

Some Fixed Point Theorems Involving α -Admissible Self-Maps And Geraghty Functions In b -Metric-Like Spaces*

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Abstract

Incited by the lately proposed exciting concept of the α -admissibility type S , in this manuscript, we come by a couple of fixed point results involving Geraghty functions in b -metric-like context. Further, our findings complement, extend and unify several known results in the existing studies and also bring forth some new theorems as consequences. Moreover, we construct suitable non-trivial numerical examples to endorse our obtained results. Finally, we claim that the notion of cyclic (α, β) -admissible mappings of type S , coined by Mongkolkeha et al. [J. Nonlinear Sci. Appl., 11(9):1056-1069, 2018], is equivalent to that of cyclic (α, β) -admissible mappings.

1 Introduction and Preliminaries

Metric fixed point theory comes out as an intensive research domain since the remarkable Banach contraction principle, [4], in 1922. Instinctively, umpteen articles came to the light where generalizations of the metric notion are studied by revising the basic metric axioms [5, 6, 8, 9, 12, 13, 14, 16, 19, 20, 22, 27]. Maybe one of such compelling extensions is the idea of b -metric-like spaces. Alghamdi et al. [2] conceived the notion of a b -metric-like space as a proper generalization of the partial metric spaces, metric-like spaces and b -metric spaces, which is as follows.

Definition 1 ([2]) Let Δ be a non-empty set and $s \geq 1$ be a given real number. Suppose the mapping $d : \Delta \times \Delta \rightarrow [0, \infty)$ satisfies:

- (i) $d(\rho, \varrho) = 0$ implies that $\rho = \varrho$;
- (ii) $d(\rho, \varrho) = d(\varrho, \rho)$, for all $\rho, \varrho \in \Delta$;
- (iii) $d(\rho, \varrho) \leq s[d(\rho, z) + d(z, \varrho)]$, for all $\rho, \varrho, z \in \Delta$.

Then (Δ, d) is called a b -metric-like space with the coefficient s .

The notions of Cauchy sequences, completeness, open subsets on b -metric-like spaces are defined in [2] and furthermore, the continuity in such spaces is defined in [7]. For more notions and topological developments in b -metric-like spaces, readers are referred to [2, 7, 18] and the references therein. However, the notion of α -admissibility was initially introduced by Samet et al. [28] in the standard metric space setting.

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Definition 2 ([28]) *Let Δ be any non-empty set and also suppose that $\alpha : \Delta \times \Delta \rightarrow [0, \infty)$ be any mapping. A self-mapping Γ defined on Δ is called an α -admissible mapping if the following condition holds:*

$$\rho, \varrho \in \Delta \text{ with } \alpha(\rho, \varrho) \geq 1 \text{ implies that } \alpha(\Gamma\rho, \Gamma\varrho) \geq 1.$$

In a recent article, Sintunavarat [30] coined a new kind of α -admissibility in the b -metric context, which is termed as α -admissibility of type S and was defined as follows:

Definition 3 ([30]) *Let $\alpha : \Delta \times \Delta \rightarrow [0, \infty)$ be any mapping where Δ is a non-empty set and also let s be a given real number such that $s \geq 1$. A mapping $\Gamma : \Delta \rightarrow \Delta$ is said to be an α -admissible mapping of type S if the following holds:*

$$\rho, \varrho \in \Delta \text{ with } \alpha(\rho, \varrho) \geq s \text{ implies that } \alpha(\Gamma\rho, \Gamma\varrho) \geq s.$$

Bringing into play this interesting concept and also altering distance functions, the author affirmed several fixed point results along with some non-trivial numerical examples. For a purposeful study, the readers are referred to [23, 30] for more terminologies, theories and results on this notion. Subsequently, we bring up the idea of a Geraghty function which was originated by Geraghty [15].

Definition 4 ([15]) *A function $\beta : [0, \infty) \rightarrow (0, 1)$ is called a Geraghty function if $(r_n) \subseteq [0, \infty)$ and $\lim_{n \rightarrow \infty} \beta(r_n) = 1^+$ implies that $\lim_{n \rightarrow \infty} r_n = 0^+$.*

In this article, before all else, we explore a couple of exciting fixed point theorems involving α -admissible self-maps and Geraghty functions in the setting of b -metric-like spaces. Our achieved results are demonstrated by competent constructive and non-trivial numerical examples. Additionally, we infer a number of fixed point results from the obtained theorems as consequences in various abstract spaces. Further, we make an important remark on the concept of cyclic (α, β) -admissible mappings of type S regarding its equivalency with the previous kind of cyclic (α, β) -admissible mappings.

2 Fixed Point Results

This section revolves around a couple of fixed point results related to α -admissible self-maps involving a Geraghty function. We construct non-trivial numerical examples to validate the achieved findings as well.

Theorem 1 *Let (Δ, d) be a complete b -metric-like space with coefficient s such that $s \geq 1$, and let Γ be any α -admissible self-map of type S . Assume that whenever $\alpha(\rho, \varrho) \geq s$ with $\rho \neq \varrho$, Γ satisfies*

$$s^3(d(\Gamma\rho, \Gamma\varrho) + l)^{\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho)} \leq \beta(M(\rho, \varrho))M(\rho, \varrho) + l \tag{1}$$

for $s \geq 1$, where

$$M(\rho, \varrho) = \max \left\{ d(\rho, \varrho), d(\rho, \Gamma\rho), d(\varrho, \Gamma\varrho), \frac{d(\rho, \Gamma\varrho) + d(\varrho, \Gamma\rho)}{4s} \right\},$$

β is a Geraghty function and $l \geq 1$. Suppose

(i) *there exists $\rho_0 \in \Delta$ such that $\alpha(\rho_0, \Gamma\rho_0) \geq s$;*

(ii) *α has transitive property of type S , that is, for all $\rho, \varrho, z \in \Delta$,*

$$\alpha(\rho, \varrho) \geq s \text{ and } \alpha(\varrho, z) \geq s \text{ implies that } \alpha(\rho, z) \geq s;$$

(iii) *Γ is continuous;*

(iiia) *if there exists a sequence (ρ_n) with $\lim_{n \rightarrow \infty} \rho_n = u$ and $\alpha(\rho_n, \rho_{n+1}) \geq s$, then*

$$\alpha(\rho_n, u) \geq s \text{ and } \alpha(u, \Gamma u) \geq \frac{1}{s}.$$

Then $Fix(\Gamma) \neq \emptyset$, with $d(u, u) = 0$ where $u \in Fix(\Gamma)$.

