Applied Mathematics E-Notes, 22(2022), 371-382 ©
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ISSN 1607-2510

Extension Of Pál Type Hermite-Fejér Interpolation Onto The Unit Circle*

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Received 7 September 2021

Abstract

The paper is devoted to the study of a Pál type $(0;1)$ interpolation problem on the unit circle considering two disjoint sets of nodes. The nodal points are obtained by projecting vertically the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ and its derivative $P_n^{(\alpha,\beta)'}(x)$, together with ± 1 onto the unit circle. The Lagrange data are prescribed on the first set of nodes, the Hermite data are prescribed on the second one and generalized Hermite-Fejér boundary conditions are prescribed at ± 1. An explicit representation of the interpolatory polynomial is given and the convergence is studied for analytic functions on the unit disk. The results are of interest to approximation theory.

1 Introduction

Interpolation problems on the unit circle have been an area of constant investigation during the past few years. A considerable amount of literature got accumulated on Lacunary, Birkhoff or Pál-type interpolation on the unit circle. Throughout this paper, we denote the Jacobi polynomial of degree n by $P_n^{(\alpha,\beta)}(x)$. Pál [14] proved that there does not exist a unique polynomial of degree $\leq 2n-2$, when values of the function are prescribed on the set of nodes with n points and those of their derivatives on another set of $(n-1)$ points. To obtain a unique solution, he imposed an extra condition and provided the explicit representation of the interpolatory polynomial. Since then, researchers look forward to more general Pál-type interpolation problems. Lénárd [13] considered a $(0,2)$ type Pál interpolation problem and obtained regularity and explicit representation for the same.

In 1960, Kiŝ [10] was the initiator of interpolation processes on the unit circle. He considered the $(0,2)$ and $(0,1,........,r-2,r)$ interpolation for an integer $r \geq 2$ on the n^{th} roots of unity. Brück [5] studied Lagrange interpolation of a function considering nodes $z_{kn}^{\alpha} = T_{\alpha}(w_{kn})$, where $w_{kn} = \exp\left(\dfrac{2\pi ik}{2n+1}\right)$, $n \geq 0$, $k = 1(1)2n$ and $T_{\alpha} = \dfrac{z-\alpha}{1-\alpha z}$, $0 < \alpha < 1$ is a Mobius transformation of the unit disk into itself.

In 2003, Dikshit [8] considered the Pál-type interpolation on non-uniformly distributed nodes on the unit circle. Bruin [6] considered Pál-type interpolation problem and studied the effect of interchanging the value nodes and the derivative nodes on the problem's regularity. Bahadur and Shukla [1] considered weighted $(0;1)$ Pál-type interpolation problem on the vertically projected zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)'}(x)$ onto the unit circle. Explicit representation and convergence was studied for analytic functions on the unit disk. Many researchers ([2, 3, 4, 11, 12]) worked in similar direction.

In the present paper, we extended Pál-type Hermite-Fejér interpolation onto the unit circle by prescribing Lagrange data on nodes obtained by vertically projecting zeros of $P_n^{(\alpha,\beta)}(x)$ as well as Hermite data on nodes obtained by vertically projecting zeros of $P_n^{(\alpha,\beta)'}(x)$ onto the unit circle. The novelty of this paper is that

*Mathematics Subject Classifications: 41A10, 97N50, 41A05, 30E10.

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we took generalized Hermite-Fejér boundary conditions at ± 1. To obtain the explicit representations of the interpolatory polynomial is our first aim, since the problem is regular. We also obtained the order of convergence of such interpolatory polynomial.

The paper has been organized in the following manner. Section 2 is assigned to preliminaries. The interpolation problem and explicit representation of the interpolatory polynomial are defined in Section 3. Sections 4 and 5 are devoted to finding estimates and establishing a convergence theorem respectively. Conclusions have been covered in Section 6.

