**Applied Mathematics E-Notes, 22(2022), 393-426 ©** ISSN 1607-2510
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# Generalized Contraction Mappings And Fixed Point Results In Orthogonal Metric Space*

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Received 9 March 2021

**Abstract**

In this manuscript, we introduce some of the contraction mappings in an orthogonal metric space. The contractions such as orthogonal $\alpha$-$\eta$-$GF$-contraction, orthogonal $\alpha$-type $F$-contraction, orthogonal $TAC$-type S-contraction, orthogonal $TAC$-contraction and orthogonal *Suzuki-Berinde* type $F$-contraction together with some of their weaker versions are discussed by the means of this manuscript. Also, various fixed point results owing to these different contraction conditions are proved which indeed generalize the results given in [7, 10, 13, 14]. In support of the outcomes obtained, many examples have been considered.

## 1 Introduction

Banach contraction principle (see [5]), given by Stefan Banach in 1922, has always been prevailing result in fixed point theory. This is not just because of its usefulness but also due to its simple approach towards the fixed point of a self map in a complete metric space. However later, it was examined that any self map satisfying the Banach contraction condition is a continuous map. Since then, numerous authors have come up with new contraction conditions which no longer required the self map to be continuous. In these conditions, some have weaken one or the other sides of the Banach contraction inequality (see [6, 8, 15, 23, 31, 32, 33]) while others have replaced the space in consideration with more general ones (see [9, 12, 17, 22]).

In [11], M. E. Gordji et al. introduced the notion of orthogonal metric space and also proved Banach fixed point theorem in this setting. Many attempts have been made, since then, to generalize the contractions in orthogonal metric space by establishing new contraction mappings (see [16, 26, 28, 34]). Additionally, some authors have worked on the concept of strong orthogonal metric space and discussed various fixed point results over them (see [1, 20, 21]).

In this manuscript, we aim to introduce some contraction mappings in an orthogonally complete metric space. The contractions viz. $\alpha$-$\eta$-$GF$-contraction (see [13]), $\alpha$-type $F$-contraction (see [10]), $TAC$- contraction (see [7]) and *Suzuki-Berinde* type $F$-contraction (see [14]) are some of the conditions which use weaker contraction principles. Motivated by the work done in them, we propose to put forward the notion of orthogonal $\alpha$-$\eta$-$GF$-contraction, orthogonal $\alpha$-type $F$-contraction, orthogonal $TAC$-type S-contraction, orthogonal $TAC$-contraction and orthogonal *Suzuki-Berinde* type $F$-contraction and hence generalizing the results given in [7, 10, 13, 14].

## 2 Preliminaries

Firstly, we discuss few of the notations used in the main section of this manuscript which are further elaborated with the help of example. Throughout the manuscript symbols $X$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{Z}$ denote a non-empty set, the set of real numbers, natural numbers, non-negative real numbers and set of integers respectively.

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*Mathematics Subject Classifications: 47H10; 54H25; 46J10; 46J15.

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**Definition 1** ([11]) *For a non-empty set $X$ along with binary relation $\perp$ is claimed to be an orthogonal set (denoted by $\perp$-set) when $\exists\ \rho_0 \in X$ implies either $[\rho \perp \rho_0\ \forall\ \rho \in X]$ or $[\rho_0 \perp \rho\ \forall\ \rho \in X]$. The element $\rho_0$ is called an orthogonal element.*

**Definition 2** ([11]) *For an orthogonal set $(X, \perp)$, a sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset X$ is claimed to be an orthogonal sequence (denoted by $\perp$-sequence) when either $\left[\rho_n \perp \rho_{n+1}\ \forall\ n \in \mathbb{N}\right]$ or $\left[\rho_{n+1} \perp \rho_n\ \forall\ n \in \mathbb{N}\right]$.*

**Definition 3** ([11]) *For a non-empty set $X$ along with metric $d$ and binary relation $\perp$ is claimed to be an orthogonal metric space (written as $(X, \perp, d)$), if*

*(i) $(X, d)$ is a metric space, and*

*(ii) $(X, \perp)$ is an orthogonal set.*

**Definition 4** ([11]) *On an orthogonal metric space $(X, \perp, d)$, a self map $g : X \to X$ at $\rho \in X$ is claimed to be orthogonally continuous (denoted by $\perp$-continuous) if for every $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ with $\rho_n \to \rho$ implies $g\rho_n \to g\rho$ as $n \to \infty$. In addition, $g$ is $\perp$-continuous on entire space $X$ if $g$ is orthogonally continuous at every point $\rho \in X$.*

**Definition 5** ([11]) *An orthogonal metric space $(X, \perp, d)$ is claimed to be an orthogonally complete metric space (denoted by $\perp$-complete), if each Cauchy $\perp$-sequence in $X$ is convergent in $X$. Also, a function $g : X \to X$ is said to be orthogonal preserving (written as $\perp$-preserving) if $\rho \perp \nu \Rightarrow g\rho \perp g\nu$ and $g$ is called weakly $\perp$-preserving if $\rho \perp \nu \Rightarrow g\rho \perp g\nu$ or $g\nu \perp g\rho$.*

**Definition 6** ([24]) *For a metric space $(X, d)$, a self map $g : X \to X$ is claimed to be an $\alpha$-admissible map with respect to $\eta$, where $\alpha, \eta : X^2 \to \mathbb{R}^+$, if for $\rho, \nu \in X$ where $\eta(\rho, \nu) \leq \alpha(\rho, \nu) \Rightarrow \eta(g\rho, g\nu) \leq \alpha(g\rho, g\nu)$.*

**Definition 7** ([25]) *For a metric space $(X, d)$, a self map $g : X \to X$ is claimed to be an $\alpha$-admissible map, where $\alpha : X^2 \to [0, +\infty)$, if for each $\rho, \nu \in X$ with $1 \leq \alpha(\rho, \nu) \Rightarrow 1 \leq \alpha(g\rho, g\nu)$.*

**Definition 8** ([29]) *For a metric space $(X, d)$, a self map $g : X \to X$ is claimed to be a weak $\alpha$-admissible map, where $\alpha : X^2 \to \mathbb{R}^+$, if for each $\rho \in X$ with $1 \leq \alpha(\rho, g\rho) \Rightarrow 1 \leq \alpha(g\rho, gg\rho)$.*

**Definition 9** ([30]) *For a metric space $(X, d)$, a self map $g : X \to X$ is claimed to be an $\alpha$-admissible map type S, where $\alpha, \eta : X^2 \to \mathbb{R}^+$ and real number $s$ with $s \geq 1$, if for $\rho, \nu \in X$ we have $s \leq \alpha(\rho, \nu) \Rightarrow s \leq \alpha(g\rho, g\nu)$.*

**Definition 10** ([30]) *For a metric space $(X, d)$, a self map $g : X \to X$ is claimed to be a weak $\alpha$-admissible map type S, where $\alpha, \eta : X^2 \to \mathbb{R}^+$ and real number $s$ with $s \geq 1$, if for $\rho \in X$ we have $s \leq \alpha(\rho, g\rho) \Rightarrow s \leq \alpha(g\rho, gg\rho)$.*

**Remark 1** *Following are few observations from [30]:*

*(i) Every $\alpha$-admissible map is weak $\alpha$-admissible map.*

*(ii) Every $\alpha$-admissible map type S is weak $\alpha$-admissible map type S.*

*(iii) However, the class of $\alpha$-admissible map is different from $\alpha$-admissible map type S.*

**Definition 11** ([2]) *For a metric space $(X, d)$, a self map $g : X \to X$ is claimed to be a cyclic $(\hat{\alpha}, \beta)$-admissible map, with $\hat{\alpha}, \beta : X \to \mathbb{R}^+$, if*

*(i) for any $\rho \in X$, $\hat{\alpha}(\rho) \geq 1$ implies $\beta(g\rho) \geq 1$;*

*(ii) for any $\rho \in X$,$\beta(\rho) \geq 1$ implies $\hat{\alpha}(g\rho) \geq 1$.*

**Definition 12** (**[18]**) *For a metric space $(X,d)$, a self map $g : X \to X$ is claimed to be a cyclic $(\hat{\alpha},\beta)$-admissible map type S, with $\hat{\alpha},\beta : X \to \mathbb{R}^+$ and real number $s$ where $s \geq 1$, if*

*(i) for any $\rho \in X$, $\hat{\alpha}(\rho) \geq s$ implies $\beta(g\rho) \geq s$;*

*(ii) for any $\rho \in X$,$\beta(\rho) \geq s$ implies $\hat{\alpha}(g\rho) \geq s$.*

**Remark 2** *The class of cyclic $(\hat{\alpha},\beta)$-admissible mappings is different from the class of cyclic $(\hat{\alpha},\beta)$-admissible mappings type S (see [18]).*

**Definition 13** (**[13]**) *Denote by $\mathfrak{G}$, the set of all maps $G : \mathbb{R}^{+^4} \to [0,+\infty)$ such that for all $(\gamma_1,\gamma_2,\gamma_3,\gamma_4) \in \mathbb{R}^{+^4}$ with $\gamma_1.\gamma_2.\gamma_3.\gamma_4 = 0$ we have $\tau > 0$, such that*

$$G(\gamma_1,\gamma_2,\gamma_3,\gamma_4) = \tau.$$

**Example 1** *Let $G(\gamma_1,\gamma_2,\gamma_3,\gamma_4) = K.(\gamma_1.\gamma_2.\gamma_3.\gamma_4) + \tau$ where $\tau > 0$ and $K$ be a non-negative real constant. Then $G \in \mathfrak{G}$.*

**Example 2** *Let $G(\gamma_1,\gamma_2,\gamma_3,\gamma_4) = \tau.e^{K.(\gamma_1.\gamma_2.\gamma_3.\gamma_4)}$ where $\tau > 0$ and $K$ be a non-negative real constant. Then $G \in \mathfrak{G}$.*

**Definition 14** (**[32]**) *Denote by $\Im$, the family of all mappings $F : (0,+\infty) \to (-\infty,+\infty)$ such that*

*(F1) for $\rho,\nu \in (0,+\infty)$ if $\rho < \nu \Rightarrow F(\rho) < F(\nu)$;*

*(F2) for each sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of positive real number such that*

$$\lim_{n\to\infty} \rho_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\rho_n) = -\infty;$$

*(F3) $\exists\, r \in (0,1)$ then $\lim_{\zeta\to 0^+} \zeta^r F(\zeta) = 0$.*

**Definition 15** (**[3]**) *Denote by $\mathcal{C}$, the family of all function, $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ called as $\mathcal{C}$-class function if $f$ is continuous map which satisfies following conditions*

*(i) $f(\rho,\nu) \leq \rho$ for each $(\rho,\nu) \in \mathbb{R}^+ \times \mathbb{R}^+$;*

*(ii) $f(\rho,\nu) = \rho$ implies either $\rho = 0$ or $\nu = 0$.*

**Definition 16** (**[19]**) *Denote by $\Omega_{\mathcal{F}}$, the family of all mappings $\mathcal{F} : \mathbb{R}^+ \to (-\infty,+\infty)$ such that*

*$(\mathcal{F}_1)$ for $\rho,\nu \in (0,+\infty)$ if $\rho \leq \nu \Rightarrow \mathcal{F}(\rho) \leq \mathcal{F}(\nu)$;*

*$(\mathcal{F}_2)$ $\inf \mathcal{F} = -\infty$;*

*$(\mathcal{F}_3)$ $\mathcal{F}$ is continuous in $(0,\infty)$.*

**Lemma 1** (**[27]**) *Define $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ an increasing mapping and let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers. Then following hold*

*(i) If $\mathcal{F}(\rho_n) \to -\infty$, implies $\rho_n \to 0$;*

*(ii) If $\inf \mathcal{F} = -\infty$ and $\rho_n \to 0$, implies $\mathcal{F}(\rho_n) \to -\infty$.*

# 3 Main Results

For the better understanding of various contractions, we divide this section into four subsections, each of which introduces a different contraction in an orthogonal metric space and consequently, explores various fixed point result owing to much weaker conditions. The definitions established at the beginning of each subsection forms basis of the results proved and also, many examples are discussed in each subsection that further substantiate the results.

## 3.1 Orthogonal $\alpha$-$\eta$-$GF$-Contraction

N. Hussain and P. Salimi were the first one to put forward the notion of $\alpha$-$\eta$-$GF$-contraction (see [13]) in a complete metric space. The main idea behind the paper was to obtain fixed point results in a complete metric space as well as in a complete partially ordered metric space with the help of more general family of mappings $\mathfrak{G}$.

In this subsection, firstly we introduce some of the basic definitions including orthogonal $\alpha$-$\eta$-$GF$-contraction and orthogonal $\alpha$-$\eta$-$GF$-weak contraction and secondly, proceed to prove few fixed point results in these settings. The definitions and results are further supported with the help of examples.

**Definition 17** *For an orthogonal metric space $(X, \perp, d)$ with two mappings $\alpha, \eta : X^2 \to \mathbb{R}^+$, we say a self map $g$ on $X$ is an orthogonal $\alpha$-$\eta$-continuous mapping (denoted by $\perp$-$\alpha$-$\eta$-continuous) if for some $\rho \in X$ and an $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $X$ where $\eta(\rho_n, \rho_{n+1}) \leq \alpha(\rho_n, \rho_{n+1})\ \forall\ n \in \mathbb{N}$ and $\lim_{n\to\infty} \rho_n = \rho$ implies*

$$\lim_{n\to\infty} g\rho_n = g\rho.$$

**Example 3** *Let $X = \mathbb{R}^+$ be equipped with usual metric. Let $\rho \perp \nu \Leftrightarrow \rho.\nu \in \{\rho, \nu\}$. Then $(X, \perp, d)$ is an orthogonal metric space. Define $g : X \to X$ as*

$$g(\rho) = \begin{cases} 0 & \rho \in [0, 1); \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

*Define $\alpha, \eta : X^2 \to \mathbb{R}^+$ where*

$$\alpha(\rho, \nu) = \begin{cases} 4 & \rho, \nu \in [0, 1); \\ \frac{1}{4} & \text{otherwise,} \end{cases}$$

*and, $\eta(\rho, \nu) = 1$ for all $\rho, \nu \in X$. Thus, for $\alpha(\rho, \nu) \geq \eta(\rho, \nu)$, we must have $\rho, \nu \in [0, 1)$. The sequence $\{\rho_n\}_{n\in\mathbb{N}}$, defined as*

$$\rho_n = \begin{cases} 0 & n = 2m-1,\ \forall\ m \in \mathbb{N}; \\ \frac{1}{2^m} & n = 2m,\ \forall\ m \in \mathbb{N}, \end{cases}$$

*is an $\perp$-sequence in $X$ and $\alpha(\rho_n, \rho_{n+1}) \geq \eta(\rho_n, \rho_{n+1})\ \forall\ n \in \mathbb{N}$. Also, since $\rho_n \to 0$ as $n \to \infty$, we see that $\lim_{n\to\infty} g\rho_n = 0 = g0$. Hence, $g$ is $\perp$-$\alpha$-$\eta$-continuous however, it is not a continuous function.*

**Definition 18** *For an orthogonal metric space $(X, \perp, d)$ with two mappings $\alpha, \eta : X^2 \to \mathbb{R}^+$, we say a self map $g : X \to X$ on $X$ is orthogonal $\alpha$-$\eta$-GF-contraction (denoted by $\perp$-$\alpha$-$\eta$-GF-contraction) if for all $\rho, \nu \in X$ with $\rho \perp \nu$, $d(g\rho, g\nu) > 0$ and $\eta(\rho, g\rho) \leq \alpha(\rho, \nu)$, we have*

$$G\big(d(\rho, g\rho), d(\nu, g\nu), d(\rho, g\nu), d(\nu, g\rho)\big) + F\big(d(g\rho, g\nu)\big) \leq F\big(d(\rho, \nu)\big),$$

*where $G \in \mathfrak{G}$ and $F \in \Im$.*

**Example 4** *Consider $X = \{0, 2, 4, ..., 2^k, ...\}$ along with usual metric space. Let $\rho \perp \nu \Leftrightarrow \rho.\nu \in \{0\}$. Then $(X, \perp, d)$ is an orthogonal metric space. Define $g : X \to X$ as*

$$g(\rho) = \begin{cases} 2^{m-1} & \text{for } \rho = 2^m \text{ where } m \in \mathbb{N} - \{1\}; \\ 0 & \rho \in \{0, 2\}. \end{cases}$$

*Define* $\alpha, \eta : X^2 \to \mathbb{R}^+$ *as,*

$$\alpha(\rho,\nu) = \begin{cases} 1 & \rho \in \{0,2\}; \\ \frac{5}{2} & otherwise, \end{cases}$$

*and,*

$$\eta(\rho,\nu) = \begin{cases} \frac{1}{2} & \rho \in \{0,2\}; \\ 1 & otherwise. \end{cases}$$

*Now by above, we have:*

*(I) for* $d(g\rho, g\nu) > 0$*, we must have either* $\rho \in \{0,2\}$ *and* $\nu = 2^m$ *where* $m \in \mathbb{N} - \{1\}$ or $\rho = 2^m$ *where* $m \in \mathbb{N} - \{1\}$ *and* $\nu \in \{0,2\}$*,*