Proof. From condition (i), there is a $\rho_0 \in \Delta$ such that $\alpha(\rho_0, \Gamma\rho_0) \geq s$ holds. Now, utilizing this, we define a sequence (ρ_n) by $\rho_n = \Gamma^n \rho_0$ for all $n \in \mathbb{N}_0$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $\rho_p = \rho_{p+1}$ for some $p \in \mathbb{N}_0$, then Γ must have a fixed point $\rho_p \in \Delta$. So, we assume $\rho_n \neq \rho_{n+1}$ for all $n \in \mathbb{N}_0$. Since Γ is an α -admissible self-map of type S , we have

$$\alpha(\rho_1, \Gamma\rho_1) \geq s.$$

Similarly, applying mathematical induction, we get

$$\alpha(\rho_n, \Gamma\rho_n) \geq s,$$

for all $n \in \mathbb{N}_0$. Hence from (1), we have

$$\begin{aligned} d(\Gamma\rho_n, \Gamma\rho_{n+1}) + l &\leq s^3(d(\Gamma\rho_n, \Gamma\rho_{n+1}) + l)^{\alpha(\rho_n, \Gamma\rho_n)\alpha(\rho_{n+1}, \Gamma\rho_{n+1})} \\ &\leq \beta(M(\rho_n, \rho_{n+1}))M(\rho_n, \rho_{n+1}) + l \end{aligned}$$

where

$$\begin{aligned} M(\rho_n, \rho_{n+1}) &= \max \left\{ d(\rho_n, \rho_{n+1}), d(\rho_n, \rho_{n+1}), d(\rho_{n+1}, \rho_{n+2}), \frac{d(\rho_n, \rho_{n+2}) + d(\rho_n, \rho_n)}{4s} \right\} \\ &\leq \max \left\{ d(\rho_n, \rho_{n+1}), d(\rho_{n+1}, \rho_{n+2}), \frac{s[d(\rho_n, \rho_{n+1}) + d(\rho_{n+1}, \rho_{n+2}) + d(\rho_n, \rho_{n+1}) + d(\rho_{n+1}, \rho_n)]}{4s} \right\} \\ &= \max \left\{ d(\rho_n, \rho_{n+1}), d(\rho_{n+1}, \rho_{n+2}), \frac{3d(\rho_n, \rho_{n+1}) + d(\rho_{n+1}, \rho_{n+2})}{4} \right\} \\ &= \max \{ d(\rho_n, \rho_{n+1}), d(\rho_{n+1}, \rho_{n+2}) \}. \end{aligned}$$

When $M(\rho_n, \rho_{n+1}) = d(\rho_{n+1}, \rho_{n+2})$, then

$$\begin{aligned} d(\rho_{n+1}, \rho_{n+2}) + l &\leq s^3(d(\rho_{n+1}, \rho_{n+2}) + l)^{\alpha(\rho_{n+1}, \Gamma\rho_{n+1})\alpha(\rho_{n+2}, \Gamma\rho_{n+2})} \\ &\leq \beta(d(\rho_{n+1}, \rho_{n+2}))d(\rho_{n+1}, \rho_{n+2}) + l \\ &< d(\rho_{n+1}, \rho_{n+2}) + l, \end{aligned}$$

a contradiction. Hence $M(\rho_n, \rho_{n+1}) = d(\rho_n, \rho_{n+1})$ and

$$\begin{aligned} d(\rho_{n+1}, \rho_{n+2}) + l &\leq s^3(d(\rho_{n+1}, \rho_{n+2}) + l)^{\alpha(\rho_{n+1}, \Gamma\rho_{n+1})\alpha(\rho_{n+2}, \Gamma\rho_{n+2})} \\ &\leq \beta(d(\rho_n, \rho_{n+1}))d(\rho_n, \rho_{n+1}) + l \\ &< d(\rho_n, \rho_{n+1}) + l. \end{aligned} \tag{2}$$

So, we have $d(\rho_{n+1}, \rho_{n+2}) < d(\rho_n, \rho_{n+1})$. Hence $(d(\rho_n, \rho_{n+1}))$ is a decreasing sequence of reals. Let

$$\lim_{n \rightarrow \infty} d(\rho_n, \rho_{n+1}) = r.$$

First we suppose, $r > 0$. Then from (2), we get

$$d(\rho_{n+1}, \rho_{n+2}) \leq \beta(d(\rho_n, \rho_{n+1}))d(\rho_n, \rho_{n+1}),$$

this implies

$$\frac{d(\rho_{n+1}, \rho_{n+2})}{d(\rho_n, \rho_{n+1})} \leq \beta(d(\rho_n, \rho_{n+1})) < 1.$$

Letting $n \rightarrow \infty$ and using the Sandwich theorem, we obtain

$$\lim_{n \rightarrow \infty} \beta(d(\rho_n, \rho_{n+1})) = 1$$

which implies that

$$\lim_{n \rightarrow \infty} d(\rho_n, \rho_{n+1}) = 0.$$

Next, we show that (ρ_n) is Cauchy. On the contrary, suppose the sequence is not Cauchy. Then for any given $\epsilon > 0$, there exist two subsequences of naturals, say (m_k) and (n_k) such that $m_k > n_k > k$ with

$$d(\rho_{m_k}, \rho_{n_k}) \geq \epsilon \quad \text{and} \quad d(\rho_{m_k}, \rho_{n_k-1}) < \epsilon.$$

Using the triangular inequality, we get

$$\epsilon \leq d(\rho_{m_k}, \rho_{n_k}) \leq s[d(\rho_{m_k}, \rho_{n_k-1}) + d(\rho_{n_k-1}, \rho_{n_k})] < s[\epsilon + d(\rho_{n_k-1}, \rho_{n_k})]. \tag{3}$$

Taking limit as $k \rightarrow \infty$, we get

$$\frac{\epsilon}{s} \leq \limsup_{k \rightarrow \infty} d(\rho_{m_k}, \rho_{n_k-1}) \leq \epsilon.$$

Again, we know,

$$\begin{aligned} d(\rho_{m_k+1}, \rho_{n_k}) &\leq s[d(\rho_{m_k+1}, \rho_{m_k}) + d(\rho_{m_k}, \rho_{n_k})] \\ &< sd(\rho_{m_k+1}, \rho_{m_k}) + s^2[d(\rho_{m_k}, \rho_{n_k-1}) + d(\rho_{n_k-1}, \rho_{n_k})]. \end{aligned}$$

Letting $k \rightarrow \infty$, we obtain

$$\limsup_{k \rightarrow \infty} d(\rho_{m_k+1}, \rho_{n_k}) \leq s^2\epsilon. \tag{4}$$

Furthermore, employing the triangle inequality,

$$\epsilon \leq d(\rho_{m_k}, \rho_{n_k}) \leq s[d(\rho_{m_k}, \rho_{m_k+1}) + d(\rho_{m_k+1}, \rho_{n_k})].$$

Utilizing (4) and letting $k \rightarrow \infty$, we have

$$\epsilon \leq s \limsup_{k \rightarrow \infty} d(\rho_{m_k+1}, \rho_{n_k})$$

and that leads to

$$\frac{\epsilon}{s} \leq \limsup_{k \rightarrow \infty} d(\rho_{m_k+1}, \rho_{n_k}) \leq s^2\epsilon.$$

Finally,

$$\frac{\epsilon}{s} \leq \limsup_{k \rightarrow \infty} d(\rho_{m_k+1}, \rho_{n_k}) \leq s \limsup_{k \rightarrow \infty} [d(\rho_{m_k+1}, \rho_{n_k+1}) + d(\rho_{n_k+1}, \rho_{n_k})].$$

As $k \rightarrow \infty$,

$$\frac{\epsilon}{s^2} \leq \limsup_{k \rightarrow \infty} d(\rho_{m_k+1}, \rho_{n_k+1}). \tag{5}$$

Moreover,

$$\begin{aligned} d(\rho_{m_k+1}, \rho_{n_k+1}) &\leq s[d(\rho_{m_k+1}, \rho_{n_k}) + d(\rho_{n_k}, \rho_{n_k+1})] \\ &\leq s^3\epsilon + sd(\rho_{n_k}, \rho_{n_k+1}). \end{aligned}$$