2 Preliminaries

This section includes the following results, which we shall use. The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is

$$(1-x^2)P_n^{(\alpha,\beta)''}(x)+[\beta-\alpha-(\alpha+\beta+2)x]P_n^{(\alpha,\beta)'}(x)+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0.$$

Using the Szegő transformation $x=\dfrac{1+z^2}{2z}$, we have

$$\begin{aligned}&(z^2-1)^4P_n^{(\alpha,\beta)''}(x)+4z(z^2-1)\big[\{(\alpha+\beta+2)z^2+1\}(z^2-1)-2z^3(\beta-\alpha)\big]P_n^{(\alpha,\beta)'}(x)\\&-16z^6n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0.\end{aligned}$$

Let Z_{2n} and T_{2n-2} be two distinct sets of nodes such that

$$Z_{2n}=\{z_k=x_k+iy_k=\cos\theta_k+i\,\sin\theta_k\,;\,z_{n+k}=\overline{z_k}\,;\,k=1,2,...,n\,;\,x_k,y_k\in R\}$$

and

$$T_{2n-2}=\{t_k=x_k^*+iy_k^*=\cos\phi_k+i\,\sin\phi_k\,;\,t_{n+k}=\overline{t_k}\,;\,k=1,2,...,(n-1)\,;\,x_k^*,y_k^*\in R\},$$

which are obtained by projecting vertically the zeros of $P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)'}(x)$ respectively on the unit circle.

The nodal polynomials $W(z)$ and $W_1(z)$ defined on Z_{2n} and T_{2n-2} are given by (1) and (2) respectively.

$$W(z)=\prod_{k=1}^{2n}(z-z_k)=K_nP_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n \tag{1}$$

and

$$W_1(z)=\prod_{k=1}^{2n-2}(z-t_k)=K_n^*P_n^{(\alpha,\beta)'}\left(\frac{1+z^2}{2z}\right)z^{n-1}, \tag{2}$$

where

$$K_n=2^{2n}n!\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)}$$

and

$$K_n^*=2^{2n-1}(n-1)!\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)}.$$

The fundamental polynomials of Lagrange interpolation on the zeros of $W(z)$ and $W_1(z)$ are respectively given by (3) and (4).

$$l_k(z)=\frac{W(z)}{(z-z_k)W'(z_k)},\qquad k=1,2,...,2n \tag{3}$$

and

$$l_k^*(z)=\frac{W_1(z)}{(z-t_k)W_1'(t_k)},\qquad k=1,2,...,(2n-2)\,. \tag{4}$$

We can write $z = x + iy$, where $x, y \in R$. If $\mid z \mid = 1$, then

$$\left|z^2 - 1\right| = 2\sqrt{1 - x^2} \tag{5}$$

and

$$|z - z_k| = \sqrt{2}\sqrt{1 - xx_k - \sqrt{1 - x^2}\sqrt{1 - x_k^2}}. \tag{6}$$

To evaluate the estimates of the fundamental polynomials formed in the next Section 3, we will use the following (refer to pg.164–166 of [16]).

For $-1 \le x \le 1$, we have

$$(1 - x^2)^{1/2} \mid P_n^{(\alpha,\beta)}(x) \mid = O(n^{\alpha-1}), \tag{7}$$

$$\left|P_n^{(\alpha,\beta)}(x)\right| = O(n^{\alpha}), \tag{8}$$

$$\left|P_n^{(\alpha,\beta)'}(x)\right| = O(n^{\alpha+2}), \tag{9}$$

$$\left|P_n^{(\alpha,\beta)''}(x)\right| = O(n^{\alpha+4}). \tag{10}$$

Considering set of nodes Z_{2n} and T_{2n-2} such that for each k, $x_k, x_k^* \in (-1, 1)$, we have

$$(1 - x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \tag{11}$$

$$\left|P_n^{(\alpha,\beta)}(x_k^*)\right| \sim k^{-\alpha-\frac{1}{2}} n^{\alpha}, \tag{12}$$

$$\left|P_n^{(\alpha,\beta)'}(x_k)\right| \sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}, \tag{13}$$

$$\left|P_n^{(\alpha,\beta)''}(x_k)\right| \sim k^{-\alpha-\frac{5}{2}} n^{\alpha+4}. \tag{14}$$

Let $f(z)$ be continuous for $|z| \le 1$, analytic for $|z| < 1$ and $f^{(r)} \in Lip\,\nu$, $\nu = 1 + \delta$, $\delta > 0$. Then, there exists a polynomial $F_n(z)$ of degree $\le 4n + 2r - 1$ satisfying Jackson's inequality (see [9]):

$$|f(z) - F_n(z)| \le C\,\omega_{r+1}(f, n^{-1}), \qquad r \ge 0 \tag{15}$$

and also an inequality by O. Kiŝ [10]

$$\left|F_n^{(m)}(z)\right| \le C\,n^m\,\omega_{r+1}(f, n^{-1}), \qquad m \in \mathbb{Z}^+, \tag{16}$$

where $\omega_r(f, n^{-1}) = O(n^{-r+1-\nu})$ denotes the r^{th} modulus of continuity of $f(z)$ as well as C is a constant independent of n and z.

