Robust Numerical Method For Singularly Perturbed Two-Parameter Differential-Difference Equations[*]

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Abstract

In this paper, singularly perturbed differential-difference equations with mixed small shift is considered. The considered problem contains delay and advance parameters on the convection and reaction terms respectively. The solution of the problem exhibits a boundary layer behavior on left or right side of the domain depending on the sign of the convection term. The terms with delay and advance are approximated using Taylor series expansion. The resulting singularly perturbed boundary value problem is solved using fitted non-polynomial spline method. The stability and uniform convergence of the schemes are proved. Numerical examples are used to validate the theoretical finding of the schemes.

1 Introduction

A differential equation is said to be singularly perturbed delay differential equation, if it includes at least one delay term, involving unknown functions occurring with different arguments, and also, the highest derivative term is multiplied by a small parameter. Such type of delay, differential equations play a very important role in the mathematical models of science and engineering, such as, the human pupil light reflex with mixed delay type [7], variational problems in control theory with small state problem [5], models of HIV infection [2], and signal transition [3].

Any system involving a feedback control almost involves time delay. The delay occurs because a finite time is required to sense the information and then react to it. Finding the solution of singularly perturbed delay differential equations, whose application mentioned above, is a challenging problem. In response to these, in recent years, there has been a growing interest in numerical methods on singularly perturbed delay differential equations. Recently, the authors of [9], [11], [4] and [6] have developed various numerical schemes on uniform meshes for singularly perturbed second order differential equations having small delay on convection term.

In most of the paper of delay and advance problems, both the delay and advance parameters are present in the reaction term but in this paper, we consider a new governing problem having small delay in convection term and an advance in reaction term. As far as the researchers' knowledge is considered numerical solution of singularly perturbed boundary value problem having this behavior via fitted non polynomial spline method is first being considered. Thus, motivated by the works of [1], the purpose of this study is to develop stable, convergent and accurate numerical method for solving singularly perturbed differential-difference equations of the problem under consideration.

2 Description of the Method

To describe the method, we consider the boundary value problems for a class of singularly perturbed differential difference equations with delay and advance parameters in the convection and the reaction terms,

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with two-point boundary value problem of the form

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x+\gamma) + c(x)y(x) = f(x), \quad x \in (0,1), \tag{1}$$

with the given interval boundary condition

$$y(x) = \phi(x), \; x \in [-\delta, 0], \; y(x) = \Phi(x), \quad x \in [1, 1+\gamma], \tag{2}$$

where ε is small parameter, $0 < \varepsilon << 1$ and δ, γ are also small shifting parameters, $0 < \delta << 1, 0 < \gamma << 1$, $a(x), b(x), c(x), \phi(x), \Phi(x)$ and $f(x)$ are sufficiently smooth functions in $(0,1)$.

By using Taylor series expansion in the neighborhood of x we have

$$\begin{cases} y'(x-\delta) = y'(x) - \delta y''(x) + O(\delta^2), \\ y(x+\gamma) = y(x) + \gamma y'(x) + \frac{\gamma^2}{2}y''(x) + O(\gamma^3). \end{cases} \tag{3}$$

Substituting (3) into (1), we obtain an asymptotically equivalent singularly perturbed two point boundary value problem

$$c_\varepsilon(x)y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \tag{4}$$

where $c_\varepsilon(x) = \varepsilon - \delta a(x) + \frac{1}{2}\gamma^2 b(x)$, $p(x) = a(x) + \gamma b(x)$, $q(x) = b(x) + c(x)$, with boundary conditions

$$y(0) = \phi(0), \quad y(1) = \Phi(0). \tag{5}$$

Consider a uniform mesh with interval $[0,1]$ in which $0 = x_0 < x_1 < ... < x_N = 1$ where $h = \frac{1}{N}$ and $x_i = ih, \quad i = 0, 1, 2, ..., N$.

For each segment $[x_i, x_{i+1}]$, $i = 1, 2, ..., N-1$ the non-polynomial cubic spline $S_\Delta(x)$ has the following form

$$S_\Delta(x) = a_i + b_i(x-x_i) + c_i(e^{w(x-x_i)} + e^{-w(x-x_i)}) + d_i(e^{w(x-x_i)} - e^{-w(x-x_i)}), \tag{6}$$

where a_i, b_i, c_i and d_i are unknown coefficients, and $w \neq 0$ arbitrary parameter which will be used to increase the accuracy of the method.