Letting $k \rightarrow \infty$,

$$\limsup_{k \rightarrow \infty} d(\rho_{m_k+1}, \rho_{n_k+1}) \leq s^3\epsilon. \tag{6}$$

From (5) and (6), we obtain

$$\frac{\epsilon}{s^2} \leq \limsup_{k \rightarrow \infty} d(\rho_{m_k+1}, \rho_{n_k+1}) \leq s^3\epsilon. \tag{7}$$

Since α has transitive property of type S , using this repeatedly we have

$$\alpha(\rho_{m_k}, \rho_{n_k}) \geq s.$$

Further, we claim that $\rho_{m_k} \neq \rho_{n_k}$. Otherwise, we have $\rho_{m_k} = \rho_{n_k}$, that is $\Gamma\rho_{m_k} = \Gamma\rho_{n_k}$, i.e., $\rho_{m_k+1} = \rho_{n_k+1}$, which implies that

$$d(\rho_{m_k}, \rho_{m_k+1}) < d(\rho_{m_k-1}, \rho_{m_k}) < \cdots < d(\rho_{n_k}, \rho_{n_k+1}) = d(\rho_{m_k}, \rho_{m_k+1}),$$

and this is impossible. Hence from (1)

$$\begin{aligned} s^3(d(\Gamma\rho_{m_k}, \Gamma\rho_{n_k}) + l) &\leq s^3(d(\Gamma\rho_{m_k}, \Gamma\rho_{n_k}) + l)^{\alpha(\rho_{m_k}, \Gamma\rho_{m_k})\alpha(\rho_{n_k}, \Gamma\rho_{n_k})} \\ &\leq \beta(M(\rho_{m_k}, \rho_{n_k}))M(\rho_{m_k}, \rho_{n_k}) + l, \end{aligned} \quad (8)$$

where

$$\begin{aligned} M(\rho_{m_k}, \rho_{n_k}) &= \max \left\{ d(\rho_{m_k}, \rho_{n_k}), d(\rho_{m_k}, \rho_{m_k+1}), d(\rho_{n_k}, \rho_{n_k+1}), \frac{d(\rho_{m_k}, \rho_{n_k+1}) + d(\rho_{n_k}, \rho_{m_k+1})}{4s} \right\} \\ &\leq \max \left\{ d(\rho_{m_k}, \rho_{n_k}), d(\rho_{m_k}, \rho_{m_k+1}), d(\rho_{n_k}, \rho_{n_k+1}), \right. \\ &\quad \left. \frac{s[d(\rho_{m_k}, \rho_{n_k}) + d(\rho_{n_k}, \rho_{n_k+1}) + d(\rho_{n_k}, \rho_{m_k}) + d(\rho_{m_k}, \rho_{m_k+1})]}{4s} \right\}. \end{aligned}$$

Letting $k \rightarrow \infty$, we have

$$\limsup_{k \rightarrow \infty} M(\rho_{m_k}, \rho_{n_k}) \leq \limsup_{k \rightarrow \infty} d(\rho_{m_k}, \rho_{n_k}). \quad (9)$$

Therefore from (3), (8) and (9),

$$\begin{aligned} s^3 \limsup_{k \rightarrow \infty} d(\Gamma\rho_{m_k}, \Gamma\rho_{n_k}) + l &\leq s^3 [\limsup_{k \rightarrow \infty} d(\Gamma\rho_{m_k}, \Gamma\rho_{n_k}) + l] \\ &\leq s^3 [\limsup_{k \rightarrow \infty} d(\Gamma\rho_{m_k}, \Gamma\rho_{n_k}) + l]^{\alpha(\rho_{m_k}, \Gamma\rho_{m_k})\alpha(\rho_{n_k}, \Gamma\rho_{n_k})} \\ &\leq \limsup_{k \rightarrow \infty} \beta(M(\rho_{m_k}, \rho_{n_k})) \limsup_{k \rightarrow \infty} M(\rho_{m_k}, \rho_{n_k}) + l, \end{aligned} \quad (10)$$

so,

$$\begin{aligned} s^3 \limsup_{k \rightarrow \infty} d(\rho_{m_k+1}, \rho_{n_k+1}) &\leq \limsup_{k \rightarrow \infty} \beta(M(\rho_{m_k}, \rho_{n_k})) \limsup_{k \rightarrow \infty} M(\rho_{m_k}, \rho_{n_k}) \\ &\leq \limsup_{k \rightarrow \infty} \beta(M(\rho_{m_k}, \rho_{n_k})) \limsup_{k \rightarrow \infty} d(\rho_{m_k}, \rho_{n_k}) \\ &\leq \limsup_{k \rightarrow \infty} \beta(M(\rho_{m_k}, \rho_{n_k})) s\epsilon \end{aligned}$$

and using (7), we obtain

$$\begin{aligned} s^3 \frac{\epsilon}{s^2} &\leq \limsup_{k \rightarrow \infty} \beta(M(\rho_{m_k}, \rho_{n_k})) s\epsilon, \\ s\epsilon &\leq \limsup_{k \rightarrow \infty} \beta(M(\rho_{m_k}, \rho_{n_k})) s\epsilon. \end{aligned}$$

Now we conclude

$$\limsup_{k \rightarrow \infty} \beta(M(\rho_{m_k}, \rho_{n_k})) \geq 1.$$

This implies

$$\limsup_{k \rightarrow \infty} \beta(M(\rho_{m_k}, \rho_{n_k})) = 1.$$

Hence we have

$$\limsup_{k \rightarrow \infty} M(\rho_{m_k}, \rho_{n_k}) = 0. \tag{11}$$

Using (11) in (10), we have

$$s^3 \limsup_{k \rightarrow \infty} d(\rho_{m_k+1}, \rho_{n_k+1}) \leq 0$$

which implies that

$$\limsup_{k \rightarrow \infty} d(\rho_{m_k+1}, \rho_{n_k+1}) = 0.$$

Now using triangle inequality, we get

$$\begin{aligned} d(\rho_{m_k}, \rho_{n_k}) &\leq s[d(\rho_{m_k}, \rho_{m_k+1}) + d(\rho_{m_k+1}, \rho_{n_k})] \\ &\leq sd(\rho_{m_k}, \rho_{m_k+1}) + s^2[d(\rho_{m_k+1}, \rho_{n_k+1}) + d(\rho_{n_k+1}, \rho_{n_k})]. \end{aligned}$$

This leads to

$$\limsup_{k \rightarrow \infty} d(\rho_{m_k}, \rho_{n_k}) = 0,$$

which is a contradiction, as for any given $\epsilon > 0$, we have

$$d(\rho_{m_k}, \rho_{n_k}) \geq \epsilon.$$

Therefore, (ρ_n) is a Cauchy sequence. Since (Δ, d) is complete, there exists $u \in \Delta$ such that

$$\lim_{n \rightarrow \infty} \rho_n = u,$$

which implies $\lim_{n \rightarrow \infty} d(\rho_n, u) = 0$. So $\lim_{m, n \rightarrow \infty} d(\rho_m, \rho_n) = 0$. In a similar manner, we can prove that $d(u, u) = 0$.