*(II) for* $\rho \perp \nu$*, either* $\rho = 0$ *or* $\nu = 0$*.*

*Thus for (I) and (II) to hold together, we have either* $\rho = 0$ *and* $\nu = 2^m$ *where* $m \in \mathbb{N} - \{1\}$ or $\rho = 2^m$ *where* $m \in \mathbb{N} - \{1\}$ *and* $\nu = 0$*.*
*Consider* $\rho = 0$ *and* $\nu = 2^m$ *where* $m \in \mathbb{N} - \{1\}$*. Then for such choice of* $\rho$ *and* $\nu$*, we have* $\eta(\rho, g\rho) < \alpha(\rho,\nu)$*. So for* $F(\beta) = \ln(\beta)$ *and* $\tau = 0.5$*, we have*

$$G\big(d(\rho,g\rho), d(\nu,g\nu), d(\rho,g\nu), d(\nu,g\rho)\big) + F\big(d(g\rho,g\nu)\big) = \tau + \ln 2^{m-1}, \tag{1}$$

*and,*

$$F\big(d(\rho,\nu)\big) = \ln(2^m). \tag{2}$$

*Thus from (1) and (2), we obtain*

$$G\big(d(\rho,g\rho), d(\nu,g\nu), d(\rho,g\nu), d(\nu,g\rho)\big) + F\big(d(g\rho,g\nu)\big) \le F\big(d(\rho,\nu)\big).$$

*Hence,* $g$ *is* $\perp$-$\alpha$-$\eta$-GF*-contraction on* $X$*.*

**Definition 19** *For an orthogonal metric space* $(X, \perp, d)$ *with two mappings* $\alpha, \eta : X^2 \to \mathbb{R}^+$*, we say a self map* $g : X \to X$ *on* $X$ *is orthogonal* $\alpha$-$\eta$-GF*-weak contraction (denoted by* $\perp$-$\alpha$-$\eta$-GF*-weak contraction) if for all* $\rho, \nu \in X$ *with* $\rho \perp \nu$*,* $d(g\rho, g\nu) > 0$ *and* $\eta(\rho, g\rho) \le \alpha(\rho,\nu)$*, we have*

$$\begin{aligned} & G\big(d(\rho,g\rho), d(\nu,g\nu), d(\rho,g\nu), d(\nu,g\rho)\big) + F\big(d(g\rho,g\nu)\big) \\ \le \quad & F\left(\max\left\{d(\rho,\nu), d(\rho,g\rho), d(\nu,g\nu), \frac{d(\rho,g\nu)+d(\nu,g\rho)}{2}\right\}\right), \end{aligned}$$

*where* $G \in \mathfrak{G}$ *and* $F \in \Im$*.*

**Remark 3** *From above definitions, we can conclude that every* $\perp$-$\alpha$-$\eta$-GF*-contraction is an* $\perp$-$\alpha$-$\eta$-GF*-weak contraction.*

**Theorem 1** *For an orthogonally complete metric space* $(X, \perp, d)$ *with* $\rho_0$ *as an orthogonal element, let* $G \in \mathfrak{G}$ *and* $F \in \Im$*. Suppose* $\alpha, \eta : X^2 \to \mathbb{R}^+$ *are two functions on* $X^2$ *and* $g : X \to X$ *is a self map such that:*

*(I)* $g$ *is* $\perp$*-preserving;*

*(II)* $g$ *is* $\alpha$*-admissible map with respect to* $\eta$*;*

*(III)* $\exists\, \rho_0 \in X$ *such that* $\eta(\rho_0, g\rho_0) \le \alpha(\rho_0, g\rho_0)$*;*

*(IV)* $g$ *is* $\perp$-$\alpha$-$\eta$*-continuous;*

*(V)* $g$ *is* $\perp$-$\alpha$-$\eta$-GF*-contraction.*

*Then $g$ possesses a fixed point. Moreover, if for all $\rho, \nu \in X$ such that $\rho \perp \nu$, $g\rho = \rho$ and $g\nu = \nu$ implies $\eta(\rho,\rho) \leq \alpha(\rho,\nu)$, then $g$ possesses a unique fixed point.*

**Proof.** Consider $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence in $X$ where $\rho_{n+1} = g\rho_n = g^{n+1}\rho_0$ for each $n \in \mathbb{N}$. Since $\eta(\rho_0, g\rho_0) \leq \alpha(\rho_0, g\rho_0)$, by using $\alpha$-admissibility of $g$ with respect to $\eta$, we get

$$\eta(\rho_1,\rho_2) = \eta(g\rho_0, g^2\rho_0) \leq \alpha(g\rho_0, g^2\rho_0) = \alpha(\rho_1,\rho_2),$$

continue applying $\alpha$-admissibility of $g$ with respect to $\eta$, we obtain

$$\eta(\rho_{n-1},\rho_n) \leq \alpha(\rho_{n-1},\rho_n) \quad \forall\, n \in \mathbb{N}.$$

Also, as $\rho_0, g\rho_0 \in X$ where $(X,\perp)$ is an $\perp$-set then repeated use of $\perp$-preserving property of $g$, gives

$$[\rho_{n-1} \perp \rho_n \;\forall\, n \in \mathbb{N}] \quad \text{or} \quad [\rho_n \perp \rho_{n+1} \;\forall\, n \in \mathbb{N}].$$

By using given contractive property of $g$, we get

$$G\big(d(\rho_{n-1}, g\rho_{n-1}), d(\rho_n, g\rho_n), d(\rho_{n-1}, g\rho_n), d(\rho_n, g\rho_{n-1})\big) + F\big(d(g\rho_{n-1}, g\rho_n)\big) \leq F\big(d(\rho_{n-1},\rho_n)\big). \quad (3)$$

Now since we have

$$d(\rho_n,\rho_{n+1}).d(\rho_{n-1},\rho_n).d(\rho_{n-1},\rho_{n+1}).d(\rho_n,\rho_n) = 0,$$

we see that $\exists\, \tau > 0$, such that

$$G(d(\rho_n,\rho_{n+1}), d(\rho_{n-1},\rho_n), d(\rho_{n-1},\rho_{n+1}), d(\rho_n,\rho_n)) = \tau. \quad (4)$$

On using (4) in (3), we obtain

$$\tau + F(d(g\rho_{n-1}, g\rho_n)) \leq F(d(\rho_{n-1},\rho_n)),$$

that is,

$$F(d(\rho_n,\rho_{n+1})) \leq F(d(\rho_{n-1},\rho_n)) - \tau \leq F(d(\rho_{n-2},\rho_{n-1})) - 2\tau \leq ... \leq F(d(\rho_0,\rho_1)) - n\tau. \quad (5)$$

Taking limit as $n \to \infty$ in (5) and by using $F2$ property of $F$, we have

$$\lim_{n\to\infty} d(\rho_n,\rho_{n+1}) = 0. \quad (6)$$

Further by $F3$ property of $F$, there exist some $0 < r < 1$, such that

$$\lim_{n\to\infty} \big(d(\rho_n,\rho_{n+1})\big)^r F(d(\rho_n,\rho_{n+1})) = 0. \quad (7)$$

Using (6) and (7) in (5), we get

$$\big(d(\rho_n,\rho_{n+1})\big)^r \big(F(d(\rho_n,\rho_{n+1})) - F(d(\rho_0,\rho_1))\big) \leq -n\tau\big(d(\rho_n,\rho_{n+1})\big)^r \leq 0.$$

On letting $n \to \infty$ in above, we have

$$\lim_{n\to\infty} n\big(d(\rho_n,\rho_{n+1})\big)^r = 0.$$

Then $\exists\, n_0 \in \mathbb{N}$, such that

$$n\big(d(\rho_n,\rho_{n+1})\big)^r \leq 1 \quad \forall\, n \geq n_0, \Rightarrow d(\rho_n,\rho_{n+1}) \leq \frac{1}{n^{1/r}} \quad \forall\, n \geq n_0.$$

Now, for $m > n > n_0$ and using triangle inequality, we obtain

$$d(\rho_n, \rho_m) \leq \sum_{i=n}^{m-1} d(\rho_i, \rho_{i+1}) \leq \sum_{i=1}^{\infty} d(\rho_i, \rho_{i+1}) \leq \sum_{i=1}^{\infty} \frac{1}{n^{1/r}}.$$

As $0 < r < 1$, so convergence of $\sum_{i=1}^{\infty} \frac{1}{n^{1/r}}$ implies $\{\rho_n\}_{n\in\mathbb{N}}$ is a Cauchy $\perp$-sequence and thus by orthogonal completeness of $X$, we have

$$\lim_{n\to\infty} \rho_n = \rho.$$

Therefore, by $\perp$-$\alpha$-$\eta$-continuity of $g$, we get

$$\lim_{n\to\infty} g\rho_n = g\rho \;\Rightarrow \lim_{n\to\infty} \rho_{n+1} = \rho \;\Rightarrow \rho = g\rho.$$

Thus $g$ possesses a fixed point.

Next, suppose $\nu$ is another fixed point of $g$ in $X$ where $\rho \perp \nu$ then by given condition $\eta(\rho, \rho) \leq \alpha(\rho, \nu)$. On using $\perp$-$\alpha$-$\eta$-$GF$-contraction of $g$ over $\rho$ and $\nu$, we obtain

$$G\big(d(\rho, g\rho), d(\nu, g\nu), d(\rho, g\nu), d(\nu, g\rho)\big) + F\big(d(g\rho, g\nu)\big) \leq F\big(d(\rho, \nu)\big).$$

Since

$$d(\rho, g\rho).d(\nu, g\nu).d(\rho, g\nu).d(\nu, g\rho) = 0,$$

we see that $\exists\, \tau > 0$, such that

$$G\big(d(\rho, g\rho), d(\nu, g\nu), d(\rho, g\nu), d(\nu, g\rho)\big) = \tau.$$

Therefore,

$$\tau + F\big(d(g\rho, g\nu)\big) \leq F\big(d(\rho, \nu)\big) \Rightarrow \tau + F\big(d(\rho, \nu)\big) \leq F\big(d(\rho, \nu)\big),$$

which holds only if $d(\rho, \nu) = 0$ i.e. $\rho = \nu$. Hence, $g$ possesses a unique fixed point. ■

**Example 5** *Consider the orthogonal metric space and $\perp$-$\alpha$-$\eta$-GF-contraction map $g$ defined in Example 4.*

*(i)* *$(X, \perp, d)$ is orthogonally complete metric space: Let $\{\rho_n\}_{n\in\mathbb{N}}$ be any Cauchy $\perp$-sequence in $X$. Then we have a subsequence $\{\rho_{n_k}\}$ of $\{\rho_n\}$ such that $\rho_{n_k} = 0\ \forall\ k \geq 1$, that is, $\rho_{n_k} \to 0$ as $n \to \infty$. Since this happens with any Cauchy $\perp$-sequence in X, we have $\{\rho_n\}_{n\in\mathbb{N}}$ convergent in $X$. Thus $(X, \perp, d)$ is orthogonally complete.*

*(ii)* *$g$ is $\perp$-preserving: Since $0 \perp y\ \forall\ y \in X$, we see that $g0 = 0 \perp gy\ \forall\ y \in X$. Thus $g$ is $\perp$-preserving.*

*(iii)* *$g$ is $\alpha$-admissible with respect to $\eta$: From the definition of $\alpha$, $\eta$ and $g$ it can be concluded that $g$ is $\alpha$-admissible with respect to $\eta$.*

*(iv)* *$g$ is $\perp$-continuous: For any convergent $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$, we have $\rho_n \to 0$ as $n \to \infty$. Then $g\rho_n \to g0 = 0$ as $n \to \infty$. Therefore, $g$ is $\perp$-continuous.*

*Since all the conditions of Theorem 1 holds, we see that $g$ possesses a fixed point viz. $\rho = 0$.*

**Corollary 1** *For an orthogonally complete metric space $(X, \perp, d)$ with $\rho_0$ as an orthogonal element, let $F \in \Im$. Suppose $\alpha, \eta : X^2 \to \mathbb{R}^+$ are two functions on $X^2$ and $g : X \to X$ is a self map such that*

*(I)* *$g$ is $\perp$-preserving;*

*(II)* *$g$ is $\alpha$-admissible map with respect to $\eta$;*

*(III)* *$\exists\, \rho_0 \in X$ such that $\eta(\rho_0, g\rho_0) \leq \alpha(\rho_0, g\rho_0)$;*

*(IV) $g$ is $\perp$-$\alpha$-$\eta$-continuous;*

*(V) for all $\rho, \nu \in X$ with $d(g\rho, g\nu) > 0$, $\rho \perp \nu$ and $\eta(\rho, g\rho) \leq \alpha(\rho, \nu)$ implies*

$$\tau + F\big(d(g\rho, g\nu)\big) \leq F\big(d(\rho, \nu)\big).$$

*Then $g$ possesses a fixed point. Moreover, if for all $\rho, \nu \in X$ such that $\rho \perp \nu$, $g\rho = \rho$ and $g\nu = \nu$ implies $\eta(\rho, \rho) \leq \alpha(\rho, \nu)$, then $g$ possesses a unique fixed point.*

**Proof.** In Theorem 1, if we consider function $G(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = K.(\gamma_1.\gamma_2.\gamma_3.\gamma_4) + \tau$, where $K \geq 0$ is a constant and $\tau > 0$ then the result follows. ■

**Theorem 2** *For an orthogonally complete metric space $(X, \perp, d)$ with $\rho_0$ as an orthogonal element, let $G \in \mathfrak{G}$ and $F \in \Im$. Suppose $\alpha, \eta : X^2 \to \mathbb{R}^+$ are two functions on $X^2$ and $g : X \to X$ is a self map such that*

*(I) $g$ is $\perp$-preserving;*

*(II) $g$ is $\alpha$-admissible map with respect to $\eta$;*

*(III) $\exists\, \rho_0 \in X$ such that $\eta(\rho_0, g\rho_0) \leq \alpha(\rho_0, g\rho_0)$;*

*(IV) $g$ is $\perp$-$\alpha$-$\eta$-continuous;*

*(V) $g$ is $\perp$-$\alpha$-$\eta$-*GF*-weak contraction.*

*Then $g$ possesses a fixed point. Moreover, if for all $\rho, \nu \in X$ such that $\rho \perp \nu$, $g\rho = \rho$ and $g\nu = \nu$ implies $\eta(\rho, \rho) \leq \alpha(\rho, \nu)$, then $g$ possesses a unique fixed point.*

**Proof.** Working on the footprints of Theorem 1, we obtain an orthogonal sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $X$, such that

$$\begin{aligned} & G\big(d(\rho_{n-1}, g\rho_{n-1}), d(\rho_n, g\rho_n), d(\rho_{n-1}, g\rho_n), d(\rho_n, g\rho_{n-1})\big) + F\big(d(g\rho_{n-1}, g\rho_n)\big) \\ \leq \;& F\left(\max\left\{d(\rho_{n-1}, \rho_n), d(\rho_{n-1}, g\rho_{n-1}), d(\rho_n, g\rho_n), \frac{d(\rho_{n-1}, g\rho_n) + d(\rho_n, g\rho_{n-1})}{2}\right\}\right). \end{aligned} \tag{8}$$

Since we have

$$d(\rho_n, \rho_{n+1}).d(\rho_{n-1}, \rho_n).d(\rho_{n-1}, \rho_{n+1}).d(\rho_n, \rho_n) = 0,$$

we see that $\exists\, \tau > 0$, such that

$$G(d(\rho_n, \rho_{n+1}), d(\rho_{n-1}, \rho_n), d(\rho_{n-1}, \rho_{n+1}), d(\rho_n, \rho_n)) = \tau. \tag{9}$$

On using (9) in (8), we obtain

$$\begin{aligned} \tau + F(d(g\rho_{n-1}, g\rho_n)) &\leq F\left(\max\left\{d(\rho_{n-1}, \rho_n), d(\rho_n, \rho_{n+1}), \frac{d(\rho_{n-1}, \rho_{n+1})}{2}\right\}\right), \\ &\leq F\left(\max\left\{d(\rho_{n-1}, \rho_n), d(\rho_n, \rho_{n+1}), \frac{d(\rho_{n-1}, \rho_n) + d(\rho_n, \rho_{n+1})}{2}\right\}\right) \\ &= F\left(\max\left\{d(\rho_{n-1}, \rho_n), d(\rho_n, \rho_{n+1})\right\}\right). \end{aligned}$$

**Case 1** *If* $\max\left\{d(\rho_{n-1}, \rho_n), d(\rho_n, \rho_{n+1})\right\} = d(\rho_n, \rho_{n+1})$*, then we have*

$$\tau + F\big(d(\rho_n, \rho_{n+1})\big) \leq F(d(\rho_n, \rho_{n+1})),$$

*which is a contradiction for $\tau > 0$.*

**Case 2** *If* $\max\left\{d(\rho_{n-1},\rho_n), d(\rho_n,\rho_{n+1})\right\} = d(\rho_{n-1},\rho_n)$, *then we have*

$$\tau + F\big(d(\rho_n,\rho_{n+1})\big) \leq F(d(\rho_{n-1},\rho_n)),$$

$$\Rightarrow F\big(d(\rho_n,\rho_{n+1})\big) \leq F(d(\rho_{n-1},\rho_n)) - \tau = F(d(\rho_{n-1},\rho_n)) - 2\tau \leq \ldots \leq F(d(\rho_0,\rho_1)) - n\tau.$$

The proof now follows on the lines of Theorem 1. ■

**Remark 4** *In the upcoming results, we drop the condition of* $\perp$-$\alpha$-$\eta$-*continuity of* $g$ *and instead consider a weaker condition.*