To determine the unknown coefficients in (6) in terms of y_i, y_{i+1}, M_i and M_{i+1} first we define

$$\begin{cases} S_\Delta(x_i) = y_i, & S_\Delta(x_{i+1}) = y_{i+1}, \\ S_\Delta''(x_i) = M_i, & S_\Delta''(x_{i+1}) = M_{i+1}. \end{cases} \tag{7}$$

The coefficients in (6) are determined as

$$\begin{cases} a_i = y_i - \frac{M_i}{w^2}, \\ b_i = \frac{y_{i+1}-y_i}{h} + \frac{M_i-M_{i+1}}{w\theta}, \\ c_i = \frac{M_{i+1}}{w^2(e^\theta-e^{-\theta})} - \frac{M_i(e^\theta+e^{-\theta})}{2w^2(e^\theta-e^{-\theta})}, \\ d_i = \frac{M_i}{2w^2}, \end{cases} \tag{8}$$

where $\theta = wh$. Using the continuity condition of the first derivative at x_i, $S_{\Delta-1}'(x_i) = S_\Delta'(x_i)$ we have

$$b_{i-1} + wc_{i-1}(e^\theta + e^{-\theta}) + wd_{i-1}(e^\theta - e^{-\theta}) = b_i + 2wc_i. \tag{9}$$

Reducing indices of (8) by one and substituting into (9), we obtain

$$\frac{y_i - y_{i-1}}{h} + \frac{M_i - M_{i+1}}{w\theta} + w\left(\frac{2M_i - (e^\theta + e^{-\theta})M_{i-1}}{2w^2(e^\theta + e^{-\theta})}\right)$$

$$= \frac{y_{i+1} - y_i}{h} + \frac{M_i - M_{i+1}}{w\theta} + 2w\left(\frac{M_{i+1}}{w^2(e^\theta - e^{-\theta})} - \frac{M_i(e^\theta + e^{-\theta})}{2w^2(e^\theta - e^{-\theta})}\right).$$

Then

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = \alpha M_{i-1} + 2\beta M_i + \alpha M_{i+1}, \tag{10}$$

where

$$\alpha = \frac{1}{\theta^2}\left(1 - \frac{2\theta}{e^\theta - e^{-\theta}}\right), \quad \beta = \frac{1}{\theta^2}\left(\frac{\theta(e^\theta + e^{-\theta})}{e^\theta - e^{-\theta}} - 1\right).$$

If $h \to 0$, then $\theta = hk \to 0$. Thus, using L'Hopitals rule we have $\lim\limits_{h\to 0}\alpha = \frac{1}{6}$ and $\lim\limits_{h\to 0}\beta = \frac{1}{3}$. Using $S''_\triangle(x_i) = y''_i = M_i$ in to (4), we get

$$\begin{cases} c_\varepsilon(x_i)M_i = f_i - p_iy'_i - q_iy_i, \\ c_\varepsilon(x_i)M_{i-1} = c_{i-1} - p_{i-1}y'_{i-1} - q_{i-1}y_{i-1}, \\ c_\varepsilon(x_i)M_{i+1} = f_{i+1} - p_{i+1}y'_{i+1} - q_{i+1}y_{i+1}. \end{cases} \tag{11}$$

Using Taylor series expansions of $y_{i-1}, y_{i+1}, y'_{i-1}$ and y'_{i+1} simplifying, we have:

$$\begin{cases} y'_i = \frac{y_{i+1} - y_{i-1}}{2h} + T_1, \\ y'_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h} + T_2, \\ y'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} + T_2, \end{cases} \tag{12}$$

where $T_1 = -\frac{h^2}{6}y'''(\xi)$ and $T_2 = \frac{h^2}{12}y'''(\xi)$, for $\xi \in (x_{i-1}, x_i)$. Substituting (12) in to (11), we get

$$\begin{cases} M_i = \frac{1}{c_\varepsilon(x_i)}\left\{f_i - p_i\left(\frac{y_{i+1} - y_{i-1}}{2h} + T_1\right) - q_iy_i\right\}, \\ M_{i-1} = \frac{1}{c_\varepsilon(x_i)}\left\{f_{i-1} - p_{i-1}\left(\frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h} + T_2\right) - q_{i-1}y_{i-1}\right\}, \\ M_{i+1} = \frac{1}{c_\varepsilon(x_i)}\left\{f_{i+1} - p_{i+1}\left(\frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} + T_2\right) - q_{i+1}y_{i+1}\right\}. \end{cases} \tag{13}$$