(iii) If Γ is continuous, then

$$\Gamma u = \lim_{n \rightarrow \infty} \Gamma \rho_n = \lim_{n \rightarrow \infty} \rho_{n+1} = u,$$

and this implies that u is a fixed point of Γ .

(iiia) Here we have the sequence (ρ_n) with $\lim_{n \rightarrow \infty} \rho_n = u$ and $\alpha(\rho_n, \rho_{n+1}) \geq s$. Hence $\alpha(\rho_n, u) \geq s$ and $\alpha(u, \Gamma u) \geq \frac{1}{s}$. Suppose that $d(u, \Gamma u) > 0$. So, there does not exist some $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$, we get

$$\rho_n = u.$$

Hence, there exists a subsequence (ρ_{r_k}) of (ρ_n) such that

$$\rho_{r_k} \neq u.$$

Taking into account this fact and $\alpha(\rho_n, u) \geq s$, we have from (1),

$$\begin{aligned} s^3(d(\Gamma \rho_n, \Gamma u) + l)^{\alpha(\rho_n, \Gamma \rho_n)\alpha(u, \Gamma u)} &\leq \beta(M(\rho_n, u))M(\rho_n, u) + l \\ \Rightarrow d(\Gamma \rho_n, \Gamma u) + l &\leq \beta(M(\rho_n, u))M(\rho_n, u) + l, \end{aligned} \tag{12}$$

for all n , where

$$M(\rho_n, u) = \max \left\{ d(\rho_n, u), d(\rho_n, \Gamma \rho_n), d(u, \Gamma u), \frac{d(\rho_n, \Gamma u) + d(u, \Gamma \rho_n)}{4s} \right\}.$$

Letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} M(\rho_n, u) = d(u, \Gamma u).$$

From (12),

$$\lim_{n \rightarrow \infty} d(\Gamma \rho_n, \Gamma u) \leq \lim_{n \rightarrow \infty} \beta(M(\rho_n, u)) \lim_{n \rightarrow \infty} M(\rho_n, u),$$

which implies

$$d(u, \Gamma u) \leq \lim_{n \rightarrow \infty} \beta(M(\rho_n, u))d(u, \Gamma u).$$

Then we have $\lim_{n \rightarrow \infty} \beta(M(\rho_n, u)) \geq 1$. So, $\lim_{n \rightarrow \infty} \beta(M(\rho_n, u)) = 1$. Now, we conclude that

$$\lim_{n \rightarrow \infty} M(\rho_n, u) = 0.$$

Hence, we have $d(u, \Gamma u) = 0$. Then u is a fixed point of Γ . Therefore in both cases, Γ has a fixed point. ■

To warrant the uniqueness of the secured fixed point, we need the following additional hypothesis along with those of Theorem 1.

Theorem 2 *Let Γ be any α -admissible self-map of type S on a complete b -metric-like space (Δ, d) . Also suppose that Γ satisfies all the hypotheses of Theorem 1. If for any*

$$u, v \in \text{Fix}(\Gamma) \text{ with } u \neq v, \text{ we have } \alpha(u, v) \geq s,$$

then Γ has a unique fixed point.

Proof. Let $u, v \in \text{Fix}(\Gamma)$. Then $u = \Gamma u$ and $v = \Gamma v$. Also $\alpha(u, v) \geq s$ and $u \neq v$. Hence from (1)

$$d(\Gamma u, \Gamma v) + l \leq s^3(d(\Gamma u, \Gamma v) + l)^{\alpha(u, \Gamma u)\alpha(v, \Gamma v)} \leq \beta(M(u, v))M(u, v) + l$$

where

$$\begin{aligned} M(u, v) &= \max \left\{ d(u, v), d(u, \Gamma u), d(v, \Gamma v), \frac{d(u, \Gamma v) + d(v, \Gamma u)}{4s} \right\} \\ &= \max \{ d(u, v), d(u, u), d(v, v) \} \\ &= d(u, v). \end{aligned}$$

Therefore,

$$d(u, v) + l \leq \beta(d(u, v))d(u, v) + l,$$

which implies

$$d(u, v) < d(u, v),$$

which is impossible. So, Γ has a unique fixed point. ■

Here we recall the definition of well-posedness of a fixed point problem.

Definition 5 (see [10]) *Let Γ be any self-map defined on a metric space (Δ, d) . Then the fixed point problem concerning Γ is said to be well-posed if the followings hold:*

- (i) Γ has a unique fixed point $z \in \Delta$;
- (ii) for any sequence (ρ_n) in Δ such that

$$\lim_{n \rightarrow \infty} d(\rho_n, \Gamma \rho_n) = 0,$$

we have

$$\lim_{n \rightarrow \infty} d(\rho_n, z) = 0.$$

The succeeding theorem ensures that our derived Theorem 1 is also well-posed.

Theorem 3 *Let $(\rho_n) \subseteq \Delta$ such that $\lim_{n \rightarrow \infty} d(\rho_n, \Gamma \rho_n) = 0$, where (Δ, d) is a b -metric-like space with coefficient $s \geq 1$. If Γ is a continuous function, then the fixed point problem is well-posed.*

Proof. Let z be the unique fixed point of Γ . Also suppose that (ρ_n) is a sequence with

$$\lim_{n \rightarrow \infty} d(\rho_n, \Gamma\rho_n) = 0.$$

Now,

$$d(\rho_n, z) = d(\rho_n, \Gamma z) \leq s[d(\rho_n, \Gamma\rho_n) + d(\Gamma\rho_n, \Gamma z)]. \tag{13}$$

Letting $n \rightarrow \infty$ in (13). Then

$$\lim_{n \rightarrow \infty} d(\rho_n, z) \leq 0,$$

which implies that

$$\lim_{n \rightarrow \infty} d(\rho_n, z) = 0.$$

Hence the fixed point problem is well-posed. ■

Now we recollect the notion of limit shadowing property of a fixed point problem.

Definition 6 (see [29]) *Let Γ be a self-map defined on a metric space (Δ, d) . Then the fixed point problem involving Γ is said to possess limit shadowing property in Δ if for any sequence (ρ_n) in Δ such that*

$$\lim_{n \rightarrow \infty} d(\rho_n, \Gamma\rho_n) = 0,$$

we have $z \in \Delta$ with

$$\lim_{n \rightarrow \infty} d(\Gamma^n z, \rho_n) = 0.$$

Employing these auxiliary criteria, we can claim that the mentioned problem also possesses limit shadowing property.

Theorem 4 *Let $(\rho_n) \subseteq \Delta$ such that $\lim_{n \rightarrow \infty} d(\rho_n, \Gamma\rho_n) = 0$ where (Δ, d) is a b-metric-like space with coefficient $s \geq 1$. Also suppose that z is a fixed point of Γ . If Γ is a continuous function, then Theorem 1 possesses limit shadowing property.*

Proof. From well-posedness property, we have

$$\lim_{n \rightarrow \infty} d(\rho_n, z) = 0.$$

Therefore, $\lim_{n \rightarrow \infty} d(z, \rho_n) = 0$ implies $\lim_{n \rightarrow \infty} d(\Gamma^n z, \rho_n) = 0$. Therefore the fixed point problem has the limit shadowing property. ■

The following constructive example validates Theorem 1.