3 The Problem & Explicit Representation of Interpolatory Polynomial

Here, we are interested in determining the convergence of interpolatory polynomial $R_n(z)$ of degree $\le 4n + 2r - 1$ on the set of nodes Z_{2n} and T_{2n-2} with Hermite-Fejér boundary conditions at ± 1 satisfying the conditions.

$$\begin{cases} R_n(z_k) = \alpha_k & \text{for } k = 1, 2, ..., 2n, \\ R_n'(t_k) = \beta_k & \text{for } k = 1, 2, ..., (2n-2), \\ R_n^{(m)}(\pm 1) = 0 & \text{for } m = 0, 1, ..., r, \end{cases} \tag{17}$$

where α_k and β_k are complex constants and $r < \infty$.

$R_n(z)$ can be written in the form given below

$$R_n(z) = \sum_{k=1}^{2n} \alpha_k A_k(z) + \sum_{k=1}^{2n-2} \beta_k B_k(z). \tag{18}$$

Here, $A_k(z)$ and $B_k(z)$ are the first and second fundamental polynomials each of degree $\leq 4n + 2r - 1$ satisfying (19) and (20) respectively. For $k = 1, 2, ..., 2n$,

$$\begin{cases} A_k(z_j) = \delta_{kj} & \text{for } j = 1, 2, ..., 2n, \\ A_k^{'}(t_j) = 0 & \text{for } j = 1, 2, ..., 2n-2, \\ A_k^{(m)}(\pm 1) = 0 & \text{for } m = 0, 1, ..., r, \end{cases} \tag{19}$$

and for $k = 1, 2, ...2n - 2$,

$$\begin{cases} B_k(z_j) = 0 & \text{for } j = 1, 2, ..., 2n, \\ B_k^{'}(t_j) = \delta_{kj} & \text{for } j = 1, 2, ..., 2n-2, \\ B_k^{(m)}(\pm 1) = 0 & \text{for } m = 0, 1, ..., r. \end{cases} \tag{20}$$

Explicit expressions of the polynomials $B_k(z)$ and $A_k(z)$ are given in Theorems 1 and 2 respectively.

Remark 1 *The equations (21) and (22) have been developed while deriving out explicit representation of the interpolatory polynomial. Readers can get the motivation to form such expression from the idea to maintain the degree of the polynomial as well as simultaneously satisfy the conditions required to form the fundamental polynomial.*

$$J_k(z) = \int_0^z z^{n+1}(z^2-1)^r\, l_k^*(z)dz \tag{21}$$

and

$$J_{1j}(z) = \int_0^z z^{n-j}(z^2-1)^r\, W_1(z)dz \qquad ; j = 0, 1\,, \tag{22}$$

where $J_{1j}(-1) = (-1)^{j+1} J_{1j}(1)$.

Theorem 1 *For* $k = 1, 2, ..., (2n-2)$, *second fundamental polynomial is given by (23)*

$$B_k(z) = z^{-n} W(z) \Big[b_k J_k(z) + b_{0k} J_{10}(z) + b_{1k} J_{11}(z) \Big], \tag{23}$$

where

$$b_k = \frac{1}{W(t_k) t_k (t_k^2 - 1)^r}, \tag{24}$$

$$b_{1k} = \frac{-b_k(J_k(1) + J_k(-1))}{2J_{11}(1)} \tag{25}$$

and

$$b_{0k} = \frac{b_k(J_k(-1) - J_k(1))}{2J_{10}(1)}. \tag{26}$$

Proof. Consider (23), where $B_k(z)$ is atmost of the degree $(4n+2r-1)$ satisfying the conditions given in (20). At $z = z_j$, $j = 1, 2, ..., n$,

$$B_k(z_j) = z_j^{-n} W(z_j) \Big[b_k J_k(z_j) + b_{0k} J_{10}(z_j) + b_{1k} J_{11}(z_j) \Big].$$