Substituting (13) into (10) and rearranging, we get

$$\begin{aligned} &\frac{c_\varepsilon(x_i)}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha p_{i-1}}{2h}(-y_{i+1} - 4y_i - 3y_{i-1}) + \frac{2\beta p_i}{2h}(y_{i+1} - y_{i-1}) \\ &+ \frac{\alpha p_{i+1}}{2h}(3y_{i+1} - 3y_i + y_{i-1}) \\ &= \alpha(f_{i-1} - q_{i-1}y_{i-1} + f_{i+1} - q_{i+1}y_{i+1}) + 2\beta(f_i - q_iy_i) + T, \end{aligned} \tag{14}$$

where, $T = (4\beta p_i - \alpha p_{i-1} - \alpha p_{i+1})\frac{h^2}{12}y'''(\xi)$ is the local truncation error.

From the theory of singular perturbations described in [8] and the Taylor series expansion of $y(x)$ about the point $'0'$ in the asymptotic solution of the problem in Eq.(4), we have

$$y(x_i) \approx y_0(x_i) + (\phi_0 - y_0(0))e^{-p(0)\frac{ih}{c_\varepsilon(x_i)}},$$

and letting $\rho = \frac{h}{c_\varepsilon(x_i)}$ we get

$$\lim_{h\to 0} y(ih) \approx y_0(ih) + (\phi_0 - y_0(0))e^{-p(0)i\rho},$$

since $x_i = x_0 + ih$. Introducing a fitting factor $\sigma(\rho)$ in to (14), we get

$$\begin{aligned} &\frac{\sigma(\rho)c_\varepsilon(x_i)}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha p_{i-1}}{2h}(-y_{i+1} - 4y_i - 3y_{i-1}) + \frac{2\beta p_i}{2h}(y_{i+1} - y_{i-1}) \\ &+ \frac{\alpha p_{i+1}}{2h}(3y_{i+1} - 3y_i + y_{i-1}) \\ &= \alpha(f_{i-1} - q_{i-1}y_{i-1} + f_{i+1} - q_{i+1}y_{i+1}) + 2\beta(f_i - b_iy_i) + T. \end{aligned} \tag{15}$$

Multiplying (15) by h and taking a limit as $h \to 0$ we get

$$\frac{\sigma(\rho)}{\rho} \lim_{h \to 0}(y_{i-1} - 2y_i + y_{i+1}) + \frac{\alpha p(0)}{h} \lim_{h \to 0}(-y_{i+1} - 4y_i - 3y_{i-1})$$

$$+\beta p(0) \lim_{h \to 0}(y_{i+1} - y_{i-1}) + \frac{\alpha p(0)}{2} \lim_{h \to 0}(3y_{i+1} - 3y_i + y_{i-1}) = 0. \tag{16}$$

Thus, we consider two cases of the boundary layers.

Case 1: For $p(x) > 0$ (Left-end boundary layer), we have

$$\begin{cases} \lim_{h \to 0}(y_{i-1} - 2y_i + y_{i+1}) = (\phi_0 - y_0(0))e^{-p(0)i\rho}(e^{p(0)\rho} + e^{-p(0)\rho} - 2), \\[2mm] \lim_{h \to 0}(-y_{i+1} - 4y_i - 3y_{i-1}) = (\phi_0 - y_0(0))e^{-p(0)i\rho}(-3e^{p(0)\rho} - e^{-p(0)\rho} + 4), \\[2mm] \lim_{h \to 0}(y_{i+1} - y_{i-1}) = (\phi_0 - y_0(0))e^{-p(0)i\rho}(e^{p(0)\rho} + 3e^{-p(0)\rho} - 4), \\[2mm] \lim_{h \to 0}(3y_{i+1} - 3y_i + y_{i-1}) = (\phi_0 - y_0(0))e^{-p(0)i\rho}(e^{-p(0)\rho} - e^{p(0)\rho}). \end{cases} \tag{17}$$