**Theorem 3** *For an orthogonally complete metric space* $(X,\perp,d)$ *with* $\rho_0$ *as an orthogonal element, let* $G \in \mathfrak{G}$ *and* $F \in \mathfrak{I}$. *Suppose* $\alpha,\eta : X^2 \to \mathbb{R}^+$ *are two functions on* $X^2$ *and* $g : X \to X$ *is a self map such that*

*(I)* $g$ *is* $\perp$-*preserving;*

*(II)* $g$ *is* $\alpha$-*admissible map with respect to* $\eta$*;*

*(III)* $\exists\, \rho_0 \in X$ *such that* $\eta(\rho_0, g\rho_0) \leq \alpha(\rho_0, g\rho_0)$*;*

*(IV) if* $\{\rho_n\}_{n\in\mathbb{N}}$ *is an* $\perp$-*sequence in* $X$ *such that* $\eta(\rho_n,\rho_{n+1}) \leq \alpha(\rho_n,\rho_{n+1})$ *and* $\rho_n \to \rho$ *as* $n \to \infty$, *then*

$$[\rho_n \perp \rho \;\forall\, n] \quad or \quad [\rho \perp \rho_n \;\forall\, n],$$

*and,*

$$[\eta(g\rho_n, g^2\rho_n) \leq \alpha(g\rho_n,\rho)] \quad or \quad [\eta(g^2\rho_n, g^3\rho_n) \leq \alpha(g^2\rho_n,\rho)] \;\forall\, n \in \mathbb{N};$$

*(V)* $g$ *is* $\perp$-$\alpha$-$\eta$-GF-*contraction.*

*Then* $g$ *possesses a fixed point. Moreover, if for all* $\rho,\nu \in X$ *with* $\rho \perp \nu$, $g\rho = \rho$ *and* $g\nu = \nu$ *implies* $\eta(\rho,\rho) \leq \alpha(\rho,\nu)$, *then* $g$ *possesses a unique fixed point.*

**Proof.** On the lines of Theorem 1, we obtain an $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ such that

$$\eta(\rho_n,\rho_{n+1}) \leq \alpha(\rho_n,\rho_{n+1}) \quad \text{and} \quad \lim_{n\to\infty} \rho_n = \rho.$$

Here we claim that $\rho$ is a fixed point of $g$ in $X$. By given condition, we have

$$[\rho_n \perp \rho \;\forall\, n \in \mathbb{N}] \quad \text{or} \quad [\rho \perp \rho_n \;\forall\, n \in \mathbb{N}],$$

and,

$$[\eta(\rho_{n+1},\rho_{n+2}) \leq \alpha(\rho_{n+1},\rho)] \quad \text{or} \quad [\eta(\rho_{n+2},\rho_{n+3}) \leq \alpha(\rho_{n+2},\rho)] \;\forall\, n \in \mathbb{N}.$$

Then $\exists$ a subsequence $\{\rho_{n_s}\}$ of $\{\rho_n\}$, such that

$$\eta(\rho_{n_s}, g\rho_{n_s}) \leq \alpha(\rho_{n_s},\rho).$$

By $\perp$-$\alpha$-$\eta$-$GF$-contraction of $g$, we have

$$F\big(d(g\rho_{n_s}, g\rho)\big) < G\big(d(\rho_{n_s}, g\rho_{n_s}), d(\rho, g\rho), d(\rho_{n_s}, g\rho), d(\rho, g\rho_{n_s})\big) + F\big(d(g\rho_{n_s}, g\rho)\big) \leq F\big(d(\rho_{n_s},\rho)\big),$$

that is, $F(d(g\rho_{n_s}, g\rho)) \leq F(d(\rho_{n_s},\rho))$. From $F1$ property of $F$, we have

$$d(g\rho_{n_s}, g\rho) < d(\rho_{n_s},\rho). \tag{10}$$

Letting $s \to \infty$ in (10), gives

$$d(\rho, g\rho) = 0.$$

Thus $g$ possesses a fixed point. Further, the uniqueness of fixed point follows on the line of Theorem 1. ■

**Corollary 2** *For an orthogonally complete metric space $(X, \perp, d)$ with $\rho_0$ as an orthogonal element, let $F \in \Im$. Suppose $\alpha, \eta : X^2 \to \mathbb{R}^+$ are two functions on $X^2$ and $g : X \to X$ is a self map such that*

*(I) $g$ is $\perp$-preserving;*

*(II) $g$ is $\alpha$-admissible map with respect to $\eta$;*

*(III) $\exists\, \rho_0 \in X$ such that $\eta(\rho_0, g\rho_0) \leq \alpha(\rho_0, g\rho_0)$ ;*

*(IV) if $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\perp$-sequence in $X$ such that $\eta(\rho_n, \rho_{n+1}) \leq \alpha(\rho_n, \rho_{n+1})$ and $\rho_n \to \rho$ as $n \to \infty$, then*

$$[\rho_n \perp \rho\ \forall\, n] \quad or \quad [\rho \perp \rho_n\ \forall\, n],$$

*and,*

$$[\eta(g\rho_n, g^2\rho_n) \leq \alpha(g\rho_n, \rho)] \quad or \quad [\eta(g^2\rho_n, g^3\rho_n) \leq \alpha(g^2\rho_n, \rho)]\ \forall\, n \in \mathbb{N};$$

*(V) for all $\rho, \nu \in X$ with $d(g\rho, g\nu) > 0$, $\rho \perp \nu$ and $\eta(\rho, g\rho) \leq \alpha(\rho, \nu)$ implies*

$$\tau + F(d(g\rho, g\nu)) \leq F(d(\rho, \nu)).$$

*Then $g$ possesses a fixed point. Moreover, if for all $\rho, \nu \in X$ with $\rho \perp \nu$, $g\rho = \rho$ and $g\nu = \nu$ implies $\eta(\rho, \rho) \leq \alpha(\rho, \nu)$, then $g$ possesses a unique fixed point.*

**Proof.** In Theorem 3, if we consider function $G(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = K.(\gamma_1.\gamma_2.\gamma_3.\gamma_4) + \tau$, where $K \geq 0$ is a constant and $\tau > 0$ then the result follows. ■

**Theorem 4** *For an orthogonally complete metric space $(X, \perp, d)$ with $\rho_0$ as an orthogonal element, let $G \in \mathfrak{G}$ and $F \in \Im$. Suppose $\alpha, \eta : X^2 \to \mathbb{R}^+$ are two functions on $X^2$ and $g : X \to X$ is a self map such that*

*(I) $g$ is $\perp$-preserving;*

*(II) $g$ is $\alpha$-admissible map with respect to $\eta$;*

*(III) $\exists\, \rho_0 \in X$ such that $\eta(\rho_0, g\rho_0) \leq \alpha(\rho_0, g\rho_0)$;*

*(IV) if $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\perp$-sequence in $X$ such that $\eta(\rho_n, \rho_{n+1}) \leq \alpha(\rho_n, \rho_{n+1})$ and $\rho_n \to \rho$ as $n \to \infty$, then*

$$[\rho_n \perp \rho\ \forall\, n] \quad or \quad [\rho \perp \rho_n\ \forall\, n],$$

*and,*

$$[\eta(g\rho_n, g^2\rho_n) \leq \alpha(g\rho_n, \rho)] \quad or \quad [\eta(g^2\rho_n, g^3\rho_n) \leq \alpha(g^2\rho_n, \rho)]\ \forall\, n \in \mathbb{N};$$

*(V) $g$ is $\perp$-$\alpha$-$\eta$-GF-weak contraction.*

*Then $g$ possesses a fixed point. Moreover, if for all $\rho, \nu \in X$ with $\rho \perp \nu$, $g\rho = \rho$ and $g\nu = \nu$ implies $\eta(\rho, \rho) \leq \alpha(\rho, \nu)$, then $g$ possesses a unique fixed point.*

**Proof.** Working on the lines of Theorem 3, we obtain a subsequence $\{\rho_{n_s}\}$ of an orthogonal sequence $\{\rho_n\}$ with $\eta(\rho_{n_s}, g\rho_{n_s}) \leq \alpha(\rho_{n_s}, \rho)$, such that

$$\begin{aligned} F\big(d(g\rho_{n_s}, g\rho)\big) &< G\big(d(\rho_{n_s}, g\rho_{n_s}), d(\rho, g\rho), d(\rho_{n_s}, g\rho), d(\rho, g\rho_{n_s})\big) + F\big(d(g\rho_{n_s}, g\rho)\big) \\ &\leq F\left(\max\left\{d(\rho_{n_s}, \rho), d(\rho_{n_s}, g\rho_{n_s}), d(\rho, g\rho), \frac{d(\rho_{n_s}, g\rho) + d(\rho, g\rho_{n_s})}{2}\right\}\right). \end{aligned} \tag{11}$$

From $F1$ property of $F$ in (11), we have

$$d(g\rho_{n_s}, g\rho) < \max\left\{d(\rho_{n_s}, \rho), d(\rho_{n_s}, g\rho_{n_s}), d(\rho, g\rho), \frac{d(\rho_{n_s}, g\rho) + d(\rho, g\rho_{n_s})}{2}\right\}. \tag{12}$$

Letting $s \to \infty$ in (12), gives

$$d(\rho, g\rho) = 0.$$

Thus $g$ possesses a fixed point. Further, the uniqueness of fixed point follows on the line of Theorem 1. ■

### 3.2 Orthogonal $\alpha$-type $F$-contraction

The concept of $\alpha$-type $F$-contraction was given by D. Gopal et al. in [10], and the results proved were generalization of the contraction results in [8, 32, 33]. Here in this subsection, we discuss some basic definitions and prove fixed point results related to orthogonal $\alpha$-type $F$-contraction and some of its weaker contraction conditions in succeeding results.

**Definition 20** *For an orthogonal metric space $(X, \perp, d)$ and for $\alpha : X^2 \to (0, \infty)$, a map $g : X \to X$ on $X$ is an orthogonal $\alpha$-type $F$-contraction (denoted by $\perp$-$\alpha$ type $F$-contraction) if there exist $\tau > 0$, $F \in \Im$ such that for all $\rho, \nu \in X$ with $\rho \perp \nu$, $d(g\rho, g\nu) > 0$, implies*

$$\tau + \alpha(\rho, \nu)F\big(d(g\rho, g\nu)\big) \leq F\big(d(\rho, \nu)\big).$$

**Example 6** *Let $X = \mathbb{R}^+$, $d(\rho, \nu) = |\rho - \nu|$ and $\rho \perp \nu \Leftrightarrow$ either $\rho = 0$ or $\nu = 0$. Then one can easily verify that $(X, \perp, d)$ is an orthogonal metric space. Define $g : X \to X$ as*

$$g(\rho) = \begin{cases} \frac{3}{2} & \rho \in [10, 20); \\ 0 & otherwise. \end{cases}$$

*Let $\alpha : X^2 \to [0, \infty)$ be defined as $\alpha(\rho, \nu) = 3/2 \quad \forall \rho, \nu \in X$. Define $F(\eta) = \ln(\eta)$. For $d(g\rho, g\nu) > 0$ and $\rho \perp \nu$ to hold simultaneously, we have either $\rho = 0$ and $\nu \in [10, 20)$ or $\rho \in [10, 20)$ and $\nu = 0$.*

**case (i)** *Let $\rho = 0$ and $\nu \in [10, 20)$. Then*

$$\tau + \alpha(0, \nu)\ln(d(g0, g\nu)) = \tau + \frac{3}{2}\ln(d(0, 3/2)) = \tau + \frac{3}{2}\ln(3/2) \tag{13}$$

*and*

$$\ln(d(0, \nu)) = \ln(\nu). \tag{14}$$

*From (13) and (14), for $\tau = 1$ we can conclude that $g$ is $\perp$-$\alpha$ type $F$-contraction. The case (ii) for $\rho \in [10, 20)$ and $\nu = 0$ holds on the lines of case(i).*

**Definition 21** *For an orthogonal metric space $(X, \perp, d)$ and for $\alpha : X^2 \to (0, \infty)$, a map $g : X \to X$ on $X$ is an orthogonal $\alpha$-type $F$-weak contraction (denoted by $\perp$-$\alpha$ type $F$-weak contraction) if there exist $\tau > 0$ and $F \in \Im$ and for all $\rho, \nu \in X$ with $\rho \perp \nu$, $d(g\rho, g\nu) > 0$, implies*

$$\tau + \alpha(\rho, \nu)F\big(d(g\rho, g\nu)\big) \leq F\bigg(max\bigg\{d(\rho, \nu), d(\rho, g\rho), d(\nu, g\nu), \frac{d(\rho, g\nu) + d(\nu, g\rho)}{2}\bigg\}\bigg).$$

**Remark 5** *From the above definitions it can be easily concluded that every $\perp$-$\alpha$ type $F$-contraction is an $\perp$-$\alpha$ type F-weak contraction.*

**Theorem 5** *For an orthogonally complete metric space $(X, \perp, d)$ with $s \geq 1$ and $\rho_0$ as an orthogonal element, $F \in \Im$. Suppose $\alpha : X^2 \to \mathbb{R}^+$ is a map on $X^2$ and $g : X \to X$ is a self map such that*

*(I) $g$ is $\perp$-preserving;*

*(II) $g$ is weak $\alpha$-admissible map type S;*

*(III) $\exists\ \rho_0 \in X$ with $s \leq \alpha(\rho_0, g\rho_0)$;*

*(IV) $g$ is $\perp$-continuous;*

*(V) $g$ is $\perp$-$\alpha$ type $F$-contraction.*

*Then $g$ possesses a fixed point. Moreover, if for all $\rho, \nu \in X$ with $\rho \perp \nu$, $g\rho = \rho$ and $g\nu = \nu$ implies $s \leq \alpha(\rho, \nu)$, then $g$ possesses a unique fixed point.*

**Proof.** On defining a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $X$ where $\rho_{n+1} = g\rho_n = g^{n+1}\rho_0$ for each $n \in \mathbb{N}$ and since $\rho_0, g\rho_0 \in X$ where $(X, \perp)$ is an $\perp$-set, and by $\perp$-preserving property of $g$, gives

$$[\rho_{n+1} \perp \rho_n \ \forall\, n \in \mathbb{N}] \quad \text{or} \quad [\rho_n \perp \rho_{n+1} \ \forall\, n \in \mathbb{N}],$$

that is, $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\perp$-sequence in $X$.

Now, by given condition $\alpha(\rho_0, \rho_1) = \alpha(\rho_0, g\rho_0) \geq s$ then as $g$ is weak $\alpha$-admissible map type S, we have $\alpha(\rho_1, \rho_2) \geq s$ continuing, we get $\alpha(\rho_{n-1}, \rho_n) \geq s$. Thus, we have

$$F\big(d(\rho_n, \rho_{n+1})\big) = F\big(d(g\rho_{n-1}, g\rho_n)\big) \leq sF\big(d(g\rho_{n-1}, g\rho_n)\big) \leq \alpha(\rho_{n-1}, \rho_n)F\big(d(g\rho_{n-1}, g\rho_n)\big).$$

Using $\perp$-$\alpha$ type $F$-contraction condition of $g$ and for $\tau > 0$, we get

$$\begin{aligned} \tau + F\big(d(\rho_n, \rho_{n+1})\big) &\leq \tau + sF\big(d(g\rho_{n-1}, g\rho_n)\big) \\ &\leq \tau + \alpha(\rho_{n-1}, \rho_n)F\big(d(g\rho_{n-1}, g\rho_n)\big) \\ &\leq F\big(d(\rho_{n-1}, g\rho_n)\big), \end{aligned}$$

that is,

$$\begin{aligned} F\big(d(\rho_n, \rho_{n+1})\big) &\leq F\big(d(\rho_{n-1}, \rho_n)\big) - \tau \\ &\leq F\big(d(\rho_{n-2}, \rho_{n-1})\big) - 2\tau \leq ... \leq F\big(d(\rho_0, \rho_1)\big) - n\tau. \end{aligned} \tag{15}$$

Taking limit as $n \to \infty$ in (15) and by $F2$ property of $F$, we have

$$\lim_{n\to\infty} d(\rho_n, \rho_{n+1}) = 0. \tag{16}$$

And further, by $F3$ property of $F$, $\exists\, r \in (0, 1)$, implies

$$\lim_{n\to\infty} \big(d(\rho_n, \rho_{n+1})\big)^r F\big(d(\rho_n, \rho_{n+1})\big) = 0. \tag{17}$$

From (15), we deduce that

$$\big(d(\rho_n, \rho_{n+1})\big)^r \big(F(d(\rho_n, \rho_{n+1})) - F(d(\rho_0, \rho_1))\big) \leq -n\big(d(\rho_n, \rho_{n+1})\big)^r \tau. \tag{18}$$

On letting $n \to \infty$ in (18) and using (16) and (17), we obtain

$$\lim_{n\to\infty} n\big(d(\rho_n, \rho_{n+1})\big)^r = 0.$$

Thus $\exists\, n_1 \in \mathbb{N}$, such that

$$d(\rho_n, \rho_{n+1}) < \frac{1}{n^{1/r}} \quad \forall\, n \geq n_1.$$

Consider $m > n > n_1$, then by triangle inequality, we obtain

$$\begin{aligned} d(\rho_n, \rho_m) &\leq d(\rho_n, \rho_{n+1}) + d(\rho_{n+1}, \rho_{n+2}) + ... + d(\rho_{m-1}, \rho_m), \\ &\leq \sum_{i=1}^{\infty} d(\rho_i, \rho_{i+1}), \\ &= \sum_{i=1}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

Since the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$ is convergent, which implies $\{\rho_n\}_{n\in\mathbb{N}}$ is a Cauchy $\perp$-sequence and thus, by orthogonal completeness of $X$, $\exists\, \rho \in X$ for which

$$\lim_{n\to\infty} \rho_n = \rho.$$

Further by $\perp$-continuity of $g$, we get

$$\lim_{n\to\infty} g\rho_n = g\rho \Rightarrow \lim_{n\to\infty} \rho_{n+1} = g\rho \Rightarrow \rho = g\rho.$$

Thus $g$ possesses a fixed point.