Example 1 *We consider the set*

$$\Delta = \{0, 2^n, 3^n : n \in \mathbb{N}, n \geq 3\},$$

and define a mapping $d : \Delta \times \Delta \rightarrow \mathbb{R}$ by

$$d(\rho, \varrho) = \begin{cases} 0, & \text{if } \rho = \varrho = 0; \\ \frac{1}{2^{n+1}}, & \text{if } \rho = 2^n, \varrho = 0 \text{ or } \rho = 0, \varrho = 2^n; \\ \frac{1}{3^{n+1}}, & \text{if } \rho = 3^n, \varrho = 0 \text{ or } \rho = 0, \varrho = 3^n; \\ \frac{1}{\rho} + \frac{1}{\varrho}, & \text{elsewhere.} \end{cases}$$

Then (Δ, d) is a complete b-metric-like space with $s = 3$. Let us define a self-map Γ on Δ by

$$\Gamma\rho = \begin{cases} 0, & \text{if } \rho = 0; \\ 2^{n+1}, & \text{if } \rho = 2^n; \\ 3^{n+1}, & \text{if } \rho = 3^n. \end{cases}$$

Define $\alpha : \Delta \times \Delta \rightarrow [0, \infty)$ as

$$\alpha(\rho, \varrho) = \begin{cases} \rho\varrho, & \text{if } \rho = 2^n, \varrho = 2^m \text{ or } \rho = 3^n, \varrho = 3^m; \\ \frac{1}{\rho\varrho}, & \text{if } \rho = 2^n, \varrho = 3^m \text{ or } \rho = 3^n, \varrho = 2^m \end{cases}$$

and

$$\alpha(0, \rho) = \alpha(\rho, 0) = 0, \text{ for } \rho \neq 0, \alpha(0, 0) = 3.$$

Also consider the Geraghty function as

$$\beta(t) = e^{-2t}, \quad t \in \mathbb{R}^+$$

and take $l = 54$. Let $\rho, \varrho \in \Delta$ with $\alpha(\rho, \varrho) \geq s$. Then either $\rho = 2^n, \varrho = 2^m$ or $\rho = 3^n, \varrho = 3^m$ and so $\alpha(\Gamma\rho, \Gamma\varrho) \geq s$. Therefore, Γ is an α -admissible mapping of type S . Now, let $\rho, \varrho \in \Delta$ such that $\alpha(\rho, \varrho) \geq s = 3$ with $\rho \neq \varrho$. Then the following two cases arise.

Case-I: When $\rho = 2^n, \varrho = 2^m, n, m \in \mathbb{N}$ with $n \neq m$. Then $\Gamma\rho = 2^{n+1}, \Gamma\varrho = 2^{m+1}$ and so

$$s^3(d(\Gamma\rho, \Gamma\varrho) + l)^{\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho)} = 27 \left\{ \frac{1}{2^{n+m+2}} + 54 \right\}^{\frac{1}{2^{2n+2m+2}}} \leq 54.$$

Also,

$$\beta(M(\rho, \varrho))M(\rho, \varrho) + l \geq 54.$$

Hence we can claim that

$$s^3(d(\Gamma\rho, \Gamma\varrho) + l)^{\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho)} \leq \beta(M(\rho, \varrho))M(\rho, \varrho) + l$$

holds for this case.

Case-II: When $\rho = 3^n, \varrho = 3^m, n, m \in \mathbb{N}$ with $n \neq m$. In this case, in a similar manner we can prove that

$$s^3(d(\Gamma\rho, \Gamma\varrho) + l)^{\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho)} \leq \beta(M(\rho, \varrho))M(\rho, \varrho) + l$$

holds. Further, $0 \in \Delta$ such that $\alpha(0, \Gamma 0) = \alpha(0, 0) \geq s = 3$ and α has transitive property of type S . Also Γ is continuous here. Hence Theorem 1 can be applied here. Then Γ has a fixed point and it is 0.

Now we state another result concerning α -admissible self-maps of type S . Since, the proof is quite analogous to that of Theorem 1, we skip the proof here.

Theorem 5 Let (Δ, d) be any complete b -metric-like space with coefficient s and Γ be any α -admissible self-map of type S . Assume that, whenever $\alpha(\rho, \varrho) \geq s$ with $\rho \neq \varrho$, Γ satisfies

$$(\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho) + 1)^{s^3 d(\Gamma\rho, \Gamma\varrho)} \leq (s^2 + 1)^{\beta(M(\rho, \varrho))M(\rho, \varrho)}$$

for $s \geq 1$, where

$$M(\rho, \varrho) = \max \left\{ d(\rho, \varrho), d(\rho, \Gamma\rho), d(\varrho, \Gamma\varrho), \frac{d(\rho, \Gamma\varrho) + d(\varrho, \Gamma\rho)}{4s} \right\},$$

and β is a Geraghty function. Suppose

- (i) there exists $\rho_0 \in \Delta$ such that $\alpha(\rho_0, \Gamma\rho_0) \geq s$;
- (ii) α has the transitive property of type S ;
- (iii) Γ is continuous;

(iii) if there exists a sequence (ρ_n) with $\lim_{n \rightarrow \infty} \rho_n = u$ and $\alpha(\rho_n, \rho_{n+1}) \geq s$, then

$$\alpha(\rho_n, u) \geq s \quad \text{and} \quad \alpha(u, \Gamma u) \geq \frac{1}{s}.$$

Then $\text{Fix}(\Gamma) \neq \emptyset$, with $d(u, u) = 0$ where $u \in \text{Fix}(\Gamma)$.

By means of the succeeding numerical example, we authenticate our aforementioned result.

Example 2 We consider the set $\Delta = \mathbb{N} \cup \{0\}$ and define a mapping $d : \Delta \times \Delta \rightarrow \mathbb{R}$ such that

$$d(\rho, \varrho) = \begin{cases} 0, & \text{if } \rho = \varrho = 0; \\ \frac{4}{n}, & \text{if } \rho = n, \varrho = 0 \text{ or } \rho = 0, \varrho = n; \\ 2\left(\frac{1}{n} + \frac{1}{m}\right), & \text{if } \rho = n, \varrho = m. \end{cases}$$

Then (Δ, d) is a complete b -metric-like space with $s = 2$. Let us define a self-map Γ on Δ by

$$\Gamma\rho = \begin{cases} 0, & \text{if } \rho = 0; \\ 8\rho + 1, & \text{if } \rho \text{ is even}; \\ 8\rho + 2, & \text{if } \rho \text{ is odd}. \end{cases}$$

Define $\alpha : \Delta \times \Delta \rightarrow [0, \infty)$ as

$$\alpha(\rho, \varrho) = \begin{cases} \rho + \varrho, & \text{if } \rho, \varrho \text{ both are even or both are odd}; \\ \frac{1}{100} \left(\frac{1}{\rho} + \frac{1}{\varrho}\right), & \text{if one of } \rho, \varrho \text{ is even and the other one is odd}; \end{cases}$$

with

$$\alpha(0, \rho) = \alpha(\rho, 0) = 0, \quad \text{for } \rho \neq 0, \quad \alpha(0, 0) = 2$$

and the Geraghty function as

$$\beta(t) = e^{-\frac{t}{4}}, \quad t \in \mathbb{R}^+.$$

Then clearly Γ is an α -admissible mapping of type S . Let $\rho, \varrho \in \Delta$ with $\alpha(\rho, \varrho) \geq s = 2$ and $\rho \neq \varrho$. Then the following two cases arise.