Since z_j's are the zeros of the polynomial $W(z)$, so $B_k(z_j) = 0$. Differentiating $B_k(z)$ with respect to z gives us

$$\begin{aligned} B_k'(z) &= \big[-nz^{-n-1} W(z) + z^{-n} W'(z) \big] \big(b_k J_k(z) + b_{0k} J_{10}(z) + b_{1k} J_{11}(z) \big) \\ &\quad + z^{-n} W(z) \big(b_k J_k'(z) + b_{0k} J_{10}'(z) + b_{1k} J_{11}'(z) \big) \\ &= \Big[-nz^{-n-1} \Big\{ K_n P_n^{(\alpha,\beta)} \Big(\frac{1+z^2}{2z} \Big) z^n \Big\} + K_n z^{-n} \Big\{ P_n^{(\alpha,\beta)'} \Big(\frac{1+z^2}{2z} \Big) z^n + P_n^{(\alpha,\beta)} \Big(\frac{1+z^2}{2z} \Big) n z^{n-1} \Big\} \Big] \\ &\quad \times \Big(b_k J_k(z) + b_{0k} J_{10}(z) + b_{1k} J_{11}(z) \Big) + z^{-n} W(z) \big(b_k J_k'(z) + b_{0k} J_{10}'(z) + b_{1k} J_{11}'(z) \big) \end{aligned}$$

and

$$\begin{aligned} B_k'(z) &= \Big[K_n P_n^{(\alpha,\beta)'} \Big(\frac{1+z^2}{2z} \Big) \Big] \Big(b_k J_k(z) + b_{0k} J_{10}(z) + b_{1k} J_{11}(z) \Big) \\ &\quad + z^{-n} W(z) \big(b_k J_k'(z) + b_{0k} J_{10}'(z) + b_{1k} J_{11}'(z) \big). \end{aligned}$$

Since t_j's are the zeros of the polynomial $W_1(z)$, we see that $B_k'(t_j) = t_j^{-n} W(t_j) b_k J_k'(t_j)$.

Using (4) and (21), we have

$$B_k\prime(t_j) = t_j^{-n} W(t_j) b_k t_j^{n+1} (t_j^2 - 1)^r l_k^*(t_j) = t_j W(t_j) b_k (t_j^2 - 1)^r \delta_{kj}.$$

Using condition $B_k^{'}(t_j) = \delta_{kj}$ given in (20), at $j = k$, we get (24). One can verify the results for $j \neq k$. Also, from $B_k^{(m)}(\pm 1) = 0$ for $m = 0, 1, ..., r$, we get (25) and (26). Hence, Theorem 1 follows. ■

Theorem 2 *For $k = 1, 2, ..., 2n$, first fundamental polynomial is given by (27)*

$$A_k(z) = \frac{(z^2-1)^{r+1} \, l_k(z) W_1(z)}{(z_k^2-1)^{r+1} W_1(z_k)} + z^{-n} W(z) \Big[S_k(z) + a_{0k} J_{10}(z) + a_{1k} J_{11}(z) \Big], \tag{27}$$

where

$$S_k(z) = -\int_0^z \frac{z^n (z^2-1)^r}{W'(z_k)(z_k^2-1)^{r+1} W_1(z_k)} \Bigg[\frac{(z^2-1)W_1^{'}(z) + c_k W_1(z)}{(z-z_k)} \Bigg] dz, \tag{28}$$

$$a_{0k} = \frac{S_k(1) - S_k(-1)}{2J_{10}(1)}, \tag{29}$$

$$a_{1k} = \frac{-(S_k(1) + S_k(-1))}{2J_{11}(1)} \tag{30}$$

and

$$c_k = \frac{(1 - {z_k}^2) W_1'(z_k)}{W_1(z_k)}. \tag{31}$$

Proof. Consider (27), where $A_k(z)$ is atmost of the degree $(4n+2r-1)$ satisfying the conditions given in (19). At $z = z_j$, $j = 1, 2, ..., 2n$, we have

$$A_k(z_j) = \frac{(z_j^2-1)^{r+1} \, l_k(z_j) W_1(z_j)}{(z_k^2-1)^{r+1} W_1(z_k)} + z_j^{-n} W(z_j) \Big[S_k(z_j) + a_{0k} J_{10}(z_j) + a_{1k} J_{11}(z_j) \Big].$$

