrak{u}(s) - \Gamma_1(s) + p_2(s))ds \Big|.
\end{aligned}$$

From (b) and (c), we get

$$\begin{aligned}
|\mathfrak{f}\mathcal{A}\mathfrak{u} - \mathcal{A}\mathfrak{f}\mathfrak{u}| &= \Big| -p_1 + p_2 + \mu \int_0^\infty k(t,s)h_1(s,\mathfrak{u}(s))ds + p_1 - p_2 \\
&\quad + \mu \int_0^\infty k(t,s)h_2(s,\mathfrak{u}(s))ds \Big| \\
&= \Big| \mu \int_0^\infty k(t,s)\Big(h_1(s,\mathfrak{u}(s)) - h_2(s,\mathfrak{u}(s))\Big)ds \Big| \\
&\leq \int_0^\infty |k(t,s)| \big| h_1(s,\mathfrak{u}(s)) - h_2(s,\mathfrak{u}(s)) \big| ds \\
&\leq K \int_0^\infty |k(t,s)|.|\mathfrak{u}(s) - \mathfrak{u}(s)|ds = 0.
\end{aligned}$$

Hence, the pair $\{\mathfrak{f}, \mathcal{A}\}$ is weakly compatible. On the same pattern, the pair $\{\mathfrak{h}, \mathcal{T}\}$ is also weakly compatible. Now,

$$\begin{aligned}
d(\mathcal{A}\mathfrak{u}, \mathcal{T}\mathfrak{v}) &= \int_0^\infty |\mathcal{A}\mathfrak{u}(s) - \mathcal{T}\mathfrak{v}(s)|ds \\
&\leq |\lambda| \int_0^\infty |m(t,s)|(|q_1(s,\mathfrak{u}(s)) - q_2(s,\mathfrak{v}(s))|ds \\
&\leq |\lambda| K_1 M_1 \int_0^\infty |\mathfrak{u}(s) - \mathfrak{v}(s)|ds \\
&= |\lambda| K_1 M_1 d(\mathfrak{u}, \mathfrak{v}).
\end{aligned}$$

Similarly,

$$\begin{aligned} d(\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{v}) &\leq |\mathfrak{u}(t) - \mathfrak{v}(t) - \mu \int_0^\infty k(t,s)(h_1(s,\mathfrak{u}(s)) - h_2(s,\mathfrak{v}(s))ds| \\ &\geq d(\mathfrak{u},\mathfrak{v}) - |\mu| K_2 M_2 d(\mathfrak{u},\mathfrak{v}) \\ &= (1 - |\mu| K_2 M_2) d(\mathfrak{u},\mathfrak{v}). \end{aligned}$$

Hence,

$$d(\mathcal{A}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq \frac{|\lambda| K_1 M_1}{1 - |\mu| K_2 M_2} d(f\mathfrak{u}, \mathfrak{h}\mathfrak{v}),$$

that is,

$$\begin{aligned} \mathcal{G}(\mathcal{A}\mathfrak{u}, \mathcal{T}\mathfrak{v}, \mathcal{T}\mathfrak{v}) &\leq \delta \mathcal{G}(\mathfrak{f}\mathfrak{u}, \mathfrak{h}\mathfrak{v}, \mathfrak{h}\mathfrak{v}) \\ &\leq \delta M(\mathfrak{u},\mathfrak{v},\mathfrak{v}) + L \min\{\mathcal{G}(\mathfrak{f}\mathfrak{u}, \mathcal{A}\mathfrak{u}, \mathfrak{u}), \mathcal{G}(\mathfrak{h}\mathfrak{v}, \mathcal{T}\mathfrak{v}, \mathcal{T}\mathfrak{v}), \mathcal{G}(\mathfrak{f}\mathfrak{u}, \mathcal{T}\mathfrak{v}, \mathcal{T}\mathfrak{v}), \mathcal{G}(\mathcal{A}\mathfrak{u}, \mathfrak{h}\mathfrak{v}, \mathfrak{h}\mathfrak{v})\}, \end{aligned}$$

where $\delta = \frac{|\lambda| K_1 M_1}{1 - |\mu| K_2 M_2} < 1$. Consequently, all the postulates of Theorem 1 are valid and $\mathfrak{f}, \mathfrak{h}, \mathcal{A}$, and $\mathcal{T}$ have a unique common fixed point and this point is the solution of a system (9). ■

Remark 3 *Pathak et al. [26] studied system (9) in a metric space, so Theorem 6 generalizes Theorem 4.1 [26] to $\mathcal{G}$-metric spaces.*

4 Conclusion

We have established the existence and uniqueness of a fixed point, coincidence point, and common fixed point for a single pair and two pairs of discontinuous self maps via generalized condition (B) in $\mathcal{G}$-metric spaces. Our theorems and corollaries are improved and enhanced versions of renowned conclusions wherein completeness and continuity have not been utilized. We have provided a novel explanation to two open problems of Abbas et al. [1] regarding the range of δ and additional assumptions, either on pair of self maps or on the domain of pair of self maps, satisfying condition (B) for the survival of common fixed points and to a problem of Rhoades [27] on the question of the existence of contractive map having a fixed point at the point of discontinuity in a non-complete $\mathcal{G}$-metric space. Further, we have introduced circle, fixed circle, common fixed circle, via novel $\mathfrak{u}_0$-generalized condition (B) to examine the geometry of a set of fixed points in a $\mathcal{G}$-metric space. It is relevant to examine some conditions which exclude the possibility of an identity map in Theorems 3, 4, and 5 in some future work. Illustrative examples and an application to find the solution of the Volterra-Hammerstein non-linear integral equation authenticate the utility of our conclusions.