Substituting (17) into (16) and simplifying, we get

$$\sigma_0 = \rho p(0)(\alpha + \beta) \coth\left(\frac{p(0)\rho}{2}\right).$$

Case 2: For $p(x) < 0$ (Right-end boundary layer), we have:

$$\begin{cases} \lim_{h \to 0}(y_{i-1} - 2y_i + y_{i+1}) = (\varphi - y_0(1))e^{-p(1)i\rho}(e^{p(1)\rho} + e^{-p(1)\rho} - 2), \\[2mm] \lim_{h \to 0}(-y_{i+1} - 4y_i - 3y_{i-1}) = (\varphi - y_0(1))e^{-p(1)i\rho}(-3e^{p(1)\rho} - e^{-p(1)\rho} + 4), \\[2mm] \lim_{h \to 0}(y_{i+1} - y_{i-1}) = (\varphi - y_0(1))e^{-p(1)i\rho}(e^{p(1)\rho} + 3e^{-p(1)\rho} - 4), \\[2mm] \lim_{h \to 0}(3y_{i+1} - 3y_i + y_{i-1}) = (\varphi - y_0(1))e^{-p(1)i\rho}(e^{-p(1)\rho} - e^{p(1)\rho}). \end{cases} \tag{18}$$

Substituting (18) into (16) and simplifying, we get

$$\sigma_N = \rho p(1)(\alpha + \beta) \coth\left(\frac{p(1)\rho}{2}\right).$$

In general, we take a variable fitting parameter as

$$\sigma_i = \rho_i p(x_i)(\alpha + \beta) \coth\left(\frac{p(x_i)\rho_i}{2}\right), \tag{19}$$

where, $\rho_i = \frac{h}{c_\varepsilon(x_i)}$. Thus, (15) can be written as

$$\left\{\frac{c_\varepsilon(x_i)\sigma_i}{h^2} - \frac{3\alpha p_{i-1}}{2h} + \alpha q_{i-1} - \frac{\beta p_i}{h} + \frac{\alpha p_{i+1}}{2h}\right\}y_{i-1}$$

$$-\left\{\frac{2c_\varepsilon(x_i)\sigma_i}{h^2} - \frac{2\alpha p_{i-1}}{h} - 2\beta q_i + \frac{2\alpha p_{i+1}}{h}\right\}y_i$$

$$+\left\{\frac{c_\varepsilon(x_i)\sigma_i}{h^2} - \frac{\alpha p_{i-1}}{2h} + \alpha q_{i+1} + \frac{\beta p_i}{h} + \frac{3\alpha p_{i+1}}{2h}\right\}y_{i+1}$$

$$= \alpha(f_{i-1} + f_{i+1}) + 2\beta f_i. \tag{20}$$

Further, (20) can be written as three term recurrence relation of the form

$$L^N \equiv E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = 1, 2, ..., N-1, \tag{21}$$

where

$$\begin{cases} E_i = \frac{c_\varepsilon(x_i)\sigma_i}{h^2} - \frac{3\alpha p_{i-1}}{2h} + \alpha q_{i-1} - \frac{\beta p_i}{h} + \frac{\alpha p_{i+1}}{2h}, \\ F_i = \frac{2c_\varepsilon(x_i)\sigma_i}{h^2} - \frac{2\alpha p_{i-1}}{h} - 2\beta q_i + \frac{2\alpha p_{i+1}}{h}, \\ G_i = \frac{c_\varepsilon(x_i)\sigma_i}{h^2} - \frac{\alpha p_{i-1}}{2h} + \alpha q_{i+1} + \frac{\beta p_i}{h} + \frac{3\alpha p_{i+1}}{2h}, \\ H_i = \alpha(f_{i-1} + f_{i+1}) + 2\beta f_i. \end{cases}$$

The tri-diagonal system in (21) can be easily solved by the method of Discrete Invariant Imbedding Algorithm.

3 Stability and Convergence Analysis

3.1 Truncation error

Let us expand the terms $y_{i\pm1}$ and $M_{i\pm1}$ from (10), using Taylor series as

$$\begin{cases} y_{i+1} = y_i + hy_i' + \frac{h^2}{2!}y_i'' + \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} + \frac{h^5}{5!}y_i^{(5)} + \frac{h^6}{6!}y_i^{(6)} + O(h^7), \\ y_{i-1} = y_i - hy_i' + \frac{h^2}{2!}y_i'' - \frac{h^3}{3!}y_i''' + \frac{h^4}{4!}y_i^{(4)} - \frac{h^5}{5!}y_i^{(5)} + \frac{h^6}{6!}y_i^{(6)} + O(h^7), \\ M_{i+1} = y_{i+1}'' = y_i'' + hy_i''' + \frac{h^2}{2!}y_i^{(4)} + \frac{h^3}{3!}y_i^{(5)} + \frac{h^4}{4!}y_i^{(6)} + O(h^7), \\ M_{i-1} = y_{i-1}'' = y_i'' - hy_i''' + \frac{h^2}{2!}y_i^{(4)} - \frac{h^3}{3!}y_i^{(5)} + \frac{h^4}{4!}y_i^{(6)} + O(h^7). \end{cases} \tag{22}$$