Since z_j's are the zeros of the polynomial $W(z)$, we see that

$$A_k(z_j) = \frac{(z_j^2-1)^{r+1}\, l_k(z_j) W_1(z_j)}{(z_k^2-1)^{r+1} W_1(z_k)} = \frac{(z_j^2-1)^{r+1}\, \delta_{kj} W_1(z_j)}{(z_k^2-1)^{r+1} W_1(z_k)} = \delta_{kj}.$$

Differentiating $A_k(z)$ with respect to z,

$$\begin{aligned} A_k'(z) &= \frac{1}{(z_k^2-1)^{r+1} W_1(z_k)} \Big[W_1(z) \big\{ 2z(r+1)(z^2-1)^r l_k(z) + (z^2-1)^{r+1} l_k'(z) \big\} \\ &\quad + (z^2-1)^{r+1} l_k(z) W_1'(z) \Big] + \Big\{ -nz^{-n-1} W(z) + z^{-n} W'(z) \Big\} \Big(S_k(z) + a_{0k} J_{10}(z) + a_{1k} J_{11}(z) \Big) \\ &\quad + z^{-n} W(z) \Big[S_k'(z) + a_{0k} J_{10}'(z) + a_{1k} J_{11}'(z) \Big]. \end{aligned}$$

At $z = t_j$, we have

$$\begin{aligned} A_k'(t_j) &= \frac{1}{(z_k^2-1)^{r+1} W_1(z_k)} \Big[W_1(t_j) \big\{ 2t_j(r+1)(t_j^2-1)^r l_k(t_j) + (t_j^2-1)^{r+1} l_k'(t_j) \big\} \\ &\quad + (t_j^2-1)^{r+1} l_k(t_j) W_1'(t_j) \Big] + \Big\{ -nt_j^{-n-1} W(t_j) + t_j^{-n} W'(t_j) \Big\} \Big(S_k(t_j) + a_{0k} J_{10}(t_j) + a_{1k} J_{11}(t_j) \Big) \\ &\quad + t_j^{-n} W(t_j) \Big[S_k'(t_j) + a_{0k} J_{10}'(t_j) + a_{1k} J_{11}'(t_j) \Big]. \end{aligned}$$

Since t_j's are the zeroes of $W_1(z)$, we see that

$$\begin{aligned} A_k'(t_j) &= \frac{(t_j^2-1)^{r+1} l_k(t_j) W_1'(t_j)}{(z_k^2-1)^{r+1} W_1(z_k)} + \Big\{ -nt_j^{-n-1} W(t_j) + t_j^{-n} W'(t_j) \Big\} \Big(S_k(t_j) + a_{0k} J_{10}(t_j) + a_{1k} J_{11}(t_j) \Big) \\ &\quad + t_j^{-n} W(t_j) S_k'(t_j). \end{aligned}$$

From the second condition given in (19), we have

$$\begin{aligned} 0 &= \frac{(t_j^2-1)^{r+1} l_k(t_j) W_1'(t_j)}{(z_k^2-1)^{r+1} W_1(z_k)} + \Big\{ -nt_j^{-n-1} W(t_j) + t_j^{-n} W'(t_j) \Big\} \Big(S_k(t_j) + a_{0k} J_{10}(t_j) + a_{1k} J_{11}(t_j) \Big) \\ &\quad + t_j^{-n} W(t_j) S_k'(t_j) \end{aligned}$$

and

$$\begin{aligned} t_j^{-n} W(t_j) S_k'(t_j) &= -\frac{(t_j^2-1)^{r+1} l_k(t_j) W_1'(t_j)}{(z_k^2-1)^{r+1} W_1(z_k)} \\ &\quad + \Big\{ nt_j^{-n-1} W(t_j) - t_j^{-n} W'(t_j) \Big\} \Big(S_k(t_j) + a_{0k} J_{10}(t_j) + a_{1k} J_{11}(t_j) \Big) \\ &= -\frac{(t_j^2-1)^{r+1} l_k(t_j) W_1'(t_j)}{(z_k^2-1)^{r+1} W_1(z_k)} \\ &\quad + \Bigg\{ nt_j^{-n-1} K_n P_n^{(\alpha,\beta)} \left(\frac{1+t_j^2}{2t_j} \right) t_j^n - t_j^{-n} \Bigg\{ K_n P_n^{(\alpha,\beta)'} \left(\frac{1+t_j^2}{2t_j} \right) t_j^n \\ &\quad + nK_n P_n^{(\alpha,\beta)} \left(\frac{1+t_j^2}{2t_j} \right) t_j^{n-1} \Bigg\} \Bigg\} \Bigg(S_k(t_j) + a_{0k} J_{10}(t_j) + a_{1k} J_{11}(t_j) \Bigg). \end{aligned}$$