Case-I: When ρ, ϱ both be even, say $\rho = 2n, \varrho = 2m, n, m \in \mathbb{N}$. Then

$$\Gamma\rho = 16n + 1, \quad \Gamma\varrho = 16m + 1$$

and

$$\alpha(\rho, \Gamma\rho) = \frac{1}{100} \left(\frac{1}{2n} + \frac{1}{16n + 1}\right), \quad \alpha(\varrho, \Gamma\varrho) = \frac{1}{100} \left(\frac{1}{2m} + \frac{1}{16m + 1}\right).$$

Also,

$$s^3 d(\Gamma\rho, \Gamma\varrho) = 16 \left(\frac{1}{16n + 1} + \frac{1}{16m + 1}\right).$$

Therefore,

$$\begin{aligned} M(\rho, \varrho) &= \max \left\{ d(\rho, \varrho), d(\rho, \Gamma\rho), d(\varrho, \Gamma\varrho), \frac{d(\rho, \Gamma\varrho) + d(\varrho, \Gamma\rho)}{4s} \right\} \\ &\geq d(\rho, \varrho) \\ &= 2 \left(\frac{1}{2n} + \frac{1}{2m}\right) \\ &\geq 16 \left(\frac{1}{16n + 1} + \frac{1}{16m + 1}\right) = s^3 d(\Gamma\rho, \Gamma\varrho). \end{aligned} \tag{14}$$

Again it is clear that $M(\rho, \varrho) \leq 2$. So

$$\beta(M(\rho, \varrho)) = e^{-\frac{M(\rho, \varrho)}{4}} \geq e^{-\frac{1}{2}},$$

i.e.,

$$e^{\frac{1}{2}}\beta(M(\rho, \varrho)) \geq 1. \quad (15)$$

Hence from (14) and (15), we get

$$s^3 d(\Gamma\rho, \Gamma\varrho) \leq e^{\frac{1}{2}}\beta(M(\rho, \varrho))M(\rho, \varrho). \quad (16)$$

Now

$$\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho) + 1 \leq 1.01.$$

This implies

$$\ln(\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho) + 1) \leq \ln(1.01),$$

so,

$$e^{\frac{1}{2}} \ln(\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho) + 1) \leq e^{\frac{1}{2}} \ln(1.01) \leq \ln(5).$$

Thus

$$\ln(\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho) + 1) \leq e^{-\frac{1}{2}} \ln(5). \quad (17)$$

From (16) and (17), we get

$$\begin{aligned} s^3 d(\Gamma\rho, \Gamma\varrho) \ln(\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho) + 1) &\leq \beta(M(\rho, \varrho))M(\rho, \varrho) \ln(5) \\ &= \beta(M(\rho, \varrho))M(\rho, \varrho) \ln(s^2 + 1). \end{aligned}$$

Then

$$(\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho) + 1)^{s^3 d(\Gamma\rho, \Gamma\varrho)} \leq (s^2 + 1)^{\beta(M(\rho, \varrho))M(\rho, \varrho)}.$$

Case-II: When ρ, ϱ both be odd. In this case, we can similarly prove that

$$(\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho) + 1)^{s^3 d(\Gamma\rho, \Gamma\varrho)} \leq (s^2 + 1)^{\beta(M(\rho, \varrho))M(\rho, \varrho)}.$$

Further, $0 \in \Delta$ such that $\alpha(0, \Gamma 0) = \alpha(0, 0) \geq s = 2$ and α has the transitive property of type S . Also Γ is continuous here. Hence by Theorem 5, Γ has a fixed point and it is 0.

Remark 1 It is worth mentioning that the previously illustrated examples (Example 1 and 2) are not applicable to validate several comparable results on various abstract spaces for example, metric spaces, metric-like spaces and b-metric spaces. More precisely the said examples are not applicable to [1, Theorem 2.1], [11, Theorem 3], [17, Theorems 4,6], [21, Theorem 6] and [26, Theorem 4].

Remark 2 The uniqueness of the fixed point attained in Theorem 5 can be obtained by imposing some additional hypotheses similar to Theorem 2.

Remark 3 Implementing some surplus conditions in the lines of Theorem 3-4, it can be easily vindicated that Theorem 5 is also well-posed and possesses limit shadowing property.

3 Consequences in Other Metric Spaces

In this section, we present several fixed point results in various abstract spaces which can be easily inferred from our explored findings.

Corollary 1 Let (Δ, d) be a complete b -metric space with coefficient $s \geq 1$ and Γ be any α -admissible self-map of type S defined on Δ . Assume that whenever $\alpha(\rho, \varrho) \geq s$ with $\rho \neq \varrho$, Γ satisfies

$$s^3(d(\Gamma\rho, \Gamma\varrho) + l)^{\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho)} \leq \beta(M(\rho, \varrho))M(\rho, \varrho) + l,$$

where

$$M(\rho, \varrho) = \max \left\{ d(\rho, \varrho), d(\rho, \Gamma\rho), d(\varrho, \Gamma\varrho), \frac{d(\rho, \Gamma\varrho) + d(\varrho, \Gamma\rho)}{4s} \right\},$$

β is a Geraghty function and $l \geq 1$. Suppose

- (i) there exists $\rho_0 \in \Delta$ such that $\alpha(\rho_0, \Gamma\rho_0) \geq s$;
- (ii) α has the transitive property of type S ;
- (iii) Γ is continuous;
- (iiia) if there exists a sequence (ρ_n) with $\lim_{n \rightarrow \infty} \rho_n = u$ and $\alpha(\rho_n, \rho_{n+1}) \geq s$, then

$$\alpha(\rho_n, u) \geq s \quad \text{and} \quad \alpha(u, \Gamma u) \geq \frac{1}{s}.$$

Then $\text{Fix}(\Gamma) \neq \emptyset$.

Proof. Since a b -metric space is also a b -metric-like space, the result holds from Theorem 1. ■

Corollary 2 Let (Δ, d) be a complete b -metric space with coefficient $s \geq 1$ and Γ be any α -admissible self-map of type S . Assume that whenever $\alpha(\rho, \varrho) \geq s$ with $\rho \neq \varrho$, Γ satisfies

$$s^3(d(\Gamma\rho, \Gamma\varrho) + l)^{\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho)} \leq \beta(d(\rho, \varrho))d(\rho, \varrho) + l$$

where β is a Geraghty function and $l \geq 1$. Suppose

- (i) there exists $\rho_0 \in \Delta$ such that $\alpha(\rho_0, \Gamma\rho_0) \geq s$;
- (ii) α has the transitive property of type S ;
- (iii) Γ is continuous;
- (iiia) if there exists a sequence (ρ_n) with $\lim_{n \rightarrow \infty} \rho_n = u$ and $\alpha(\rho_n, \rho_{n+1}) \geq s$, then

$$\alpha(\rho_n, u) \geq s \quad \text{and} \quad \alpha(u, \Gamma u) \geq \frac{1}{s}.$$

Then Γ has a fixed point.