Next, let $\nu$ be another fixed point of $g$ such that $\rho \perp \nu$. Then by given condition, we have $\alpha(\rho,\nu) \geq s$. Using $\perp$-$\alpha$ type $F$-contraction property of $g$, we have

$$\begin{aligned}\tau + F\big(d(g\rho, g\nu)\big) &\leq \tau + sF\big(d(g\rho, g\nu)\big)\\ &\leq \tau + \alpha(\rho,\nu)F\big(d(g\rho, g\nu)\big) \leq F\big(d(\rho,\nu)\big),\end{aligned}$$

that is,

$$\tau + F\big(d(\rho,\nu)\big) \leq F\big(d(\rho,\nu)\big). \tag{19}$$

Now, (19) holds only if $\rho = \nu$. Hence, $g$ possesses a unique fixed point. ■

**Example 7** *The self map $g$ defined in Example 6 satisfies all conditions of above theorem and thus has a fixed point $\rho = 0$.*

**Theorem 6** *For an orthogonally complete metric space $(X,\perp,d)$ with $s \geq 1$ and $\rho_0$ as an orthogonal element, $F \in \Im$. Suppose $\alpha : X^2 \to \mathbb{R}^+$ is a map on $X^2$ and $g : X \to X$ is a self map such that*

*(I) $g$ is $\perp$-preserving;*

*(II) $g$ is weak $\alpha$-admissible map type S;*

*(III) $\exists\, \rho_0 \in X$ with $s \leq \alpha(\rho_0, g\rho_0)$;*

*(IV) $g$ is $\perp$-continuous;*

*(V) $g$ is $\perp$-$\alpha$ type $F$-weak contraction.*

*Then $g$ possesses a fixed point. Moreover, if for all $\rho,\nu \in X$ with $\rho \perp \nu$, $g\rho = \rho$ and $g\nu = \nu$ implies $s \leq \alpha(\rho,\nu)$, then $g$ possesses a unique fixed point.*

**Proof.** Working on the lines of Theorem 5, we obtain an $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $X$ with $\alpha(\rho_n,\rho_{n+1}) \geq s \quad \forall\, n \in \mathbb{N}$.

$$F\big(d(\rho_n,\rho_{n+1})\big) = F\big(d(g\rho_{n-1}, g\rho_n)\big) \leq sF\big(d(g\rho_{n-1}, g\rho_n)\big) \leq \alpha(\rho_{n-1},\rho_n)F\big(d(g\rho_{n-1}, g\rho_n)\big).$$

As $g$ is $\perp$-$\alpha$ type $F$-weak contraction, so we have

$$\begin{aligned}\tau + F\big(d(\rho_n,\rho_{n+1})\big) &\leq \tau + \alpha(\rho_{n-1},\rho_n)F\big(d(g\rho_{n-1}, g\rho_n)\big)\\ &\leq F\left(\max\left\{d(\rho_{n-1},\rho_n), d(\rho_{n-1}, g\rho_{n-1}), d(\rho_n, g\rho_n), \frac{d(\rho_{n-1}, g\rho_n) + d(\rho_n, g\rho_{n-1})}{2}\right\}\right)\\ &= F\left(\max\left\{d(\rho_{n-1},\rho_n), d(\rho_n,\rho_{n+1}), \frac{d(\rho_{n-1},\rho_{n+1})}{2}\right\}\right)\\ &\leq F\left(\max\left\{d(\rho_{n-1},\rho_n), d(\rho_n,\rho_{n+1}), \frac{d(\rho_{n-1},\rho_n) + d(\rho_n,\rho_{n+1})}{2}\right\}\right).\end{aligned}$$

Then

$$\tau + F\big(d(\rho_n,\rho_{n+1})\big) \leq F\Big(\max\Big\{d(\rho_{n-1},\rho_n), d(\rho_n,\rho_{n+1})\Big\}\Big). \tag{20}$$

**Case 1** $\max\Big\{d(\rho_{n-1},\rho_n), d(\rho_n,\rho_{n+1})\Big\} = d(\rho_n,\rho_{n+1})$. Then by (20), we have

$$\tau + F\big(d(\rho_n,\rho_{n+1})\big) \leq F(d(\rho_n,\rho_{n+1})),$$

which is a contradiction for $\tau > 0$.

**Case 2** If $\max\Big\{d(\rho_{n-1},\rho_n), d(\rho_n,\rho_{n+1})\Big\} = d(\rho_{n-1},\rho_n)$. Then by (20), we have

$$\tau + F\big(d(\rho_n,\rho_{n+1})\big) \leq F(d(\rho_{n-1},\rho_n)).$$

Then

$$\begin{aligned} F\big(d(\rho_n,\rho_{n+1})\big) &\leq F(d(\rho_{n-1},\rho_n)) - \tau \\ &= F(d(\rho_{n-1},\rho_n)) - 2\tau \leq ... \leq F(d(\rho_0,\rho_1)) - n\tau. \end{aligned}$$

The proof now follows on the lines of Theorem 5. ■

**Remark 6** *In the upcoming result, we weaken the condition of $\perp$-continuity of $g$.*

**Theorem 7** *For an orthogonally complete metric space $(X,\perp,d)$ with $s \geq 1$ and $\rho_0$ as an orthogonal element, $F \in \Im$. Suppose $\alpha : X^2 \to \mathbb{R}^+$ is a map on $X^2$ and $g : X \to X$ is a self map such that*

*(I) $g$ is $\perp$-preserving;*

*(II) $g$ is weak $\alpha$-admissible map type S;*

*(III) $\exists\, \rho_0 \in X$ with $s \leq \alpha(\rho_0, g\rho_0)$;*

*(IV) if there exists an $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ with $\alpha(\rho_n,\rho_{n+1}) \geq s$ and $\rho_n \to \rho$ as $n \to \infty$, then $\alpha(\rho_n,\rho) \geq s$ and* either *$[\rho_n \perp \rho\ \forall\, n \in \mathbb{N}]$* or *$[\rho \perp \rho_n\ \forall\, n \in \mathbb{N}]$;*

*(V) $g$ is $\perp$-$\alpha$ type $F$-contraction.*

*Then $g$ possesses a fixed point. Moreover, if for all $\rho,\nu \in X$ with $\rho \perp \nu$, $g\rho = \rho$ and $g\nu = \nu$ implies $s \leq \alpha(\rho,\nu)$, then $g$ possesses a unique fixed point.*

**Proof.** Proceeding on the lines of Theorem 5, one can obtain $\{\rho_n\}_{n\in\mathbb{N}}$ an $\perp$-sequence where $\rho_n \to \rho$ as $n \to \infty$ and $\alpha(\rho_n,\rho_{n+1}) \geq s$. Then by given condition we have, $\alpha(\rho_n,\rho) \geq s$ and

$$[\rho_n \perp \rho\ \forall\, n \in \mathbb{N}] \quad \text{or} \quad [\rho \perp \rho_n\ \forall\, n \in \mathbb{N}].$$

Using $\perp$-preserving property of $g$, we get

$$[g\rho_n \perp g\rho\ \forall\, n \in \mathbb{N}] \quad \text{or} \quad [g\rho \perp g\rho_n\ \forall\, n \in \mathbb{N}].$$

Since $g$ is an $\perp$-$\alpha$ type $F$-contraction, we have

$$\begin{aligned} F\big(d(\rho_{n+1},g\rho)\big) &\leq \tau + F\big(d(\rho_{n+1},g\rho)\big) = \tau + F\big(d(g\rho_n,g\rho)\big) \\ &\leq \tau + sF\big(d(g\rho_n,g\rho)\big) \\ &\leq \tau + \alpha(\rho_n,\rho)F\big(d(g\rho_n,g\rho)\big) \leq F\big(d(\rho_n,\rho)\big). \end{aligned} \tag{21}$$

Using $F1$ property of $F$ in (21), we obtain

$$d(\rho_{n+1},g\rho) < d(\rho_n,\rho).$$

Taking limit as $n \to \infty$, we obtain

$$d(\rho,g\rho) = 0.$$

Thus $g$ possesses a fixed point. Further, the uniqueness of the fixed point of $g$ follows on the line of Theorem 5. ■

**Theorem 8** *For an orthogonally complete metric space $(X, \perp, d)$ with $s \geq 1$ and $\rho_0$ as an orthogonal element, $F \in \Im$. Suppose $\alpha : X^2 \to \mathbb{R}^+$ is a map on $X^2$ and $g : X \to X$ is a self map such that*

*(I)* *$g$ is $\perp$-preserving;*

*(II)* *$g$ is weak $\alpha$-admissible map type S;*

*(III)* *$\exists\, \rho_0 \in X$ with $s \leq \alpha(\rho_0, g\rho_0)$;*

*(IV)* *if there exists an $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ with $\alpha(\rho_n, \rho_{n+1}) \geq s$ and $\rho_n \to \rho$ as $n \to \infty$, then $\alpha(\rho_n, \rho) \geq s$ and* either $[\rho_n \perp \rho\, \forall\, n \in \mathbb{N}]$ or $[\rho \perp \rho_n\, \forall\, n \in \mathbb{N}]$;

*(V)* *$g$ is $\perp$-$\alpha$ type $F$-weak contraction.*

*Then $g$ possesses a fixed point. Moreover, if for all $\rho, \nu \in X$ with $\rho \perp \nu$, $g\rho = \rho$ and $g\nu = \nu$ implies $s \leq \alpha(\rho, \nu)$, then $g$ possesses a unique fixed point.*

**Proof.** Working on the lines of Theorem 7, we have

$$\begin{aligned} F\big(d(\rho_{n+1}, g\rho)\big) &\leq \tau + F\big(d(\rho_{n+1}, g\rho)\big) = \tau + F\big(d(g\rho_n, g\rho)\big) \\ &\leq \tau + sF\big(d(g\rho_n, g\rho)\big) \\ &\leq \tau + \alpha(\rho_n, \rho)F\big(d(g\rho_n, g\rho)\big), \\ &\leq F\left(\max\left\{d(\rho_n, \rho), d(\rho_n, g\rho_n), d(\rho, g\rho), \frac{d(\rho_n, g\rho) + d(\rho, g\rho_n)}{2}\right\}\right). \end{aligned} \tag{22}$$

Using $F1$ property of $F$ in (22), we obtain

$$d(\rho_{n+1}, g\rho) < \max\left\{d(\rho_n, \rho), d(\rho_n, g\rho_n), d(\rho, g\rho), \frac{d(\rho_n, g\rho) + d(\rho, g\rho_n)}{2}\right\}.$$

Taking limit as $n \to \infty$, we obtain

$$d(\rho, g\rho) = 0.$$

Thus $g$ possesses a fixed point. Further, the uniqueness of the fixed point of $g$ follows on the line of Theorem 7. ■

**Remark 7** *It should be noted that Theorem 5, Theorem 6, Theorem 7 and Theorem 8 proved above are valid even if g is considered as an $\alpha$-admissible map type S.*

**Theorem 9** *For an orthogonally complete metric space $(X, \perp, d)$ with $\rho_0$ as an orthogonal element, $F \in \Im$. Suppose $\alpha : X^2 \to \mathbb{R}^+$ is a map on $X^2$ and $g : X \to X$ is a self map such that*

*(I)* *$g$ is $\perp$-preserving;*

*(II)* *$g$ is weak $\alpha$-admissible map;*

*(III)* *$\exists\, \rho_0 \in X$ with $1 \leq \alpha(\rho_0, g\rho_0)$;*

*(IV)* *$g$ is $\perp$-continuous;*

*(V)* *$g$ is $\perp$-$\alpha$ type $F$-contraction.*

*Then $g$ possesses a fixed point. Moreover, if for all $\rho, \nu \in X$ with $\rho \perp \nu$, $g\rho = \rho$ and $g\nu = \nu$ implies $1 \leq \alpha(\rho, \nu)$, then $g$ possesses a unique fixed point.*

**Proof.** On defining a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $X$ where $\rho_{n+1} = g\rho_n = g^{n+1}\rho_0$ for each $n \in \mathbb{N}$ and since $\rho_0, g\rho_0 \in X$ where $(X,\perp)$ is an $\perp$-set, and by $\perp$-preserving property of $g$, gives

$$[\rho_{n-1} \perp \rho_n \; \forall \, n \in \mathbb{N}] \quad \text{or} \quad [\rho_n \perp \rho_{n+1} \; \forall \, n \in \mathbb{N}],$$

that is, $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\perp$-sequence in $X$.

Now, by given condition $\alpha(\rho_0,\rho_1) = \alpha(\rho_0, g\rho_0) \geq 1$, then by weak $\alpha$-admissibility of $g$, we have $\alpha(g\rho_0, gg\rho_0) = \alpha(\rho_1,\rho_2) \geq 1$ continuing, we get $\alpha(\rho_{n-1},\rho_n) \geq 1$. Thus, we have

$$F\big(d(\rho_n,\rho_{n+1})\big) = F\big(d(g\rho_{n-1}, g\rho_n)\big) \leq \alpha(\rho_{n-1},\rho_n)F\big(d(g\rho_{n-1}, g\rho_n)\big).$$

Using $\perp$-$\alpha$ type $F$-contraction condition of $g$ and for $\tau > 0$, we get

$$\tau + F\big(d(\rho_n,\rho_{n+1})\big) \leq \tau + \alpha(\rho_{n-1},\rho_n)F\big(d(g\rho_{n-1}, g\rho_n)\big) \leq F\big(d(\rho_{n-1}, g\rho_n)\big),$$

that is,

$$\begin{aligned} F\big(d(\rho_n,\rho_{n+1})\big) &\leq F\big(d(\rho_{n-1},\rho_n)\big) - \tau \\ &\leq F\big(d(\rho_{n-2},\rho_{n-1})\big) - 2\tau \leq ... \leq F\big(d(\rho_0,\rho_1)\big) - n\tau. \end{aligned} \tag{23}$$

Taking limit as $n \to \infty$ in (23) and by $F2$ property of $F$, we have

$$\lim_{n\to\infty} d(\rho_n,\rho_{n+1}) = 0. \tag{24}$$

And further, by $F3$ property of $F$, $\exists \, r \in (0,1)$, implies

$$\lim_{n\to\infty} \big(d(\rho_n,\rho_{n+1})\big)^r F\big(d(\rho_n,\rho_{n+1})\big) = 0. \tag{25}$$

From (23), we deduce that

$$\big(d(\rho_n,\rho_{n+1})\big)^r \big(F(d(\rho_n,\rho_{n+1})) - F(d(\rho_0,\rho_1))\big) \leq -n\big(d(\rho_n,\rho_{n+1})\big)^r \tau. \tag{26}$$

On letting $n \to \infty$ in (26) and using (24) and (25), we obtain

$$\lim_{n\to\infty} n\big(d(\rho_n,\rho_{n+1})\big)^r = 0.$$

Thus $\exists \, n_1 \in \mathbb{N}$, such that

$$d(\rho_n,\rho_{n+1}) < \frac{1}{n^{1/r}} \quad \forall \, n \geq n_1.$$

Consider $m > n > n_1$, then by triangle inequality, we obtain

$$\begin{aligned} d(\rho_n,\rho_m) &\leq d(\rho_n,\rho_{n+1}) + d(\rho_{n+1},\rho_{n+2}) + ... + d(\rho_{m-1},\rho_m), \\ &\leq \sum_{i=1}^{\infty} d(\rho_i,\rho_{i+1}) = \sum_{i=1}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

Since the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$ is convergent, which implies $\{\rho_n\}_{n\in\mathbb{N}}$ is a Cauchy $\perp$-sequence and thus, by orthogonal completeness of $X$, $\exists \, \rho \in X$ for which

$$\lim_{n\to\infty} \rho_n = \rho.$$

Further by $\perp$-continuity of $g$, we get

$$\lim_{n\to\infty} g\rho_n = g\rho \Rightarrow \lim_{n\to\infty} \rho_{n+1} = g\rho \Rightarrow \rho = g\rho.$$

Thus $g$ possesses a fixed point.