Acknowledgement. The authors thank the referee for his / her constructive comments, in particular, for calling our attention to the references [20] and [25].

References

[1] M. Abbas, G. V. R. Babu and G. N. Alemayehu, On common fixed points of weakly compatible mappings satisfying 'generalized condition (B)', Filomat, 25(2011), 9–19.

[2] M. Abbas and D. Ilic, Common fixed points of generalized almost nonexpansive mappings, Filomat, 24(2010), 11–18.

[3] M. Abbas, S. H. Khan and T. Nazir, Common fixed points of R-weakly commuting maps in generalized metric space, Fixed Point Theory and Appl., 2011(2011), 1–11.

[4] M. Abbas, T. Nazir and D. Doric, Common fixed point of mappings satisfying (EA) property in generalized metric spaces, Appl. Math. Comput., 218(2012), 7665–7670.

[5] G. V. R. Babu, M. L. Sandhya and M. V. R. Kameshwari, A note on a fixed point theorem of Berinde on weak contractions, Carpathian J. Math., 9(2008), 8–12.

[6] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équation intégrales, Fund. Math., 3(1992), 133–181.

[7] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum, 9(2004), 43–53.

[8] V. Berinde, General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces, Carpathian J. Math., 24(2008), 10–19.

[9] S. K. Chatterjea, Fixed point theorems, C. R. Acad. Bulgare Sci., 25(1972), 727–730.

[10] Lj. B. Ćirić, A generalization of Banach's contraction principle, Proc. Am. Math. Soc., 45(1974), 267–273.

[11] G. Jungck and R. E. Rhoades, Fixed point for set valued functions without continuity, Indian J. Pure Appl. Math., 29(1998), 227–238.

[12] M. Joshi, A. Tomar, H. A. Nabwey and R. George, On unique and nonunique fixed points and fixed circles in M_v^b-metric space and application to cantilever beam problem, J. Funct. Spaces 2021, Article ID 6681044, (2021), 15 pages.

[13] M. Joshi and A. Tomar, On unique and non-unique fixed points in metric spaces and application to chemical sciences, J. Funct. Spaces, 2021, Article ID 5525472, (2021), 11 pages.

[14] M. Joshi, A. Tomar and S. K. Padaliya, On geometric properties of non-unique fixed points in b-metric spaces, Chapter in a book "Fixed Point Theory and its Applications to Real World Problem" Nova Science Publishers, New York, USA, (2021), 33–50.

[15] M. Joshi, A. Tomar and S. K. Padaliya, Fixed point to fixed disc and application in partial metric spaces, Chapter in a book "Fixed point theory and its applications to real world problem" Nova Science Publishers, New York, USA, (2021), 391–406.

[16] M. Joshi, A. Tomar and S. K. Padaliya, Fixed point to fixed ellipse in metric spaces and discontinuous activation function, Appl. Math. E-Notes, 1(2021), 225–237.

[17] A. Kaewcharoen and T. Yuyin, Unique common fixed point theorems on partial metric spaces, J. Nonlinear Sci. App., 7(2014), 90–101.

[18] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc., 60(1968), 71–76.

[19] L. Kikina, K. Kikina, and I. Vardhami, Fixed point theorems for almost contractions in generalized metric spaces, Creative Math. Inf., 23(2014), 65–72.

[20] N. Mlaiki, N. Taş and N. Y. Özgür, On the fixed-circle problem and Khan type contractions, Axioms, 7(2018), 80.

[21] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Conv. Anal., 7(2006), 289–297.

[22] N. Y. Özgür, N. Taş and U. Çelik, New fixed-circle results on $\mathcal{S}$-metric spaces, Bull. Math. Anal. Appl., 9(2017), 10–23.

[23] N. Y. Özgür and N. Taş, Some fixed-circle theorems on metric spaces, Bull. Malays. Math. Sci. Soc., 4(2019), 1433–1449.

[24] N. Y. Özgür and N. Taş, Some fixed-circle theorems and discontinuity at fixed circle, AIP Conf. Proc., 1926(2018), 020048 pp.

[25] N. Özgür, Fixed-disc results via simulation functions, Turkish J. Math., 43(2019), 2794–2805.

[26] H. K. Pathak, M. S. Khanb and R. Tiwari, A common fixed point theorem and its application to nonlinear integral equations, Computer. Math. Appl., (2007), 961–971.

[27] B. E. Rhoades, Contractive definitions and continuity, Fixed point theory and its applications (Berkeley, CA, 1986), 233–245,Contemp. Math., 72, Amer. Math. Soc., Providence, RI, 1988.

[28] S. L. Singh and A. Tomar, Weaker forms of commuting maps and existence of fixed points, J. Korea. Soc. Math. Educ. ser. B Pure Appl. Math., 10(2003), 145–161.

[29] A. Tomar and E. Karapinar, On variants of continuity and existence of fixed point via Meir-Keeler contractions in MC-spaces, J. Adv. Math. Stud., 9(2016), 348–359.

[30] A. Tomar, M. Joshi, and S. K. Padaliya, Fixed point to fixed circle and activation function in partial metric space, J. Appl. Anal., 28(2022), 57–66.

[31] A. Tomar and M. Joshi, Near fixed point, near fixed interval circle and near fixed interval disc in metric interval space, Chapter in a book "Fixed Point Theory and its Applications to Real World Problem" Nova Science Publishers, New York, USA, (2021), 131–150.

[32] T. Zamfirescu, Fix point theorems in metric spaces, Arch. Math., 23(1972), 292–298.