Proof. If $M(\rho, \varrho) = d(\rho, \varrho)$, we can claim this result from Corollary 1. ■

Corollary 3 ([17]) Let (Δ, d) be a complete metric space and Γ be any α -admissible self-map of type S on Δ . Assume that whenever $\alpha(\rho, \varrho) \geq 1$ with $\rho \neq \varrho$, Γ satisfies

$$(d(\Gamma\rho, \Gamma\varrho) + l)^{\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho)} \leq \beta(d(\rho, \varrho))d(\rho, \varrho) + l,$$

where β is a Geraghty function and $l \geq 1$. Suppose

- (i) there exists $\rho_0 \in \Delta$ such that $\alpha(\rho_0, \Gamma\rho_0) \geq 1$;
- (ii) α has the transitive property;

(iii) Γ is continuous;

(iiia) if there exists a sequence (ρ_n) with $\lim_{n \rightarrow \infty} \rho_n = u$ and $\alpha(\rho_n, \rho_{n+1}) \geq 1$, then

$$\alpha(\rho_n, u) \geq 1 \quad \text{and} \quad \alpha(u, \Gamma u) \geq 1.$$

Then $\text{Fix}(\Gamma) \neq \emptyset$.

Proof. This can be readily concluded from Corollary 2 by taking $s = 1$. ■

Corollary 4 Let (Δ, d) be a complete b -metric space with coefficient $s \geq 1$ and Γ be any α -admissible self-map of type S on Δ . Assume that, whenever $\alpha(\rho, \varrho) \geq s$ with $\rho \neq \varrho$, Γ satisfies

$$(\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho) + 1)^{s^3 d(\Gamma\rho, \Gamma\varrho)} \leq (s^2 + 1)^{\beta(M(\rho, \varrho))M(\rho, \varrho)},$$

where

$$M(\rho, \varrho) = \max \left\{ d(\rho, \varrho), d(\rho, \Gamma\rho), d(\varrho, \Gamma\varrho), \frac{d(\rho, \Gamma\varrho) + d(\varrho, \Gamma\rho)}{4s} \right\},$$

and β is a Geraghty function. Suppose

(i) there exists $\rho_0 \in \Delta$ such that $\alpha(\rho_0, \Gamma\rho_0) \geq s$;

(ii) α has the transitive property of type S ;

(iii) Γ is continuous;

(iiia) if there exists a sequence (ρ_n) with $\lim_{n \rightarrow \infty} \rho_n = u$ and $\alpha(\rho_n, \rho_{n+1}) \geq s$, then

$$\alpha(\rho_n, u) \geq s \quad \text{and} \quad \alpha(u, \Gamma u) \geq \frac{1}{s}.$$

Then $\text{Fix}(\Gamma) \neq \emptyset$.

Proof. This can be easily deduced from Theorem 5 with the fact that a b -metric space is always a b -metric-like space. ■

Corollary 5 Let (Δ, d) be any complete b -metric space with coefficient $s \geq 1$ and Γ be any α -admissible self-map of type S . Assume that, whenever $\alpha(\rho, \varrho) \geq s$ with $\rho \neq \varrho$, Γ satisfies

$$(\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho) + 1)^{s^3 d(\Gamma\rho, \Gamma\varrho)} \leq (s^2 + 1)^{\beta(d(\rho, \varrho))d(\rho, \varrho)}$$

where β is a Geraghty function and $s \geq 1$. Suppose

(i) there exists $\rho_0 \in \Delta$ such that $\alpha(\rho_0, \Gamma\rho_0) \geq s$;

(ii) α has the transitive property of type S ;

(iii) Γ is continuous;

(iiia) for any sequence (ρ_n) with $\lim_{n \rightarrow \infty} \rho_n = u$ and $\alpha(\rho_n, \rho_{n+1}) \geq s$, then

$$\alpha(\rho_n, u) \geq s \quad \text{and} \quad \alpha(u, \Gamma u) \geq \frac{1}{s}.$$

Then $\text{Fix}(\Gamma) \neq \emptyset$.

Proof. Considering $M(\rho, \varrho) = d(\rho, \varrho)$, we can obtain this corollary from Corollary 4. ■

Corollary 6 *Let (Δ, d) be any complete metric space and Γ be any α -admissible self-map of type S . Assume that, whenever $\alpha(\rho, \varrho) \geq 1$ with $\rho \neq \varrho$, Γ satisfies*

$$(\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho) + 1)^{d(\Gamma\rho, \Gamma\varrho)} \leq 2^{\beta(d(\rho, \varrho))d(\rho, \varrho)}$$

where β is a Geraghty function. Suppose

- (i) there exists $\rho_0 \in \Delta$ such that $\alpha(\rho_0, \Gamma\rho_0) \geq 1$;
- (ii) α has the transitive property;
- (iii) Γ is continuous;
- (iiia) for any sequence (ρ_n) with $\lim_{n \rightarrow \infty} \rho_n = u$ and $\alpha(\rho_n, \rho_{n+1}) \geq 1$, then

$$\alpha(\rho_n, u) \geq 1 \quad \text{and} \quad \alpha(u, \Gamma u) \geq 1.$$

Then $\text{Fix}(\Gamma) \neq \emptyset$.

Proof. When $s = 1$, this result is obvious from Corollary 5. ■

4 Remark on Cyclic (α, β) -Admissibility of Type S

Lately, in their research article, Mongkolkeha and Sintunavarat [23] coined the notion of a cyclic (α, β) -admissible mapping of type S in a b -metric framework. First of all, we note down the definition of such a kind of mapping.

Definition 7 ([23]) *Suppose that Δ is a non-empty set, s is any real number such that $s \geq 1$ and $\Gamma : \Delta \rightarrow \Delta$ be any self-map. Also suppose that $\alpha, \beta : \Delta \rightarrow [0, \infty)$ be two mappings. Then Γ is a cyclic (α, β) -admissible mapping of type S if*

- (i) $\alpha(\rho) \geq s$ for some $\rho \in \Delta$ implies $\beta(\Gamma\rho) \geq s$;
- (ii) $\beta(\rho) \geq s$ for some $\rho \in \Delta$ implies $\alpha(\Gamma\rho) \geq s$.

Now, if we consider a couple of mappings $\alpha_1, \beta_1 : \Delta \rightarrow [0, \infty)$ such that

$$\alpha_1(\rho) = \frac{1}{s}\alpha(\rho) \quad \text{and} \quad \beta_1(\rho) = \frac{1}{s}\beta(\rho),$$

then, it can be easily verified that, whenever $\alpha(\rho) \geq s$ for some $\rho \in \Delta$, then

$$\alpha_1(\rho) = \frac{1}{s}\alpha(\rho) \geq 1$$

too. Similarly

$$\beta_1(\rho) = \frac{1}{s}\beta(\rho) \geq 1$$

whenever $\beta(\rho) \geq s$ for some $\rho \in \Delta$. Hence the Definition 7 reduces to that of [3, Definition 2.1]. Therefore, whenever a self-map Γ is a cyclic (α, β) -admissible mapping of type S , then Γ is a cyclic (α_1, β_1) -admissible mapping. We illustrate the above discussion with the help of the example presented in [23, Example 3.2].