The local truncation error $T_i(h)$ obtained from (10) is

$$T_i(h) = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - \alpha(M_{i-1} + M_{i+1}) - 2\beta M_i. \tag{23}$$

Substituting the series of $y_{i\pm1}$ and $M_{i\pm1}$ from (22) into (23) and collecting like terms gives

$$T_i(h) = (1 - 2(\alpha + \beta))y_i'' + h^2(\frac{1}{12} - \alpha)y_i^{(4)} + O(h^4). \tag{24}$$

But from the values of $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{3}$, (24) becomes

$$T_i(h) = h^2(-\frac{1}{12})y_i^{(4)} + O(h^4),$$

which implies

$$||T_i(h)|| \leq Ch^2, \tag{25}$$

where $C = \frac{1}{12}|y_i^{(4)}|$. This establishes that the developed method is second order accurate or its order of convergence is $O(h^2)$.

3.2 Convergence Analysis

Local truncation errors refer to the differences between the original differential equation and its finite difference approximation at a mesh points. Finite difference scheme is called consistent if the limit of truncation error $(T_i(h))$ is equal to zero as the mesh size h goes to zero. Hence, the proposed method in (21) with local truncation error in (25) satisfies the definition of consistency as

$$\lim_{h\to0} T_i(h) = \lim_{h\to0} Ch^2 = 0. \tag{26}$$

Thus, the proposed scheme is consistent.

3.3   Stability Analysis

Consider the developed scheme in (21),

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \tag{27}$$

where the coefficients E_i, F_i and G_i are as in (21). If we multiply both sides of (21) by h^2 and consider the values of E_i, F_i and G_i for sufficiently small h, we get

$$E_i = G_i = c_\varepsilon(x_i)\sigma_i, \quad F_i = 2c_\varepsilon(x_i)\sigma_i, \tag{28}$$

Considering (28) into (21) the one which is multiplied by h^2 the developed scheme can be written in a matrix form

$$AY = B, \tag{29}$$

where the matrices

$$A = \begin{pmatrix} -2c_\varepsilon(x_i)\sigma_i & c_\varepsilon(x_i)\sigma_i & 0 & \ldots & 0 \\ c_\varepsilon(x_i)\sigma_i & -2c_\varepsilon(x_i)\sigma_i & c_\varepsilon(x_i)\sigma_i & \ldots & 0 \\ 0 & - & - & & 0 \\ \vdots & & & & c_\varepsilon(x_i)\sigma_i \\ 0 & - & - & c_\varepsilon(x_i)\sigma_i & -2c_\varepsilon(x_i)\sigma_i \end{pmatrix},$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{pmatrix}$$

and

$$B = \begin{pmatrix} h^2 H_1 - E_1 y_0 \\ h^2 H_2 \\ \vdots \\ h^2 H_{N-2} \\ h^2 H_{N-1} - G_{N-1} y_N \end{pmatrix}.$$

Here, the coefficient matrix A is a tri-diagonal matrix with size $(N-1) \times (N-1)$. Matrix A is irreducible if its co-diagonals contain non-zero elements only. The co-diagonals contains E_i and G_i. It is clearly seen that, for sufficiently small h both $E_i \neq 0$ and $G_i \neq 0$, for $i = 1, 2, ..., N-1$. Hence, A is irreducible.

Again we can see that all $|E_i|$, $|F_i|$, $|G_i| > 0$, for $i = 1, 2, ..., N-1$ and in each row of A, the modulus of diagonal element is greater than or equal to the sum of the modulus of the two co-diagonal elements (i.e., $|F_i| \geq |E_i| + |G_i|$). This implies that A is diagonally dominant. Under this condition, the Thomas Algorithm is stable for sufficiently small h.