Next, let $\nu$ be another fixed point of $g$ such that $\rho \perp \nu$. Then by given condition, we have $\alpha(\rho,\nu) \geq 1$. Using $\perp$-$\alpha$ type $F$-contraction property of $g$, we have

$$\tau + F\big(d(g\rho, g\nu)\big) \leq \tau + \alpha(\rho,\nu)F\big(d(g\rho, g\nu)\big) \leq F\big(d(\rho,\nu)\big),$$

that is,

$$\tau + F\big(d(\rho,\nu)\big) \leq F\big(d(\rho,\nu)\big). \tag{27}$$

Now, (27) holds only if $\rho = \nu$. Hence, $g$ possesses a unique fixed point. ■

**Example 8** *Let $X = \mathbb{R}$ along with usual metric space and define $\rho \perp \nu \Leftrightarrow \rho = k\nu \ \forall \ \nu \in X$ and for some fixed $k \in \mathbb{Z}$. Then $(X, \perp, d)$ is an orthogonal metric space. Define $g : X \to X$ as*

$$g(\rho) = \begin{cases} \frac{22}{25} & \text{for } \rho \in X - [-1,1]; \\ 0 & \text{otherwise.} \end{cases}$$

*Define $\alpha : X^2 \to \mathbb{R}^+$ as $\alpha(\rho,\nu) = 1 \ \forall \ \rho,\nu \in X$. Then for $\rho \perp \nu$ and $d(g\rho, g\nu) > 0$ to hold together, we must have either $\rho = 0$ and $\nu \in X - [-1,1]$* or *$\rho \in X - [-1,1]$ and $\nu = 0$.*

*Consider $\rho = 0$ and $\nu \in X - [-1,1]$ along with $F = \ln(\beta)$ and $\tau = -\ln(22/25) > 0$, we have*

$$\tau + \alpha(\rho,\nu)F\big(d(g\rho, g\nu)\big) = \tau + \ln(22/25) = 0 \tag{28}$$

*and*

$$F\big(d(\rho,\nu)\big) = \ln(|\nu|) \quad \text{where } \nu \in X - [-1,1]. \tag{29}$$

*Then from (28), (29) we can conclude that $g$ is $\perp$-$\alpha$ type $F$-contraction although, $g$ is not continuous. Also, the space $(X, \perp, d)$ is orthogonally complete metric space (since $(X, d)$ is complete metric space) and the self-map $g$ is weak $\alpha$-admissible and $\perp$-preserving. Next to check $\perp$-continuity of $g$, let $\{\rho_n\}_{n\in\mathbb{N}}$ be an $\perp$-sequence in $X$ which is convergent. Then we have $\rho_n \to 0$ as $n \to \infty$, that is, $\lim_{n\to\infty} g\rho_n = 0 = g0$. Thus $g$ is $\perp$-continuous. Since all the hypothesis of Theorem 9 hold, we see that $g$ has a fixed point viz. $\rho = 0$.*

**Theorem 10** *For an orthogonally complete metric space $(X, \perp, d)$ with $\rho_0$ as an orthogonal element, $F \in \Im$. Suppose $\alpha : X^2 \to \mathbb{R}^+$ is a map on $X^2$ and $g : X \to X$ is a self map such that*

*(I) $g$ is $\perp$-preserving;*

*(II) $g$ is weak $\alpha$-admissible map;*

*(III) $\exists \ \rho_0 \in X$ with $1 \leq \alpha(\rho_0, g\rho_0)$;*

*(IV) $g$ is $\perp$-continuous;*

*(V) $g$ is $\perp$-$\alpha$ type $F$-weak contraction.*

*Then $g$ possesses a fixed point. Moreover, if for all $\rho,\nu \in X$ with $\rho \perp \nu$, $g\rho = \rho$ and $g\nu = \nu$ implies $1 \leq \alpha(\rho,\nu)$, then $g$ possesses a unique fixed point.*

**Proof.** Working on the lines of Theorem 9, we obtain an $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $X$ with $\alpha(\rho_n, \rho_{n+1}) \geq 1 \quad \forall \, n \in \mathbb{N}$.

$$F\big(d(\rho_n, \rho_{n+1})\big) = F\big(d(g\rho_{n-1}, g\rho_n)\big) \leq \alpha(\rho_{n-1}, \rho_n)F\big(d(g\rho_{n-1}, g\rho_n)\big).$$

The proof now follows on the lines of Theorem 6. ■

**Remark 8** *In the upcoming result, we weaken the condition of $\perp$-continuity of $g$.*

**Theorem 11** *For an orthogonally complete metric space $(X, \perp, d)$ with $\rho_0$ as an orthogonal element, $F \in \Im$. Suppose $\alpha : X^2 \to \mathbb{R}^+$ is a map on $X^2$ and $g : X \to X$ is a self map such that:*

*(I)* $g$ *is* $\perp$*-preserving;*

*(II)* $g$ *is weak* $\alpha$*-admissible map;*

*(III)* $\exists\, \rho_0 \in X$ *with* $1 \leq \alpha(\rho_0, g\rho_0)$*;*

*(IV)* *if there exists an* $\perp$*-sequence* $\{\rho_n\}_{n\in\mathbb{N}}$ *with* $\alpha(\rho_n, \rho_{n+1}) \geq 1$ *and* $\rho_n \to \rho$ *as* $n \to \infty$*, then* $\alpha(\rho_n, \rho) \geq 1$ *and* either $[\rho_n \perp \rho\, \forall\, n \in \mathbb{N}]$ or $[\rho \perp \rho_n\, \forall\, n \in \mathbb{N}]$*;*

*(V)* $g$ *is* $\perp$*-*$\alpha$ *type* $F$*-contraction.*

*Then* $g$ *possesses a fixed point. Moreover, if for all* $\rho, \nu \in X$ *with* $\rho \perp \nu$*,* $g\rho = \rho$ *and* $g\nu = \nu$ *implies* $1 \leq \alpha(\rho, \nu)$*, then* $g$ *possesses a unique fixed point.*

**Proof.** Proceeding on the lines of Theorem 9, one can obtain $\{\rho_n\}_{n\in\mathbb{N}}$ an $\perp$-sequence where $\rho_n \to \rho$ as $n \to \infty$ and $\alpha(\rho_n, \rho_{n+1}) \geq 1$. Then by given condition we have, $\alpha(\rho_n, \rho) \geq 1$ and

$$[\rho_n \perp \rho\, \forall\, n \in \mathbb{N}] \quad \text{or} \quad [\rho \perp \rho_n\, \forall\, n \in \mathbb{N}].$$

Using $\perp$-preserving property of $g$, we get

$$[g\rho_n \perp g\rho\, \forall\, n \in \mathbb{N}] \quad \text{or} \quad [g\rho \perp g\rho_n\, \forall\, n \in \mathbb{N}].$$

Since $g$ is an $\perp$-$\alpha$ type $F$-contraction, we have

$$\begin{aligned} F\big(d(\rho_{n+1}, g\rho)\big) \leq \tau + F\big(d(\rho_{n+1}, g\rho)\big) &= \tau + F\big(d(g\rho_n, g\rho)\big) \\ &\leq \tau + \alpha(\rho_n, \rho) F\big(d(g\rho_n, g\rho)\big) \leq F\big(d(\rho_n, \rho)\big). \end{aligned} \tag{30}$$

Using $F1$ property of $F$ in (30), we obtain

$$d(\rho_{n+1}, gp) \leq d(\rho_n, \rho).$$

Taking limit as $n \to \infty$, we obtain

$$d(\rho, g\rho) = 0.$$

Thus $g$ possesses a fixed point. Further, the uniqueness of the fixed point of $g$ follows on the line of Theorem 9. ■

**Theorem 12** *For an orthogonally complete metric space* $(X, \perp, d)$ *with* $\rho_0$ *as an orthogonal element,* $F \in \Im$*. Suppose* $\alpha : X^2 \to \mathbb{R}^+$ *is a map on* $X^2$ *and* $g : X \to X$ *is a self map such that:*

*(I)* $g$ *is* $\perp$*-preserving;*

*(II)* $g$ *is weak* $\alpha$*-admissible map;*

*(III)* $\exists\, \rho_0 \in X$ *with* $1 \leq \alpha(\rho_0, g\rho_0)$*;*

*(IV)* *if there exists an* $\perp$*-sequence* $\{\rho_n\}_{n\in\mathbb{N}}$ *with* $\alpha(\rho_n, \rho_{n+1}) \geq 1$ *and* $\rho_n \to \rho$ *as* $n \to \infty$*, then* $\alpha(\rho_n, \rho) \geq 1$ *and* either $[\rho_n \perp \rho\, \forall\, n \in \mathbb{N}]$ or $[\rho \perp \rho_n\, \forall\, n \in \mathbb{N}]$*;*

*(V)* $g$ *is* $\perp$*-*$\alpha$ *type* $F$*-weak contraction.*

*Then* $g$ *possesses a fixed point. Moreover, if for all* $\rho, \nu \in X$ *with* $\rho \perp \nu$*,* $g\rho = \rho$ *and* $g\nu = \nu$ *implies* $1 \leq \alpha(\rho, \nu)$*, then* $g$ *possesses a unique fixed point.*

**Proof.** The proof follows from the working of Theorem 11 followed by working done in Theorem 8. ■

**Remark 9** *It should be noted that Theorem 9, Theorem 10, Theorem 11 and Theorem 12 proved above are valid even if* $g$ *is considered as an* $\alpha$*-admissible map.*

### 3.3 Orthogonal *TAC*-Contraction

$TAC$-type contractive map was introduced by S. Chandok et al. in [7]. Inspired by work done, here in this subsection we put forward the notion of orthogonal $TAC$-type S-contraction map, orthogonal weak $TAC$-type S-rational contraction, orthogonal $TAC$-contraction map and orthogonal weak $TAC$-rational contraction that further extends our approach towards contraction principles and fixed point results in orthogonal metric space.

Let $\Psi$ denotes the set of maps $\psi : \mathbb{R}^+ \to [0, \infty)$ which are continuous and monotonically increasing with $\psi^{-1}(\{0\}) = 0$ and let $\Phi$ denotes the set of maps $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ which are continuous where $\lim_{n\to\infty} \phi(\rho_n) = 0 \Rightarrow \lim_{n\to\infty} \rho_n = 0$.

**Definition 22** *For an orthogonal metric space $(X, \perp, d)$, a self map $g : X \to X$ is said to be an orthogonal* TAC*-type S-contraction (denoted by $\perp$-$TAC$-type S-contraction) if for $\rho, \nu \in X$ with $\rho \perp \nu$, $s \geq 1$ and $\hat{\alpha}(\rho).\beta(\nu) \geq s$ implies*

$$\psi\big(d(g\rho, g\nu)\big) \leq f\big(\psi(d(\rho, \nu)), \phi(d(\rho, \nu))\big),$$

*where $\hat{\alpha}, \beta : X \to [0, \infty)$, $f \in \mathcal{C}$, $\psi \in \Psi$ and $\phi \in \Phi$.*

**Example 9** *Let $X = \mathbb{R}$, $d(\rho, \nu) = |\rho - \nu|$ and $\rho \perp \nu \Leftrightarrow \rho.\nu = 0$. Then one can easily verify that $(X, \perp, d)$ is an orthogonal metric space. Define $g : X \to X$ as*

$$g(\rho) = \begin{cases} -\frac{\rho}{7} & \rho \in [0, \infty); \\ 0 & otherwise. \end{cases}$$

*Let $\hat{\alpha}, \beta : X \to [0, \infty)$ be defined as*

$$\hat{\alpha}(\rho) = \begin{cases} 2 & \rho \in [0, \infty); \\ 0 & otherwise; \end{cases}$$

*and*

$$\beta(\rho) = \begin{cases} 2 & \rho \in (-\infty, 0]; \\ 0 & otherwise. \end{cases}$$

*Also, $f : [0, \infty)^2 \to \mathbb{R}$ be defined as $f(\rho, \nu) = \rho - \nu$ and $\psi, \phi : [0, \infty) \to [0, \infty)$ as $\psi(\rho) = \frac{3\rho}{2}$ and $\phi(\rho) = \frac{3\rho}{4}$. Now, for $\rho \perp \nu$ and $\hat{\alpha}(\rho)\beta(\nu) \geq s = 2$ to hold simultaneously, we must have either $\rho = 0$, $\nu \in (-\infty, 0]$* or *$\rho \in [0, \infty)$, $\nu = 0$.*

**Case (i)** *for $\rho = 0$ and $\nu \in (-\infty, 0]$, we have*

$$\psi\big(d(g0, g\nu)\big) = 0 \tag{31}$$

*and*

$$f\big(\psi(d(0, \nu)), \phi(d(0, \nu))\big) = f\big(\psi(|\nu|), \phi(|\nu|)\big) = f\big(\frac{3|\nu|}{2}, \frac{3|\nu|}{4}\big) = \frac{3|\nu|}{4}. \tag{32}$$

**Case (ii)** *for $\rho \in [0, \infty)$ and $\nu = 0$, we have*

$$\psi\big(d(g\rho, g0)\big) = \psi\big(d(g\rho, 0)\big) = \psi(|\rho|/7) = \frac{3|\rho|}{14}, \tag{33}$$

*and*

$$f\big(\psi(d(\rho, 0)), \phi(d(\rho, 0))\big) = f\big(\psi(|\rho|), \phi(|\rho|)\big) = f\big(\frac{3|\rho|}{2}, \frac{3|\rho|}{4}\big) = \frac{3|\rho|}{4}. \tag{34}$$

*From (31)–(34), we have $g$ as $\perp$-$TAC$-type S-contraction.*

**Definition 23** *For an orthogonal metric space $(X,\perp,d)$, a self map $g:X\to X$ is said to be an orthogonal weak* TAC*-type $S$-rational contraction (denoted by $\perp$-weak* TAC*-type $S$-rational contraction) if for $\rho,\nu\in X$ with $\rho\perp\nu$, $s\geq 1$ and $\hat{\alpha}(\rho).\beta(\nu)\geq s$, implies*

$$d(g\rho,g\nu)\leq f\big(M(\rho,\nu),\phi(M(\rho,\nu))\big),$$

*where $\hat{\alpha},\beta:X\to[0,\infty)$, $f\in\mathcal{C}$, $\phi\in\Phi$ and*

$$M(\rho,\nu)=\max\left\{d(\rho,\nu),\frac{\big(1+d(\rho,g\rho)\big)d(\nu,g\nu)}{1+d(\rho,\nu)}\right\}.$$

**Definition 24** *For an orthogonal metric space $(X,\perp,d)$, a self map $g:X\to X$ is said to be an orthogonal* TAC*-contraction (denoted by $\perp$-TAC-contraction) if for $\rho,\nu\in X$ with $\rho\perp\nu$ and $\hat{\alpha}(\rho).\beta(\nu)\geq 1$ implies*

$$\psi\big(d(g\rho,g\nu)\big)\leq f\big(\psi(d(\rho,\nu)),\phi(d(\rho,\nu))\big),$$

*where $\hat{\alpha},\beta:X\to[0,\infty)$, $f\in\mathcal{C}$, $\psi\in\Psi$ and $\phi\in\Phi$.*

**Example 10** *Let $X=\mathbb{R}^+$, $d(\rho,\nu)=|\rho-\nu|$ and $\rho\perp\nu\Leftrightarrow\rho.\nu\in\{\frac{\rho}{2},\frac{\nu}{2}\}$. Then one can easily verify that $(X,\perp,d)$ is an orthogonal metric space.*
*Define $g:X\to X$ as*

$$g(\rho)=\begin{cases}\frac{\rho}{3} & \rho\in[0,2];\\ \frac{5}{7} & otherwise.\end{cases}$$

*Let $\hat{\alpha},\beta:X\to[0,\infty)$ be defined as*

$$\hat{\alpha}(\rho)=\begin{cases}1 & \rho\in[0,2];\\ 0 & otherwise;\end{cases}$$

*and*

$$\beta(\rho)=\begin{cases}2 & \rho\in[0,2];\\ 0 & otherwise.\end{cases}$$

*Also, let $f:[0,\infty)^2\to\mathbb{R}$ be defined as $f(\rho,\nu)=\rho-\nu$ and $\psi,\phi:[0,\infty)\to[0,\infty)$ as $\psi(\rho)=\rho$ and $\phi(\rho)=\rho/3$. Now, for $\rho\perp\nu$ and $\hat{\alpha}(\rho)\beta(\nu)\geq 1$ to hold simultaneously, we must have either $\rho=0$ and $\nu\in[0,2]$* or *$\rho\in[0,2]$ and $\nu=0$. Considering $\rho\in[0,2]$ and $\nu=0$, we have*

$$\psi\big(d(g\rho,g0)\big)=d(g\rho,0)=\rho/3,\tag{35}$$

*and*

$$f\big(\psi(d(\rho,0)),\phi(d(\rho,0))\big)=f\big(\psi(\rho),\phi(\rho)\big)=\psi(\rho)-\phi(\rho)=\rho-\rho/3=\frac{2}{3}\rho.\tag{36}$$

*From (35) and (36), we have $g$ as $\perp$-$TAC$-contraction which is clearly not continuous.*

**Definition 25** *For an orthogonal metric space $(X,\perp,d)$, a self map $g:X\to X$ is claimed to be an orthogonal weak* TAC*-rational contraction (denoted by $\perp$-weak* TAC*-rational contraction) if for $\rho,\nu\in X$ with $\rho\perp\nu$ and $\hat{\alpha}(\rho).\beta(\nu)\geq 1$, implies*

$$d(g\rho,g\nu)\leq f\big(M(\rho,\nu),\phi(M(\rho,\nu))\big),$$

*where $\hat{\alpha},\beta:X\to[0,\infty)$, $f\in\mathcal{C}$, $\phi\in\Phi$ and*

$$M(\rho,\nu)=\max\left\{d(\rho,\nu),\frac{\big(1+d(\rho,g\rho)\big)d(\nu,g\nu)}{1+d(\rho,\nu)}\right\}.$$

**Theorem 13** *For an orthogonally complete metric space $(X,\perp,d)$ with $s\geq 1$ and $\rho_0$ as an orthogonal element, suppose $\hat{\alpha},\beta:X\to[0,\infty)$ are defined on $X$ and $g:X\to X$ is a cyclic $(\hat{\alpha},\beta)$-admissible map type $S$ such that following holds:*

*(I) $g$ is $\perp$-preserving;*

*(II) $\exists$ some $\rho_0$ in $X$ with $\hat{\alpha}(\rho_0) \geq s$ and $\beta(\rho_0) \geq s$;*

*(III) $g$ is $\perp$-continuous;*

*(IV) $g$ is $\perp$-$TAC$-type S-contraction.*

*Then $g$ possesses a fixed point. In addition, if $\hat{\alpha}(\rho) \geq s$ and $\beta(\nu) \geq s$ for all $\rho, \nu \in X$ where $g\rho = \rho$ and $g\nu = \nu$ with $\rho \perp \nu$, then $g$ possesses a unique fixed point.*

**Proof.** Since $(X, \perp, d)$ is an $\perp$-set,then for $\rho_0, g\rho_0 \in X$, we have

$$[\rho_0 \perp g\rho_0] \quad \text{or} \quad [g\rho_0 \perp \rho_0]. \tag{37}$$

Define a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $X$ where $\rho_{n+1} = g\rho_n = g^{n+1}(\rho_0)\ \forall\ n \in \mathbb{N}$. Using $\perp$-preserving property of $g$ in (37), we obtain that $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\perp$-sequence in $X$.