Example 3 Let $\Delta = [0, \infty)$ and $\Gamma : \Delta \rightarrow \Delta$ be defined by

$$\Gamma\rho = \begin{cases} 3 + |\sin \rho|, & \text{if } \rho \in [3, 4], \\ \frac{1}{\rho+1}, & \text{otherwise.} \end{cases}$$

Now we consider two mappings $\alpha, \beta : \Delta \rightarrow [0, \infty)$ as

$$\alpha(\rho) = \begin{cases} \rho^2, & \text{if } \rho \in [3, 4], \\ 1, & \text{otherwise;} \end{cases}$$

and

$$\beta(\rho) = \begin{cases} 2\rho, & \text{if } \rho \in [3, 4], \\ \frac{1}{\rho+1}, & \text{otherwise.} \end{cases}$$

It can be easily checked that Γ is a cyclic (α, β) -admissible mapping of type S and also not a cyclic (α, β) -admissible mapping. Now we define two mappings $\alpha_1, \beta_1 : \Delta \rightarrow [0, \infty)$ as

$$\alpha_1(\rho) = \begin{cases} \frac{\rho^2}{2}, & \text{if } \rho \in [3, 4], \\ \frac{1}{2}, & \text{otherwise;} \end{cases}$$

and

$$\beta_1(\rho) = \begin{cases} \rho, & \text{if } \rho \in [3, 4], \\ \frac{1}{2(\rho+1)}, & \text{otherwise.} \end{cases}$$

Whenever $\alpha_1(\rho) \geq 1$, then we have $\rho \in [3, 4]$ and hence $\Gamma\rho = 3 + |\sin \rho|$, $\rho \in [3, 4]$. Therefore $\beta_1(\Gamma\rho) \geq 1$. Again if $\beta_1(\rho) \geq 1$, then we have $\rho \in [3, 4]$ and so $\Gamma\rho = 3 + |\sin \rho|$, $\rho \in [3, 4]$. Consequently $\alpha_1(\Gamma\rho) \geq 1$. Then Γ is a cyclic (α_1, β_1) -admissible mapping.

Remark 4 The previous discussion and Example 3 reveal that the set of cyclic (α, β) -admissible mappings of type S is contained in the collection of cyclic (α, β) -admissible mappings defined in [3] and therefore the idea of cyclic (α, β) -admissible mapping of type S adds nothing new to the literature.

5 Application

The fixed point theorems proved here pave the way for an application concerning the necessary conditions for the existence and uniqueness of the solutions to the following kind of integral equations:

$$\mu(\iota) = f \left(\iota, \int_0^{\varrho(\iota)} g(\iota, \kappa, \mu(\rho(\kappa))) d\kappa \right) \quad (18)$$

where $\iota \in [0, \infty)$. We will ensure such an existence by applying Theorem 1. Let $BC[0, \infty)$ be the space of all real, bounded and continuous functions on the interval $[0, \infty)$. We endow it with the b -metric like

$$d(\iota, \kappa) = \sup\{|\iota(t)| + |\kappa(t)| : t \in [0, \infty)\}.$$

Theorem 6 Suppose that the following assumptions are satisfied:

(i) $\rho, \varrho : [0, \infty) \rightarrow [0, \infty)$ are continuous functions so that

$$\Lambda = \sup\{|\varrho(t)| : t \in [0, \infty)\} < \frac{1}{2};$$

(ii) the function $f : [0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous so that

$$|f(\iota, \mu)| \leq |\mu|$$

for all $\iota \in [0, \infty)$ and $\mu \in \mathbb{R}$;

(iii) the function $g : [0, \infty)^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous so that

$$|g(\iota, \kappa, \mu(\rho(\kappa)))| \leq |\mu(\rho(\kappa))|$$

for all $\iota, \kappa \in [0, \infty)$;

(iv) $M = \max\{f(\iota, 0) : \iota \in [0, \infty)\} < \infty$ and $G = \sup\{|g(\iota, \kappa, 0)| : \iota \in [0, \infty)\} < \infty$.

Then the integral equation (18) admits at least one solution in the space $BC[0, \infty)$.

Proof. Let us consider the operator $\Upsilon : BC[0, \infty) \longrightarrow BC[0, \infty)$ defined by

$$\Upsilon(\mu)(\iota) = f\left(\iota, \int_0^{\varrho(\iota)} g(\iota, \kappa, \mu(\rho(\kappa)))d\kappa\right).$$

In view of the given assumptions, we infer that the function $\Upsilon(\mu)$ is continuous for arbitrary $\mu \in BC[0, \infty)$. Now, we show that $\Upsilon(\mu)$ is bounded in $BC[0, \infty)$. As

$$\begin{aligned} |\Upsilon(\mu)(\iota)| &= \left| f\left(\iota, \int_0^{\varrho(\iota)} g(\iota, \kappa, \mu(\rho(\kappa)))d\kappa\right) \right| \\ &\leq \left| \int_0^{\varrho(\iota)} g(\iota, \kappa, \mu(\rho(\kappa)))d\kappa \right| \\ &\leq \int_0^{\varrho(\iota)} |g(\iota, \kappa, \mu(\rho(\kappa)))| d\kappa \\ &\leq \int_0^{\varrho(\iota)} |\mu(\rho(\kappa))| d\kappa \\ &\leq \int_0^{\varrho(\iota)} \|\mu\| d\kappa \\ &= \|\mu\| \varrho(\iota) \leq \Lambda \|\mu\|. \end{aligned}$$

Due to the above inequality, the function Υ is bounded. Now, we show that Υ satisfies all the conditions of Theorem 1. Let μ_1, μ_2 be some elements of $BC[0, \infty)$. Then we have

$$\begin{aligned}
& |\Upsilon(\mu_1)(t)| + |\Upsilon(\mu_2)(t)| \\
& \leq \left| f(t, \int_0^{\varrho(t)} g(t, \kappa, \mu_1(\rho(\kappa)))d\kappa) \right| + \left| f(t, \int_0^{\varrho(t)} g(t, \kappa, \mu_2(\rho(\kappa)))d\kappa) \right| \\
& \leq \left| \int_0^{\varrho(t)} g(t, \kappa, \mu_1(\rho(\kappa)))d\kappa \right| + \left| \int_0^{\varrho(t)} g(t, \kappa, \mu_2(\rho(\kappa)))d\kappa \right| \\
& \leq \int_0^{\varrho(t)} [|\mu_1(\rho(\kappa))| + |\mu_2(\rho(\kappa))|]d\kappa \\
& \leq \varrho(t)(d(\mu_1, \mu_2)) \\
& \leq \Lambda d(\mu_1, \mu_2) \\
& \leq \frac{1}{2}M(\rho, \varrho) + 1 - 1 \\
& \leq \frac{1}{1 + e^{-3M(\rho, \varrho)+1} \tanh(M(\rho, \varrho))} M(\rho, \varrho) + 1 - 1 \\
& = \beta(M(\rho, \varrho))M(\rho, \varrho) + 1 - 1,
\end{aligned}$$

where $\beta(t) = \frac{1}{1 + e^{-3x+1} \tanh(x)}$, $l = 1$ and $\alpha(\rho, \varrho) = 1$. Thus, we obtain that

$$s^3(d(\Gamma\rho, \Gamma\varrho) + l)^{\alpha(\rho, \Gamma\rho)\alpha(\varrho, \Gamma\varrho)} \leq \beta(M(\rho, \varrho))M(\rho, \varrho) + l.$$

Using Theorem 1, we obtain that the operator Υ admits a fixed point. Thus, the functional integral equation (18) admits at least one solution in $BC[0, \infty)$. ■

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