As discussed in [12] the eigenvalues of a tri-diagonal matrix A are given by

$$\lambda_s = -2c_\varepsilon(x_i)\sigma_i + 2\{\sqrt{(c_\varepsilon(x_i)\sigma_i)(c_\varepsilon(x_i)\sigma_i)}\} \cos \frac{s\pi}{N}, s = 1(1)N - 1. \tag{30}$$

Hence, the eigenvalues of matrix A in (29) are

$$\lambda_s = -2c_\varepsilon(x_i)\sigma_i + 2\{\sqrt{(c_\varepsilon(x_i)\sigma_i)^2}\} \cos \frac{s\pi}{N} = -2c_\varepsilon(x_i)\sigma_i(1 - \cos \frac{s\pi}{N}), s = 1(1)N - 1. \tag{31}$$

But from trigonometric identity, we have $1 - \cos \frac{s\pi}{N} = 2\sin^2 \frac{s\pi}{2N}$. Thus, the eigenvalues of A

$$\lambda_s = -2c_\varepsilon(x_i)\sigma_i(2\sin^2 \frac{s\pi}{2N}) = -4c_\varepsilon(x_i)\sigma_i \sin^2 \frac{s\pi}{2N} \leq -4c_\varepsilon(x_i)\sigma_i. \tag{32}$$

A finite difference method for the boundary value problems is stable if A is non-singular and $||A^{-1}|| \leq C$, for $0 < h < h_0$, where, C and h_0 are two constants that are independent of h.

Since A is real and symmetric it follows that A^{-1} is also real and symmetric so that, its eigenvalues are real and given by $\frac{1}{\lambda_s}$. Hence, as [10] the stability condition of the method will be satisfied when

$$||A^{-1}|| = |\frac{1}{\lambda_s}| = |\frac{-1}{4c_\varepsilon(x_i)\sigma_i}| = \frac{1}{4c_\varepsilon(x_i)\sigma_i} \leq C,$$

where, C is independent of h . Thus, the developed scheme in (21) is stable. A consistent and stable finite difference method is convergent by [12]. Hence, as we have shown above, the proposed method is satisfying both the criteria of consistency and stability which are equivalent to convergence of the method.

4   Numerical Examples and Results

To demonstrate the applicability of the method, two model examples have been considered. The numerical results are presented for $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{3}$. The exact solutions of the test problem is not known. Therefore, we use the double mesh principle to estimate the error and compute the experiment rate of convergence to the computed solution. For this we put

$$E_\varepsilon^N = \max_{0 \leq i \leq 2N} |Y_i^N - Y_{2i}^{2N}|, \tag{33}$$

where Y_i^N and Y_{2i}^{2N} are the ith components of the numerical solutions on meshes of N and $2N$ respectively. We compute the uniform error and the rate of convergence as

$$E^N = \max_\varepsilon E_\varepsilon^N, \quad R^N = \log_2\left(\frac{E^N}{E^{2N}}\right). \tag{34}$$

The numerical results are presented for the values of the perturbation parameter $\varepsilon \in \{ 10^{-4}, 10^{-8}, ..., 10^{-20}\}$.

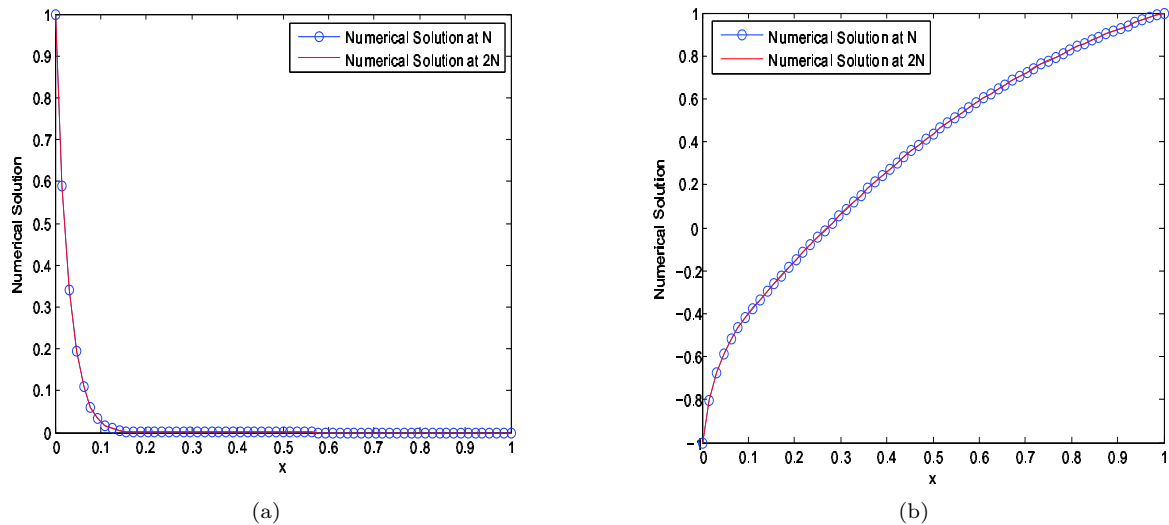(a)   (b)