Next, by given condition, $\hat{\alpha}(\rho_0) \geq s$ and as $g$ is cyclic $(\hat{\alpha}, \beta)$-admissible map type S, we have $\beta(\rho_1) = \beta(g\rho_0) \geq s$. Continuing in similar way, we get $\hat{\alpha}(\rho_{n-1}) \geq s$ and $\beta(\rho_n) \geq s$ for each $n \in \mathbb{N} \cup \{0\}$. Then $\hat{\alpha}(\rho_{n-1})\beta(\rho_n) \geq s$. Let us denote $\zeta_n = d(\rho_n, \rho_{n+1})$. Using $\perp$-$TAC$-type S-contraction of $g$, we have

$$\psi(\zeta_n) = \psi\big(d(\rho_n, \rho_{n+1})\big) = \psi\big(d(g\rho_{n-1}, g\rho_n)\big) \leq f\big(\psi(\zeta_{n-1}), \phi(\zeta_{n-1})\big) \leq \psi(\zeta_{n-1}). \tag{38}$$

Since $\psi$ is monotonically increasing function, we see that

$$\zeta_n \leq \zeta_{n-1} \quad \forall\, n \in \mathbb{N},$$

thus, $\{\zeta_n\}_{n\in\mathbb{N}}$ is decreasing and as each $\zeta_n \in \mathbb{R}^+$, so $\exists$ some $\zeta \in \mathbb{R}^+$, such that

$$\lim_{n\to\infty} \zeta_n = \zeta.$$

On taking limit as $n \to \infty$ in (38), we obtain

$$\psi(\zeta) \leq f\big(\psi(\zeta), \phi(\zeta)\big) \leq \psi(\zeta),$$

that is, $f(\psi(\zeta), \phi(\zeta)) = \psi(\zeta)$. By using definition of $f$, we obtain either $\psi(\zeta) = 0$ or $\phi(\zeta) = 0$. From either of the cases we have $\zeta = 0$, that is,

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(\rho_n, \rho_{n+1}) = 0.$$

So, for some $l = \epsilon/m > 0\ \exists\ n_l \in \mathbb{N}$ such that

$$d(\rho_n, \rho_{n+1}) < l \quad \forall\, n > n_l. \tag{39}$$

Let $n, m \in \mathbb{N}$ where $n > n_l$. Using triangle inequality and (39), we get

$$\begin{aligned} d(\rho_n, \rho_{n+m}) &\leq d(\rho_n, \rho_{n+1}) + d(\rho_{n+1}, \rho_{n+2}) + \ldots + d(\rho_{n+m-1}, \rho_{n+m}) \\ &< ml = \epsilon. \end{aligned}$$

Thus we have, $\{\rho_n\}_{n\in\mathbb{N}}$ as a Cauchy $\perp$-sequence in $X$. By orthogonal completeness of $X$, $\exists\, \rho \in X$ such that

$$\lim_{n\to\infty} \rho_n = \rho.$$

As $g$ is an $\perp$-continuous map, so

$$\lim_{n\to\infty} g\rho_n = g\rho \Rightarrow \lim_{n\to\infty} \rho_{n+1} = g\rho \Rightarrow \rho = g\rho.$$

Hence, $g$ possesses a fixed point.

Next, for uniqueness, let $\nu$ be another fixed point of $g$ such that $\rho \perp \nu$ then by given condition $\hat{\alpha}(\rho)\beta(\nu) \geq s$. Using $\perp$-$TAC$-type S-contraction condition of $g$, we obtain

$$\psi\big(d(\rho,\nu)\big) = \psi\big(d(g\rho,g\nu)\big) \leq f\big(\psi(d(\rho,\nu)),\phi(d(\rho,\nu))\big) \leq \psi\big(d(\rho,\nu)\big).$$

Then

$$f\big(\psi(d(\rho,\nu)),\phi(d(\rho,\nu))\big) = \psi(d(\rho,\nu)).$$

On using definition of $\mathcal{C}$-class function $f$ we obtain, either $\psi(d(\rho,\nu)) = 0$ or $\phi(d(\rho,\nu)) = 0$. From both the cases, we get $d(\rho,\nu) = 0$. Hence, $g$ possesses a unique fixed point. ■

**Example 11** *Example 9, satisfies all of the conditions of Theorem 13 and thus possesses a fixed point viz.* $\rho = 0$.

**Corollary 3** *For an orthogonally complete metric space* $(X,\perp,d)$ *with* $s \geq 1$ *and* $\rho_0$ *as an orthogonal element, suppose* $\hat{\alpha},\beta : X \to [0,\infty)$ *are defined on* $X$ *and* $g : X \to X$ *is a cyclic* $(\hat{\alpha},\beta)$*-admissible map type S such that following holds*

*(I)* $g$ *is* $\perp$*-preserving;*

*(II)* $\exists$ *some* $\rho_0$ *in* $X$ *with* $\hat{\alpha}(\rho_0) \geq s$ *and* $\beta(\rho_0) \geq s$*;*

*(III)* $\{\rho_n\}_{n\in\mathbb{N}}$ *is an* $\perp$*-sequence where* $\rho_n \to \rho$ *as* $n \to \infty$ *along with* $\beta(\rho_n) \geq s$ *for each* $n \in \mathbb{N}$*, implies* $\beta(\rho) \geq s$ *and* either $[\rho_n \perp \rho\, \forall\, n]$ or $[\rho \perp \rho_n\, \forall\, n]$ *;*

*(IV)* $g$ *is* $\perp$-TAC-*type S contraction.*

*Then* $g$ *possesses a fixed point. In addition, if* $\hat{\alpha}(\rho) \geq s$ *and* $\beta(\nu) \geq s$ *for all* $\rho,\nu \in X$ *where* $g\rho = \rho$ *and* $g\nu = \nu$ *with* $\rho \perp \nu$*, then* $g$ *possesses a unique fixed point.*

**Proof.** Working on the footprints of Theorem 13, we obtain $\{\rho_n\}_{n\in\mathbb{N}}$ an $\perp$-sequence in $X$ such that $\rho_n \to \rho$ as $n \to \infty$ and also, $\beta(\rho_n) \geq s$ for each $n \in \mathbb{N}$. Then, by given condition, we obtain $\beta(\rho) \geq s$ and *either* $[\rho_n \perp \rho\, \forall\, n]$ *or* $[\rho \perp \rho_n\, \forall\, n]$ . Thus $\hat{\alpha}(\rho_n)\beta(\rho) \geq s$, implies

$$\psi\big(d(\rho_{n+1},g\rho)\big) = \psi\big(d(g\rho_n,g\rho)\big) \leq f\big(\psi(d(\rho_n,\rho)),\phi(d(\rho_n,\rho))\big) \leq \psi\big(d(\rho_n,\rho)\big).$$

Taking limit as $n \to \infty$ and using continuity of $f$, $\psi$ and $\phi$, we have

$$d(\rho,g\rho) = 0.$$

Thus $g$ possesses a fixed point in $X$. Also, Theorem 13 can be used to prove the uniqueness of fixed point. ■

**Theorem 14** *For an orthogonally complete metric space* $(X,\perp,d)$ *with* $s \geq 1$ *and* $\rho_0$ *as an orthogonal element, suppose* $\hat{\alpha},\beta : X \to [0,\infty)$ *are defined on* $X$ *and* $g : X \to X$ *is a cyclic* $(\hat{\alpha},\beta)$*-admissible map of type S such that following holds:*

*(I)* $g$ *is* $\perp$*-preserving;*

*(II)* $\exists$ *some* $\rho_0$ *in* $X$ *with* $\hat{\alpha}(\rho_0) \geq s$ *and* $\beta(\rho_0) \geq s$*;*

*(III)* $g$ *is* $\perp$*-continuous;*

*(IV)* $g$ *is* $\perp$-weak TAC-*type S-rational contraction.*

*Then* $g$ *possesses a fixed point. In addition, if* $\hat{\alpha}(\rho) \geq s$ *and* $\beta(\nu) \geq s$ *for all* $\rho,\nu \in X$ *where* $g\rho = \rho$ *and* $g\nu = \nu$ *with* $\rho \perp \nu$*, then* $g$ *possesses a unique fixed point.*

**Proof.** Working on the lines of Theorem 13, one can obtain an $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ with $\hat{\alpha}(\rho_{n-1})\beta(\rho_n) \geq s$ for every $n \in \mathbb{N}$. By using $\perp$-*weak* $TAC$-type S-rational contraction of $g$, we get

$$\zeta_n = d(\rho_n, \rho_{n+1}) = d(g\rho_{n-1}, g\rho_n) \leq f\big(M(\rho_{n-1}, \rho_n), \phi(M(\rho_{n-1}, \rho_n))\big) \leq M(\rho_{n-1}, \rho_n), \tag{40}$$

where

$$\begin{aligned} M(\rho_{n-1}, \rho_n) &= \max\left\{d(\rho_{n-1}, \rho_n), \frac{\big(1 + d(\rho_{n-1}, \rho_n)\big)d(\rho_n, \rho_{n+1})}{1 + d(\rho_{n-1}, \rho_n)}\right\} \\ &= \max\{\zeta_{n-1}, \zeta_n\}. \end{aligned}$$

Suppose for some $n_0 \in \mathbb{N}$, we have

$$M(\rho_{n_0-1}, \rho_{n_0}) = \zeta_{n_0},$$

that is,

$$\zeta_{n_0} > \zeta_{n_0-1}. \tag{41}$$

Then by (40), we have

$$\zeta_{n_0} \leq f\big(\zeta_{n_0}, \phi(\zeta_{n_0})\big) \leq \zeta_{n_0}.$$

Then

$$f\big(\zeta_{n_0}, \phi(\zeta_{n_0})\big) = \zeta_{n_0}.$$

By using definition of function $f$, we have either $\zeta_{n_0} = 0$ or $\phi(\zeta_{n_0}) = 0$. From either of the cases, we get $\zeta_{n_0} = 0$ which is a contradiction to (41). Thus for all $n \in \mathbb{N}$, we have

$$\zeta_n \leq \zeta_{n-1},$$

thus, $\{\zeta_n\}_{n\in\mathbb{N}}$ is decreasing and as each $\zeta_n \in \mathbb{R}^+$, so $\exists$ some $\zeta \in \mathbb{R}^+$, such that

$$\lim_{n\to\infty} \zeta_n = \zeta.$$

On taking limit as $n \to \infty$ in (40), we get

$$\zeta \leq f\big(\zeta, \phi(\zeta)\big) \leq \zeta.$$

Then

$$f\big(\zeta, \phi(\zeta)\big) = \zeta \Rightarrow \text{either } \zeta = 0 \text{ or } \phi(\zeta) = 0,$$

which gives, $\zeta = 0$, that is,

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(\rho_n, \rho_{n+1}) = 0.$$

Now, for some $l = \epsilon/m > 0$, $\exists\, n_l \in \mathbb{N}$ such that,

$$d(\rho_n, \rho_{n+1}) < l \quad \forall\, n > n_l. \tag{42}$$

Let $n, m \in \mathbb{N}$ where $n > n_l$. Using triangle inequality and (42), we get

$$\begin{aligned} d(\rho_n, \rho_{n+m}) &\leq d(\rho_n, \rho_{n+1}) + d(\rho_{n+1}, \rho_{n+2}) + \dots + d(\rho_{n+m-1}, \rho_{n+m}) \\ &< ml = \epsilon. \end{aligned}$$

Thus we have, $\{\rho_n\}_{n\in\mathbb{N}}$ as Cauchy $\perp$-sequence in $X$. By orthogonal completeness of $X$, $\exists\, \rho \in X$, such that

$$\lim_{n\to\infty} \rho_n = \rho.$$

As $g$ is an $\perp$-continuous map, so

$$\lim_{n\to\infty} g\rho_n = g\rho \Rightarrow \lim_{n\to\infty} \rho_{n+1} = g\rho \Rightarrow \rho = g\rho.$$

Hence, $g$ possesses a fixed point.

For uniqueness, let $\nu$ be another fixed point of $g$ such that $\rho \perp \nu$ then by given condition $\hat{\alpha}(\rho)\beta(\nu) \geq s$. Using $\perp$-*weak TAC*-type S-rational contraction of $g$, we obtain

$$d(\rho,\nu) = d(g\rho, g\nu) \leq f\big(M(\rho,\nu), \phi(M(\rho,\nu))\big), \tag{43}$$

where,

$$\begin{aligned} M(\rho,\nu) &= \max\left\{d(\rho,\nu), \frac{\big(1+d(\rho,g\rho)\big)d(\nu,g\nu)}{1+d(\rho,\nu)}\right\} \\ &= d(\rho,\nu). \end{aligned}$$

Thus from (43), we get

$$d(\rho,\nu) \leq f\big(d(\rho,\nu), \phi(d(\rho,\nu))\big) \leq d(\rho,\nu),$$

which implies $d(\rho,\nu) = 0$. Hence $g$ possesses a unique fixed point. ■

**Corollary 4** *For an orthogonally complete metric space $(X, \perp, d)$ with $s \geq 1$ and $\rho_0$ as an orthogonal element, suppose $\hat{\alpha}, \beta : X \to [0,\infty)$ are defined on $X$ and $g : X \to X$ is a cyclic $(\hat{\alpha}, \beta)$-admissible map of type S such that following holds*

*(I) $g$ is $\perp$-preserving;*

*(II) $\exists$ some $\rho_0$ in $X$ with $\hat{\alpha}(\rho_0) \geq s$ and $\beta(\rho_0) \geq s$;*

*(III) $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\perp$-sequence where $\rho_n \to \rho$ as $n \to \infty$ along with $\beta(\rho_n) \geq s$ for each $n \in \mathbb{N}$, implies $\beta(\rho) \geq s$ and* either *$[\rho_n \perp \rho \; \forall \, n]$* or *$[\rho \perp \rho_n \; \forall \, n]$;*

*(IV) $g$ is $\perp$-*weak TAC*-type S-rational contraction.*

*Then $g$ possesses a fixed point. In addition, if $\hat{\alpha}(\rho) \geq s$ and $\beta(\nu) \geq s$ for all $\rho, \nu \in X$ where $g\rho = \rho$ and $g\nu = \nu$ with $\rho \perp \nu$, then $g$ possesses a unique fixed point.*

**Proof.** With reference to the working of Theorem 14, one can obtain an $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $X$ where $\rho_n \to \rho$ as $n \to \infty$ and $\beta(\rho_n) \geq s$ for each $n \in \mathbb{N}$. By given condition, $\beta(\rho) \geq s$ and *either* $[\rho_n \perp \rho \; \forall \, n]$ *or* $[\rho \perp \rho_n \; \forall \, n]$ which implies $\hat{\alpha}(\rho_n)\beta(\rho) \geq s$. On using $\perp$-*weak TAC*-type S-rational contraction of $g$, we get

$$d(\rho_{n+1}, g\rho) = d(g\rho_n, g\rho) \leq f\big(M(\rho_n,\rho), \phi(M(\rho_n,\rho))\big), \tag{44}$$

where

$$M(\rho_n,\rho) = \max\left\{d(\rho_n,\rho), \frac{\big(1+d(\rho_n, g(\rho_n))\big)d(\rho, g(\rho))}{1+d(\rho_n,\rho)}\right\}.$$

Taking limit as $n \to \infty$ in (44), we obtain

$$d(\rho, g\rho) = 0.$$

Thus $\rho$ is a fixed point of $g$ and the uniqueness of the fixed point follows on the lines of Theorem 14. ■

**Remark 10** *In the upcoming results, we consider $g$ to be a cyclic $(\hat{\alpha}, \beta)$-admissible map.*

**Theorem 15** *For an orthogonally complete metric space $(X, \perp, d)$ with $\rho_0$ as an orthogonal element, suppose $\hat{\alpha}, \beta : X \to [0,\infty)$ are defined on $X$ and $g : X \to X$ is a cyclic $(\hat{\alpha}, \beta)$-admissible map on $X$ such that following holds*

*(I) $g$ is $\perp$-preserving;*

*(II) $\exists$ some $\rho_0$ in $X$ with $\alpha(\rho_0) \geq 1$ and $\beta(\rho_0) \geq 1$;*

*(III) $g$ is $\perp$-continuous;*

*(IV) $g$ is $\perp$-$TAC$-contraction.*

*Then $g$ possesses a fixed point. In addition, if $\hat{\alpha}(\rho) \geq 1$ and $\beta(\nu) \geq 1$ for all $\rho, \nu \in X$ where $g\rho = \rho$ and $g\nu = \nu$ with $\rho \perp \nu$, then $g$ possesses a unique fixed point.*

**Proof.** Since $(X, \perp, d)$ is an $\perp$-set,then for $\rho_0, g\rho_0 \in X$, we have

$$[\rho_0 \perp g\rho_0] \quad or \quad [g\rho_0 \perp \rho_0]. \tag{45}$$

Define a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $X$ where $\rho_{n+1} = g\rho_n = g^{n+1}(\rho_0)\ \forall\ n \in \mathbb{N}$. Using $\perp$-preserving property of $g$ in (45), we obtain that $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\perp$-sequence in $X$.

Next, by given condition, $\hat{\alpha}(\rho_0) \geq 1$ then by cyclic $(\hat{\alpha}, \beta)$-admissibility of $g$, we have $\beta(\rho_1) = \beta(g\rho_0) \geq 1$. Continue using cyclic $(\hat{\alpha}, \beta)$-admissibility of $g$, we get $\hat{\alpha}(\rho_{n-1}) \geq 1$ and $\beta(\rho_n) \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$. Then $\hat{\alpha}(\rho_{n-1})\beta(\rho_n) \geq 1$. Let us denote $\zeta_n = d(\rho_n, \rho_{n+1})$. Using $\perp$-$TAC$-contraction of $g$, we have

$$\psi(\zeta_n) = \psi\big(d(\rho_n, \rho_{n+1})\big) = \psi\big(d(g\rho_{n-1}, g\rho_n)\big) \leq f\big(\psi(\zeta_{n-1}), \phi(\zeta_{n-1})\big) \leq \psi(\zeta_{n-1}). \tag{46}$$

Since $\psi$ is monotonically increasing function, we see that

$$\zeta_n \leq \zeta_{n-1} \quad \forall\, n \in \mathbb{N}.$$

Thus, $\{\zeta_n\}_{n\in\mathbb{N}}$ is decreasing and as each $\zeta_n \in \mathbb{R}^+$, so $\exists$ some $\zeta \in \mathbb{R}^+$, such that

$$\lim_{n\to\infty} \zeta_n = \zeta.$$

On taking limit as $n \to \infty$ in (46), we obtain

$$\psi(\zeta) \leq f\big(\psi(\zeta), \phi(\zeta)\big) \leq \psi(\zeta),$$

that is, $f(\psi(\zeta), \phi(\zeta)) = \psi(\zeta)$. By using definition of $f$, we obtain either $\psi(\zeta) = 0$ or $\phi(\zeta) = 0$. From either of the cases we have $\zeta = 0$, that is,

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(\rho_n, \rho_{n+1}) = 0.$$

So, for some $l = \epsilon/m > 0\ \exists\ n_l \in \mathbb{N}$ such that

$$d(\rho_n, \rho_{n+1}) < l \quad \forall\, n > n_l. \tag{47}$$

Let $n, m \in \mathbb{N}$ where $n > n_l$. Using triangle inequality and (47), we get

$$\begin{aligned} d(\rho_n, \rho_{n+m}) &\leq d(\rho_n, \rho_{n+1}) + d(\rho_{n+1}, \rho_{n+2}) + ... + d(\rho_{n+m-1}, \rho_{n+m}) \\ &< ml = \epsilon. \end{aligned}$$

Thus we have, $\{\rho_n\}_{n\in\mathbb{N}}$ as a Cauchy $\perp$-sequence in $X$. By orthogonal completeness of $X$, $\exists\ \rho \in X$ such that

$$\lim_{n\to\infty} \rho_n = \rho.$$

As $g$ is an $\perp$-continuous map, so

$$\lim_{n\to\infty} g\rho_n = g\rho \Rightarrow \lim_{n\to\infty} \rho_{n+1} = g\rho \Rightarrow \rho = g\rho.$$

Hence, $g$ possesses a fixed point.

Next, for uniqueness, let $\nu$ be another fixed point of $g$ such that $\rho \perp \nu$ then by given condition $\hat{\alpha}(\rho)\beta(\nu) \geq 1$. Using $\perp$-$TAC$-contraction of $g$, we obtain

$$\psi\big(d(\rho,\nu)\big) = \psi\big(d(g\rho, g\nu)\big) \leq f\big(\psi(d(\rho,\nu)), \phi(d(\rho,\nu))\big) \leq \psi\big(d(\rho,\nu)\big).$$

Then

$$f\big(\psi(d(\rho,\nu)), \phi(d(\rho,\nu))\big) = \psi(d(\rho,\nu)).$$

On using definition of $\mathcal{C}$-class function $f$ we obtain, either $\psi(d(\rho,\nu)) = 0$ or $\phi(d(\rho,\nu)) = 0$. From both the cases, we get $d(\rho,\nu) = 0$. Hence, $g$ possesses a unique fixed point. ■

**Example 12** *Consider the space defined in Example 10. Then the orthogonal completeness of $(X, \perp, d)$ is well evident and also such $g$ is $\perp$-preserving. Next we have:*

*(i) Cyclic $(\hat{\alpha}, \beta)$-admissibility of $g$: Since for $\rho \in [0,2]$ we get $\hat{\alpha}(\rho) \geq 1$ implies $\beta(g\rho) = \beta(\rho/3) \geq 1$, similarly, for $\rho \in [0,2]$ we get $\beta(\rho) \geq 1$ implies $\hat{\alpha}(g\rho) = \hat{\alpha}(\rho/3) \geq 1$.*

*(ii) $\perp$-continuity of $g$: Since for $\{\rho_n\}_{n\in\mathbb{N}}$ an $\perp$-sequence in $X$, we see that $\rho_n \to 0$. So we have, $\{g\rho_n\} \to 0 = g0$.*

*Since all condition of Theorem 15 holds, $g$ possesses a fixed point which is $\rho = 0$.*

**Remark 11** *The above theorem holds good even if instead of taking $g$ as an $\perp$-continuous map we consider a weaker condition as discussed in the following result.*

**Corollary 5** *For an orthogonally complete metric space $(X, \perp, d)$ with $\rho_0$ as an orthogonal element, suppose $\hat{\alpha}, \beta : X \to [0, \infty)$ are defined on $X$ and $g : X \to X$ is a cyclic $(\hat{\alpha}, \beta)$-admissible map on $X$ such that following holds:*

*(I) $g$ is $\perp$-preserving;*

*(II) $\exists$ some $\rho_0$ in $X$ with $\hat{\alpha}(\rho_0) \geq 1$ and $\beta(\rho_0) \geq 1$;*

*(III) $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\perp$-sequence where $\rho_n \to \rho$ as $n \to \infty$ along with $\beta(\rho_n) \geq 1$ for each $n \in \mathbb{N}$, implies $\beta(\rho) \geq 1$ and* either *$[\rho_n \perp \rho \; \forall \, n]$* or *$[\rho \perp \rho_n \; \forall \, n]$;*

*(IV) $g$ is $\perp$-*TAC*-contraction.*

*Then $g$ possesses a fixed point. In addition, if $\hat{\alpha}(\rho) \geq 1$ and $\beta(\nu) \geq 1$ for all $\rho, \nu \in X$ where $g\rho = \rho$ and $g\nu = \nu$ with $\rho \perp \nu$, then $g$ possesses a unique fixed point.*

**Proof.** Working on the footprints of Theorem 15, we obtain $\{\rho_n\}_{n\in\mathbb{N}}$ an $\perp$-sequence in $X$ such that $\rho_n \to \rho$ as $n \to \infty$ and also, $\beta(\rho_n) \geq 1$ for each $n \in \mathbb{N}$. Then, by given condition, we obtain $\beta(\rho) \geq 1$ and *either* $[\rho_n \perp \rho \; \forall \, n]$ *or* $[\rho \perp \rho_n \; \forall \, n]$ . Thus $\hat{\alpha}(\rho_n)\beta(\rho) \geq 1$, implies

$$\psi\big(d(g\rho_n, g\rho)\big) \leq f\big(\psi(d(\rho_n, \rho)), \phi(d(\rho_n, \rho))\big) \leq \psi\big(d(\rho_n, \rho)\big).$$

Taking limit as $n \to \infty$ and using continuity of $f$, $\psi$ and $\phi$, we have

$$d(\rho, g\rho) = 0.$$

Thus $g$ possesses a fixed point in $X$. Also, Theorem 15 can be used to prove the uniqueness of fixed point. ■

**Theorem 16** *For an orthogonally complete metric space $(X, \perp, d)$ with $\rho_0$ as an orthogonal element, suppose $\hat{\alpha}, \beta : X \to [0, \infty)$ are defined on $X$ and $g : X \to X$ is a cyclic $(\hat{\alpha}, \beta)$-admissible map on $X$ such that following holds:*

*(I) g is $\perp$-preserving;*

*(II) $\exists$ some $\rho_0$ in $X$ with $\hat{\alpha}(\rho_0) \geq 1$ and $\beta(\rho_0) \geq 1$;*

*(III) g is $\perp$-continuous;*

*(IV) g is $\perp$-*weak TAC*-rational contraction.*

*Then g possesses a fixed point. In addition, if $\hat{\alpha}(\rho) \geq 1$ and $\beta(\nu) \geq 1$ for all $\rho, \nu \in X$ where $g\rho = \rho$ and $g\nu = \nu$ with $\rho \perp \nu$, then g possesses a unique fixed point.*

**Proof.** Working on the lines of Theorem 15, one can obtain an $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ with $\hat{\alpha}(\rho_{n-1})\beta(\rho_n) \geq 1$ for every $n \in \mathbb{N}$. By using $\perp$-*weak TAC*-rational contraction of $g$, we get

$$\zeta_n = d(\rho_n, \rho_{n+1}) = d(g\rho_{n-1}, g\rho_n) \leq f\big(M(\rho_{n-1}, \rho_n), \phi(M(\rho_{n-1}, \rho_n))\big) \leq M(\rho_{n-1}, \rho_n), \tag{48}$$

where

$$\begin{aligned} M(\rho_{n-1}, \rho_n) &= \max\left\{d(\rho_{n-1}, \rho_n), \frac{\big(1 + d(\rho_{n-1}, \rho_n)\big)d(\rho_n, \rho_{n+1})}{1 + d(\rho_{n-1}, \rho_n)}\right\} \\ &= \max\{\zeta_{n-1}, \zeta_n\}. \end{aligned}$$

Suppose for some $n_0 \in \mathbb{N}$, we have

$$M(\rho_{n_0-1}, \rho_{n_0}) = \zeta_{n_0},$$

that is,

$$\zeta_{n_0} > \zeta_{n_0-1}. \tag{49}$$

Then by (48), we have

$$\zeta_{n_0} \leq f\big(\zeta_{n_0}, \phi(\zeta_{n_0})\big) \leq \zeta_{n_0}.$$

Then

$$f\big(\zeta_{n_0}, \phi(\zeta_{n_0})\big) = \zeta_{n_0}.$$

By using definition of function $f$, we have either $\zeta_{n_0} = 0$ or $\phi(\zeta_{n_0}) = 0$.

From either of the cases, we get $\zeta_{n_0} = 0$ which is a contradiction to (49). Hence for all $n \in \mathbb{N}$, we have

$$\zeta_n \leq \zeta_{n-1},$$

thus, $\{\zeta_n\}_{n\in\mathbb{N}}$ is decreasing and as each $\zeta_n \in \mathbb{R}^+$, so $\exists$ some $\zeta \in \mathbb{R}^+$, such that $\lim_{n\to\infty} \zeta_n = \zeta$.

On taking limit as $n \to \infty$ in (48), we get

$$\zeta \leq f\big(\zeta, \phi(\zeta)\big) \leq \zeta \Rightarrow f\big(\zeta, \phi(\zeta)\big) = \zeta \Rightarrow \text{either } \zeta = 0 \text{ or } \phi(\zeta) = 0$$

which gives, $\zeta = 0$, that is,

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} d(\rho_n, \rho_{n+1}) = 0.$$

Now, for some $l = \epsilon/m > 0$, $\exists\, n_l \in \mathbb{N}$ such that,

$$d(\rho_n, \rho_{n+1}) < l \quad \forall\, n > n_l. \tag{50}$$

Let $n, m \in \mathbb{N}$ where $n > n_l$. Using triangle inequality and (50), we get

$$\begin{aligned} d(\rho_n, \rho_{n+m}) &\leq d(\rho_n, \rho_{n+1}) + d(\rho_{n+1}, \rho_{n+2}) + \ldots + d(\rho_{n+m-1}, \rho_{n+m}) \\ &< ml = \epsilon. \end{aligned}$$

Thus we have, $\{\rho_n\}_{n\in\mathbb{N}}$ as Cauchy $\perp$-sequence in $X$. By orthogonal completeness of $X$, $\exists\, \rho \in X$, such that

$$\lim_{n\to\infty} \rho_n = \rho.$$

As $g$ is an $\perp$-continuous map, so

$$\lim_{n\to\infty} g\rho_n = g\rho \Rightarrow \lim_{n\to\infty} \rho_{n+1} = g\rho \Rightarrow \rho = g\rho.$$

Hence, $g$ possesses a fixed point.

For uniqueness, let $\nu$ be another fixed point of $g$ such that $\rho \perp \nu$ then by given condition $\hat{\alpha}(\rho)\beta(\nu) \geq 1$. Using $\perp$-*weak TAC*-rational contraction of $g$, we obtain

$$d(\rho,\nu) = d(g\rho, g\nu) \leq f\big(M(\rho,\nu), \phi(M(\rho,\nu))\big), \tag{51}$$

where,

$$\begin{aligned} M(\rho,\nu) &= \max\left\{d(\rho,\nu), \frac{\big(1+d(\rho,g\rho)\big)d(\nu,g\nu)}{1+d(\rho,\nu)}\right\} \\ &= d(\rho,\nu). \end{aligned}$$

Thus from (51), we get

$$d(\rho,\nu) \leq f\big(d(\rho,\nu), \phi(d(\rho,\nu))\big) \leq d(\rho,\nu),$$

which implies $d(\rho,\nu) = 0$. Hence $g$ possesses a unique fixed point. ■

**Remark 12** *The above theorem also holds good if we drop $\perp$-continuity of $g$ and instead consider a weaker condition as discussed in the following corollary.*

**Corollary 6** *For an orthogonally complete metric space $(X,\perp,d)$ with $\rho_0$ as an orthogonal element, suppose $\hat{\alpha},\beta : X \to [0,\infty)$ are defined on $X$ and $g : X \to X$ is a cyclic $(\hat{\alpha},\beta)$-admissible map on $X$ such that following holds*

- *(I) $g$ is $\perp$-preserving;*
- *(II) $\exists$ some $\rho_0$ in $X$ with $\hat{\alpha}(\rho_0) \geq 1$ and $\beta(\rho_0) \geq 1$;*
- *(III) $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\perp$-sequence where $\rho_n \to \rho$ as $n \to \infty$ along with $\beta(\rho_n) \geq 1$ for each $n \in \mathbb{N}$, implies $\beta(\rho) \geq 1$ and* either $[\rho_n \perp \rho \;\forall\, n]$ or $[\rho \perp \rho_n \;\forall\, n]$*;*
- *(IV) $g$ is* $\perp$-weak TAC-*rational contraction.*

*Then $g$ possesses a fixed point. In addition, if $\hat{\alpha}(\rho) \geq 1$ and $\beta(\nu) \geq 1$ for all $\rho,\nu \in X$ where $g\rho = \rho$ and $g\nu = \nu$ with $\rho \perp \nu$, then $g$ possesses a unique fixed point.*

**Proof.** With reference to working of Theorem 16, one can obtain an $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $X$ where $\rho_n \to \rho$ as $n \to \infty$ and $\beta(\rho_n) \geq 1$ for each $n \in \mathbb{N}$. By given condition, $\beta(\rho) \geq 1$ and *either* $[\rho_n \perp \rho \;\forall\, n]$ *or* $[\rho \perp \rho_n \;\forall\, n]$ which implies $\hat{\alpha}(\rho_n)\beta(\rho) \geq 1$. On using $\perp$-*weak TAC*-rational contraction of $g$, we get

$$d(\rho_{n+1}, g\rho) = d(g\rho_n, g\rho) \leq f\big(M(\rho_n,\rho), \phi(M(\rho_n,\rho))\big), \tag{52}$$

where

$$M(\rho_n,\rho) = \max\left\{d(\rho_n,\rho), \frac{\big(1+d(\rho_n, g(\rho_n))\big)d(\rho, g(\rho))}{1+d(\rho_n,\rho)}\right\}.$$

Taking limit as $n \to \infty$ in (52), we obtain

$$d(\rho, g\rho) = 0.$$

Thus $\rho$ is a fixed point of $g$ and the uniqueness of the fixed point follows on the lines of Theorem 16. ■

**Example 13** *Consider $X = [0, \infty)$ with usual metric space and let $\rho \perp \nu \Leftrightarrow \rho \leq \nu \; \forall \; \nu \in X$. Then $(X, \perp, d)$ is an orthogonally complete metric space. Define $g : X \to X$ as*

$$g(\rho) = \begin{cases} \frac{17\rho}{19} & \rho \in [0, 1/2); \\ 15\rho^2 & \text{otherwise.} \end{cases}$$

*Clearly, here $g$ is $\perp$-preserving and $\perp$-continuous function but not a continuous function. Define $\hat{\alpha}, \beta : X \to \mathbb{R}^+$ as*

$$\hat{\alpha}(\rho) = \begin{cases} \frac{3}{2} & \rho \in [0, 1/2); \\ 0 & \text{otherwise;} \end{cases}$$

*and,*

$$\beta(\rho) = \begin{cases} \frac{5}{4} & \rho \in [0, 1/2); \\ 0 & \text{otherwise.} \end{cases}$$

*Define $f : [0, \infty)^2 \to \mathbb{R}$ as $f(\rho, \nu) = \rho - \nu$ and $\phi : [0, \infty) \to \mathbb{R}^+$ as $\phi(\rho) = \rho/2$. For $\rho \in [0, 1/2)$, $\hat{\alpha}(\rho) \geq 1$ implies $\beta(g\rho) = \beta(\frac{17\rho}{19}) \geq 1$ and vice-versa, thus $g$ is a cyclic $(\hat{\alpha}, \beta)$-admissible mapping.*

*Next, for $\hat{\alpha}(\rho)\beta(\nu) \geq 1$ and $\rho \perp \nu$ to hold simultaneously, we must have either*

$$[\rho = 0 \text{ and } \nu \in [0, 1/2)] \quad \text{or} \quad [\nu = 0 \text{ and } \rho \in [0, 1/2)].$$

*Considering $\rho = 0$ and $\nu \in [0, 1/2)$, we get*

$$d(g0, g\nu) = d\left(0, \frac{17\nu}{19}\right) = \frac{17\nu}{19}, \tag{53}$$

*and,*

$$f\big(M(0, \nu), \phi(M(0, \nu))\big) = f(\nu, \nu/2) = \nu/2 \tag{54}$$

*From (53) and (54), we can conclude that $\nu$ is an $\perp$-*weak TAC*-rational contraction. Thus by Theorem 16 we conclude that $g$ possesses a fixed point viz. $\rho = 0$.*

### 3.4 Orthogonal Suzuki-Berinde type $F$-Contraction

Recently, N. Hussain and J. Ahmad gave the notion of *Suzuki-Berinde* type $F$-contraction in [14] and proved certain fixed point result, which is a generalisation of [19]. In this final subsection of manuscript, we put forward the notion of orthogonal *Suzuki-Berinde* type $F$-contraction and explore the fixed point result.