Figure 1: The behavior of the Numerical Solution for Example 1 and Example 2 at $\varepsilon = 10^{-12}$, $\delta = \gamma = 0.5\varepsilon$ and $N = 32$ respectively.
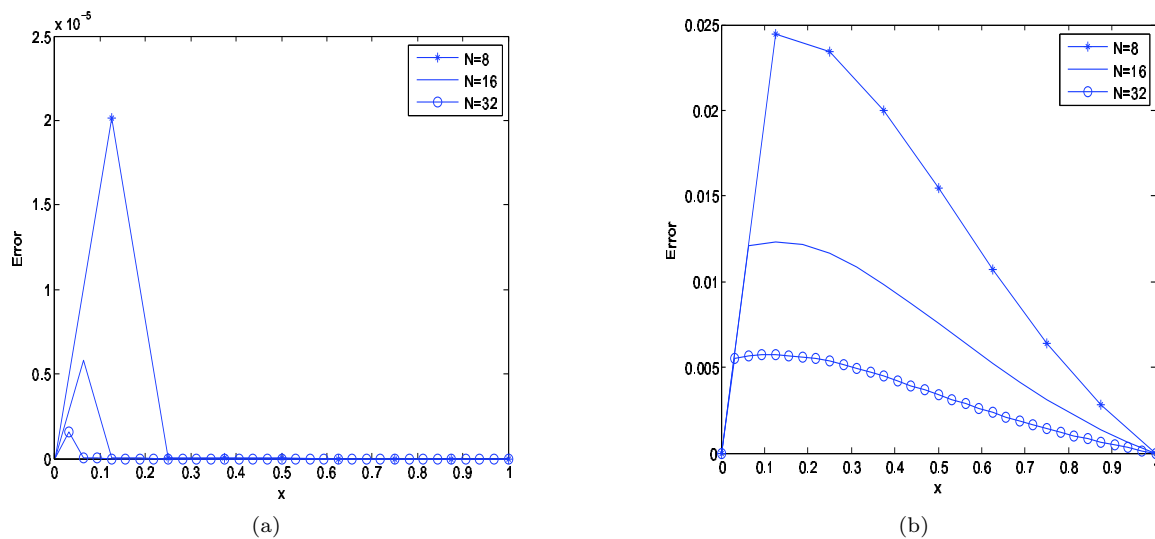
Figure 2: Point wise absolute error of Example 1 and Example 2 at $\varepsilon = 10^{-12}$ and $\delta = \gamma = 0.5\varepsilon$ with different mesh point N respectively.

Example 1 *Consider the model singularly perturbed boundary value problem*

$$\begin{cases} \varepsilon y''(x) + (1+x)y'(x-\delta) + \exp(-2x)y(x+\gamma) - 2\exp(-x)y(x) = 0, \\ y(x) = 1, \quad -\delta \le x \le 0, \quad y(1) = 0. \end{cases}$$

Table 1: Maximum absolute errors and rate of convergence for Example 1 at different number of mesh points for $\delta = \gamma = 0.5\varepsilon$.

ε	N=16	N=32	N=64	N=128	N=256
10^{-4}	5.8394e-06	1.5727e-06	4.0819e-07	1.0399e-07	2.6243e-08
10^{-8}	5.8393e-06	1.5727e-06	4.0818e-07	1.0398e-07	2.6242e-08
10^{-12}	5.8393e-06	1.5727e-06	4.0818e-07	1.0398e-07	2.6242e-08
10^{-16}	5.8393e-06	1.5727e-06	4.0818e-07	1.0398e-07	2.6242e-08
10^{-20}	5.8393e-06	1.5727e-06	4.0818e-07	1.0398e-07	2.6242e-08
E^N	5.8394e-06	1.5727e-06	4.0819e-07	1.0399e-07	2.6243e-08
R^N	1.8926	1.9460	1.9729	1.9864	