**Definition 26** *For an orthogonal metric space $(X, \perp, d)$ and for $\mathcal{F} \in \Omega_{\mathcal{F}}$, a self map $g : X \to X$ is claimed to be an orthogonal* Suzuki-Berinde *type $F$-contraction map (denoted by $\perp$-S-B type $F$-contraction) if there exist $\tau > 0$ and $L \geq 0$ such that for all $\rho, \nu \in X$ with $d(g\rho, g\nu) > 0$ and $\rho \perp \nu$, we have*

$$\frac{1}{2}d(\rho, g\rho) < d(\rho, \nu) \Rightarrow \tau + \mathcal{F}\big(d(g\rho, g\nu)\big) \leq \mathcal{F}\big(d(\rho, \nu)\big) + L.\min\big\{d(\rho, g\rho), d(\rho, g\nu), d(\nu, g\rho)\big\}.$$

**Example 14** *Let $X = [0, 7/2]$ with $d(\rho, \nu) = |\rho - \nu|$ and $\rho \perp \nu \Leftrightarrow \rho.\nu = \nu \; \forall \; \nu \in X$. Then $(X, \perp, d)$ is an orthogonal metric space (with $\rho = 1$ as an orthogonal element). Define $g : X \to X$ as*

$$g(\rho) = \begin{cases} 1 & \rho \in [0, 7/2); \\ \frac{2}{7} & \text{otherwise.} \end{cases}$$

*For $\rho \perp \nu$ and $d(g\rho, g\nu) > 0$ we must have either $\rho = 1$ and $\nu = 7/2$ or $\rho = 7/2$ and $\nu = 1$.*

*Consider $\rho = 1$ and $\nu = 7/2$. Then for $\mathcal{F}(\beta) = \ln(\beta)$ and $0 < \tau < 1$, we have*

$$\tau + \mathcal{F}\big(d(g\rho, g\nu)\big) = \tau + \ln(2/7), \tag{55}$$

*and,*

$$\mathcal{F}\big(d(\rho, \nu)\big) + L.\min\big\{d(\rho, g\rho), d(\rho, g\nu), d(\nu, g\rho)\big\} = \ln(5/2). \tag{56}$$

*From (55) and (56), we can conclude that $g$ is $\perp$-S-B type $F$-contraction.*

**Theorem 17** *For an orthogonally complete metric space $(X, \perp, d)$ with $\rho_0$ as an orthogonal element, suppose $\mathcal{F} \in \Omega_{\mathcal{F}}$ and $g : X \to X$ is a self map on $X$ satisfying following:*

*(I) $g$ is $\perp$-preserving;*

*(II) $g$ is $\perp$-$S$-$B$ type $F$-contraction;*

*(III) $g$ is $\perp$-continuous.*

*Then $g$ possesses a fixed point. Moreover, if $\rho \perp \nu$ for all $\rho, \nu \in X$ where $g\rho = \rho$ and $g\nu = \nu$, then $g$ possesses a unique fixed point.*

**Proof.** Define a sequence $\{\rho\}_{n\in\mathbb{N}}$ in $X$, where

$$\rho_{n+1} = g\rho_n = g^{n+1}\rho_0 \quad \forall\, n \in \mathbb{N}.$$

Since $\rho_0$ is an orthogonal element, we have

$$[\rho_0 \perp g\rho_0] \quad \text{or} \quad [g\rho_0 \perp \rho_0].$$

On repeated use of $\perp$- preserving property of $g$, we obtain $\{\rho\}_{n\in\mathbb{N}}$ as an $\perp$-sequence in $X$. If for some $n_0 \in \mathbb{N}$ we have $\rho_{n_0} = \rho_{n_0+1} = g\rho_{n_0}$ then we are done.

Suppose $\rho_n \neq \rho_{n+1}\ \forall\, n \in \mathbb{N}$, that is, $d(\rho_n, \rho_{n+1}) > 0$.
Since

$$\frac{1}{2}d(\rho_n, \rho_{n+1}) = \frac{1}{2}d(\rho_n, g\rho_n) < d(\rho_n, g\rho_{n+1}),$$

and since $g$ is an $\perp$-$S$-$B$ type $F$-contraction map, we see that

$$\begin{aligned} \tau + \mathcal{F}\big(d(\rho_n, \rho_{n+1})\big) &= \tau + \mathcal{F}\big(d(g\rho_{n-1}, g\rho_n)\big) \\ &\leq \mathcal{F}\big(d(\rho_{n-1}, \rho_n)\big) + L.\min\big\{d(\rho_{n-1}, \rho_n), d(\rho_{n-1}, \rho_{n+1}), d(\rho_n, \rho_n)\big\} \end{aligned}$$

Then

$$\begin{aligned} \mathcal{F}\big(d(\rho_n, \rho_{n+1})\big) &\leq \mathcal{F}\big(d(\rho_{n-1}, \rho_n)\big) - \tau \\ &\leq \mathcal{F}\big(d(\rho_{n-2}, \rho_{n-1})\big) - 2\tau \leq ... \leq \mathcal{F}\big(d(\rho_0, \rho_1)\big) - n\tau. \end{aligned} \tag{57}$$

Taking limit as $n \to \infty$ in (57) and using $(\mathcal{F}2)$ and Lemma 1, gives

$$\lim_{n\to\infty} d(\rho_n, \rho_{n+1}) = 0.$$

Thus, for some $\epsilon/m = l > 0$ there exists $n_l \in \mathbb{N}$, such that

$$d(\rho_n, \rho_{n+1}) < l \quad \forall\, n > n_l. \tag{58}$$

Let $n, m \in \mathbb{N}$ where $n > n_l$. Using triangle inequality and (58), we have

$$\begin{aligned} d(\rho_n, \rho_{n+m}) &\leq d(\rho_n, \rho_{n+1}) + d(\rho_{n+1}, \rho_{n+2}) + ... + d(\rho_{n+m-1}, \rho_{n+m}) \\ &< ml = \epsilon. \end{aligned}$$

So we conclude that $\{\rho_n\}_{n\in\mathbb{N}}$ is a Cauchy $\perp$-sequence in $X$. Therefore, by orthogonal completeness of $X$, $\exists\, \rho \in X$ where

$$\lim_{n\to\infty} \rho_n = \rho.$$

Now, since $g$ is an $\perp$-continuous map, we see that

$$\lim_{n\to\infty} g\rho_n = g\rho \Rightarrow \lim_{n\to\infty} \rho_{n+1} = g\rho \Rightarrow \rho = g\rho.$$

Thus $g$ possesses a fixed point.

Let $\nu$ be another fixed point of $g$ in $X$ then by given condition $\rho \perp \nu$. Suppose $\rho \neq \nu$, that is, $d(\rho, \nu) > 0$. Since

$$\frac{1}{2}d(\rho, \rho) = 0 = \frac{1}{2}d(\rho, g\rho) < d(\rho, \nu),$$

and by given $\perp$-$S$-$B$ type $F$-contraction of $g$, we obtain

$$\begin{aligned} \mathcal{F}\big(d(\rho, \nu)\big) = \mathcal{F}\big(d(g\rho, g\nu)\big) &< \tau + \mathcal{F}\big(d(g\rho, g\nu)\big) \\ &\leq \mathcal{F}\big(d(\rho, \nu)\big) + L.\min\big\{d(\rho, g\rho), d(\rho, g\nu), d(\nu, g\rho)\big\}. \end{aligned}$$

Then

$$\mathcal{F}\big(d(\rho, \nu)\big) < \mathcal{F}\big(d(\rho, \nu)\big),$$

which is a contradiction. Hence $g$ possesses a unique fixed point. ■

**Example 15** *Consider the orthogonal metric space and map discussed in Example 14. Then we have:*

- *(i) $(X, \perp, d)$ is an orthogonally complete metric space: For any Cauchy $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $X$, there exist a subsequence $\{\rho_{n_k}\}$ where $\rho_{n_k} = 1 \; \forall \; k \geq 1$, that is, $\{\rho_{n_k}\}$ is convergent. Thus, we have $(X, \perp, d)$ as an orthogonally complete metric space.*
- *(ii) $g$ is $\perp$-preserving: Since $1 \perp \nu \; \forall \; \nu \in X$, we see that $g(1) = 1 \perp g\nu; \forall \; \nu \in X$.*
- *(iii) $g$ is $\perp$-continuous: For any $\perp$-sequence $\{\rho_n\} \to \rho$, then $\rho = 1$. Thus $\{g\rho_n\} \to g(1) = 1$.*

*Since all the hypothesis of Theorem 17 holds, $g$ has a fixed point in $X$ viz. $\rho = 1$.*

**Corollary 7** *For an orthogonally complete metric space $(X, \perp, d)$ with $\rho_0$ as an orthogonal element, suppose $g : X \to X$ is a self map on $X$ satisfying following:*

- *(I) $g$ is $\perp$-preserving;*
- *(II) $g$ is $\perp$-$S$-$B$ type $F$-contraction;*
- *(III) If $\{\rho_n\}_{n\in\mathbb{N}}$ is an $\perp$-sequence in $X$ with $\rho_n \to \rho$ as $n \to \infty$ implies*

$$[\rho_n \perp \rho \; \forall \; n] \quad or \quad [\rho \perp \rho_n \; \forall \; n].$$

*Then $g$ possesses a fixed point. Moreover, if $\rho \perp \nu$ for all $\rho, \nu \in X$ where $g\rho = \rho$ and $g\nu = \nu$, then $g$ possesses a unique fixed point.*

**Proof.** On the footprints of Theorem 17, we obtain an $\perp$-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $X$ with $\rho_n \to \rho$ as $n \to \infty$. Thus, by given hypothesis, either $[\rho_n \perp \rho \; \forall \; n]$ or $[\rho \perp \rho_n \; \forall \; n]$. We claim that $\rho$ is a fixed point of $g$ in $X$. Suppose for some $n_0 \in \mathbb{N}$,

$$\frac{1}{2}d(\rho_{n_0}, g\rho_{n_0}) \geq d(\rho_{n_0}, \rho) \quad \text{or} \quad \frac{1}{2}d(g\rho_{n_0}, g^2\rho_{n_0}) \geq d(g\rho_{n_0}, \rho), \tag{59}$$

implies,

$$2d(\rho_{n_0}, \rho) \leq d(\rho_{n_0}, g\rho_{n_0}) \leq d(\rho_{n_0}, \rho) + d(\rho, g\rho_{n_0}),$$

that is,

$$d(\rho_{n_0}, \rho) \leq d(\rho, g\rho_{n_0}). \tag{60}$$

From (59) and (60), we get

$$d(\rho_{n_0}, \rho) \leq d(\rho, g\rho_{n_0}) \leq \frac{1}{2}d(g\rho_{n_0}, g^2\rho_{n_0}). \tag{61}$$

Since

$$\frac{1}{2}d(\rho_{n_0}, g\rho_{n_0}) < d(\rho_{n_0}, g\rho_{n_0}),$$

and by contraction condition of $g$, we have

$$\begin{aligned}\mathcal{F}\big(d(g\rho_{n_0}, g^2\rho_{n_0})\big) &< \tau + \mathcal{F}\big(d(g\rho_{n_0}, g^2\rho_{n_0})\big) \\ &\leq \mathcal{F}\big(d(\rho_{n_0}, g\rho_{n_0})\big) + L.\min\big\{d(\rho_{n_0}, g\rho_{n_0}), d(\rho_{n_0}, g^2\rho_{n_0}), d(g\rho_{n_0}, g\rho_{n_0})\big\}.\end{aligned}$$

Then

$$\mathcal{F}\big(d(g\rho_{n_0}, g^2\rho_{n_0})\big) < \tau + \mathcal{F}\big(d(g\rho_{n_0}, g^2\rho_{n_0})\big) \leq \mathcal{F}\big(d(\rho_{n_0}, g\rho_{n_0})\big).$$

By ($\mathcal{F}1$), we obtain

$$d(g\rho_{n_0}, g^2\rho_{n_0}) < d(\rho_{n_0}, g\rho_{n_0}). \tag{62}$$

Using triangle inequality and (61) in (62), we get

$$\begin{aligned}d(g\rho_{n_0}, g^2\rho_{n_0}) &< d(\rho_{n_0}, \rho) + d(\rho, g\rho_{n_0}) \\ &\leq \frac{1}{2}d(g\rho_{n_0}, g^2\rho_{n_0}) + \frac{1}{2}d(g\rho_{n_0}, g^2\rho_{n_0}) \\ &= d(g\rho_{n_0}, g^2\rho_{n_0}),\end{aligned}$$

which is a contradiction. Thus, we have

$$\frac{1}{2}d(\rho_{n_0}, g\rho_{n_0}) < d(\rho_{n_0}, \rho) \quad \text{or} \quad \frac{1}{2}d(g\rho_{n_0}, g^2\rho_{n_0}) < d(g\rho_{n_0}, \rho) \quad \forall\, n \in \mathbb{N}.$$

Since $g$ is an $\perp$-$S$-$B$ type $F$-contraction map, we obtain

$$\tau + \mathcal{F}\big(d(g\rho_n, g\rho)\big) \leq \mathcal{F}\big(d(\rho_n, \rho)\big) + L.\min\big\{d(\rho_n, g\rho_n), d(\rho_n, g\rho), d(\rho, g\rho_n)\big\}. \tag{63}$$

On taking limit as $n \to \infty$ in (63) and using ($\mathcal{F}2$) along with Lemma 1, we get

$$\lim_{n\to\infty} d(g\rho_n, g\rho) = 0 \Rightarrow d(\rho, g\rho) = 0.$$

Thus $g$ possesses a fixed point. The uniqueness of fixed point follows on the lines of Theorem 17. ■

**Conclusion:** Under some specific condition, the results proved in main section of this manuscript, reduce to many well-known fixed point results of the literature. Consider the binary relation $\rho \perp \nu \Leftrightarrow \rho, \nu \in X, \quad \forall\, \rho, \nu \in X$ then $(X, \perp, d)$ is an orthogonal metric space (for any metric $d$ on $X$) with every element in $X$ as an orthogonal element. In fact, in such case the orthogonal metric space $(X, \perp, d)$ reduces to metric space $(X, d)$.

(I) With the above condition, Theorem 1, Corollary 1, Theorem 3 and Corollary 2 of the main section reduces to Theorem 2.1, Corollary 2.1, Theorem 2.2 and Corollary 2.2 respectively of [13].

(II) Theorem 3.8 of [10] can be deduced from Theorem 10 of the main result under specific condition as mentioned above along with $g$ as an $\alpha$-admissible map.

(III) Theorem 8, Theorem 12 of [7] are particular case of Theorem 15 and Corollary 5, Theorem 16 and Corollary 6 respectively of the main section with respect to above orthogonal metric space.

(IV) Theorem 2.1 of [14] can be deduced from Corollary 7 of the main result along with specific condition as mentioned above.

**Acknowledgment.** Authors would like to express sincere gratitude to anonymous referee(s) for their careful reading and valuable suggestions that helped in improvement of the manuscript.

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