Example 2 *Consider the model singularly perturbed boundary value problem*

$$\begin{cases} \varepsilon y''(x) + (1+x)y'(x-\delta) + \sin(2x)y(x+\gamma) - \exp(-x)y(x) = \sin(2x) + 3\exp(-x), \\ y(x) = -1, \quad -\delta \le x \le 0, \quad y(1) = 1. \end{cases}$$

5   Discussion and Conclusion

This study introduces a fitted non-polynomial spline method for singularly perturbed differential-difference equations having two parameters. The numerical scheme is developed on uniform mesh using fitted operator
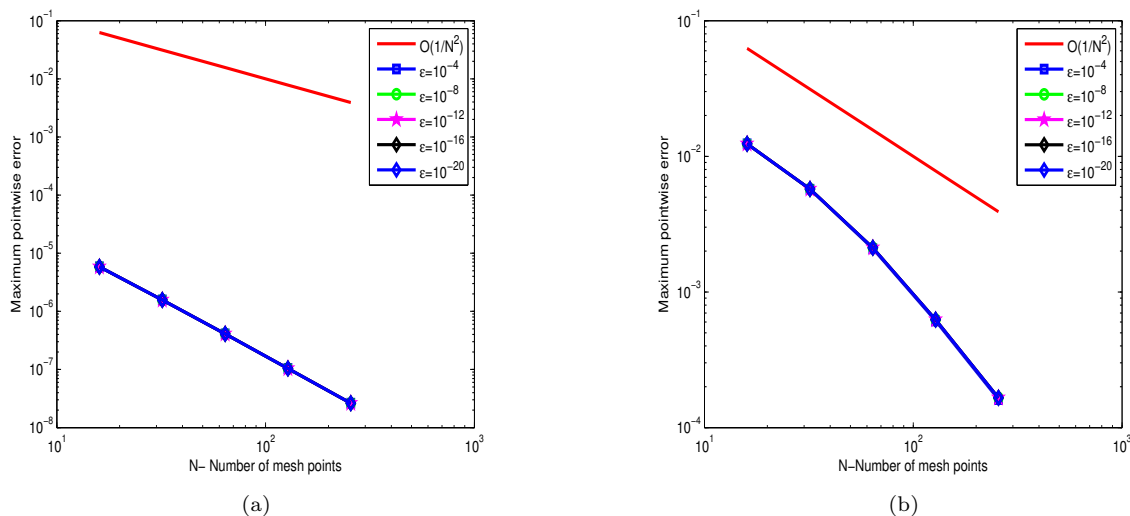
Figure 3: ε-uniform convergence with fitted operator in log-log scale for Example 1 and Example 2 respectively.

Table 2: Maximum absolute errors and rate of convergence for Example 2 at different number of mesh points for $\delta = \gamma = 0.5\varepsilon$.

ε	N=16	N=32	N=64	N=128	N=256
10^{-4}	1.2347e-02	5.7119e-03	2.0848e-03	6.1122e-04	1.6119e-04
10^{-8}	1.2349e-02	5.7422e-03	2.1152e-03	6.2507e-04	1.6565e-04
10^{-12}	1.2349e-02	5.7422e-03	2.1152e-03	6.2507e-04	1.6565e-04
10^{-16}	1.2349e-02	5.7422e-03	2.1152e-03	6.2507e-04	1.6565e-04
10^{-20}	1.2349e-02	5.7422e-03	2.1152e-03	6.2507e-04	1.6565e-04
E^N	1.2349e-02	5.7422e-03	2.1152e-03	6.2507e-04	1.6565e-04
R^N	1.1047	1.4408	1.7587	1.9159	

in the given differential equation. The stability of the developed numerical method is established and its uniform convergence is proved. To validate the applicability of the method, two model problems are considered for numerical experimentation for different values of the perturbation parameter and mesh points. The numerical results are tabulated in terms of maximum absolute errors, numerical rate of convergence and uniform errors (see Tables 1 and 2). Further, behavior of the numerical solution (Figure 1), point-wise absolute error (Figure 2) and the ε-uniform convergence of the method is shown by the log-log plot (Figure 3). The method is shown to be ε-uniformly convergent with order of convergence $O(h^2)$. The proposed method gives more accurate, stable and ε-uniform numerical result.

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