After a little computation, we get

$$S_k'(t_j) = -\frac{(t_j^2-1)^{r+1} l_k(t_j) W_1'(t_j)}{t_j^{-n} W(t_j) (z_k^2-1)^{r+1} W_1(z_k)}.$$

We can write above equation as

$$S_k'(z) = -\frac{z^n(z^2-1)^r}{W'(z_k)(z_k^2-1)^{r+1}W_1(z_k)}\left[\frac{(z^2-1)W_1'(z)+c_kW_1(z)}{(z-z_k)}\right]. \tag{32}$$

Integrating (32) provides us with a polynomial $S_k(z)$ of degree $(3n+2r-1)$ given by (28).
To establish the validity of $S_k(z)$, we must have $[(z^2-1)W_1'(z)+c_kW_1(z)]_{|z=z_k} = 0$, which in turn gives (31). Similarly, the constants a_{0k} and a_{1k} can be found out by satisfying the condition

$$A_k^{(m)}(\pm 1) = 0 \ \text{ for } m = 0, 1, ..., r.$$

Hence, Theorem 2 follows. ■

4 Estimates of Fundamental Polynomials

We need to calculate estimates in order to obtain the rate of convergence of interpolatory polynomials.

Lemma 1 *Let $A_k(z)$ be given by (27). Then for $|z| \leq 1$,*

$$\sum_{k=1}^{2n} |A_k(z)| = \mathbf{O}\Big((1-x^2)^{r/2} n^{r+1} \log n\Big), \tag{33}$$

where $-1 < \alpha \leq \frac{r}{2}$.

Lemma 2 *Let $B_k(z)$ be given by (23). Then for $|z| \leq 1$,*

$$\sum_{k=1}^{2n-2} |B_k(z)| = \mathbf{O}\Big((1-x^2)^{r/2} n^{r} \log n\Big), \tag{34}$$

where $-1 < \alpha \leq \dfrac{r-1}{2}$.

Proof of Lemma 1. From (27) we have

$$\begin{aligned}\sum_{k=1}^{2n}|A_k(z)| \quad &\leq \quad \underbrace{\sum_{k=1}^{2n}\left|\frac{(z^2-1)^{r+1}l_k(z)W_1(z)}{(z_k^2-1)^{r+1}W_1(z_k)}\right|}_{I_1} + \underbrace{\sum_{k=1}^{2n}\left|z^{-n}W(z)S_k(z)\right|}_{I_2} \\ &\quad + \underbrace{\sum_{k=1}^{2n}\left|z^{-n}W(z)(a_{0k}J_{10}(z)+a_{1k}J_{11}(z))\right|}_{I_3}.\end{aligned}$$

We can write as

$$\sum_{k=1}^{2n}|A_k(z)| \leq I_1 + I_2 + I_3. \tag{35}$$

Using (3) we have

$$\begin{aligned} I_1 &= \sum_{k=1}^{2n}\left|\frac{(z^2-1)^{r+1}W(z)W_1(z)}{(z_k^2-1)^{r+1}(z-z_k)W'(z_k)W_1(z_k)}\right| \\ &= \sum_{k=1}^{2n}\left|\frac{(z^2-1)^{r+1}\Big\{K_nP_n^{(\alpha,\beta)}\Big(\frac{1+z^2}{2z}\Big)z^n\Big\}\Big\{K_n^*P_n^{(\alpha,\beta)'}\Big(\frac{1+z^2}{2z}\Big)z^{n-1}\Big\}}{(z_k^2-1)^{r+1}(z-z_k)\Big\{K_nP_n^{(\alpha,\beta)}\Big(\frac{1+z^2}{2z}\Big)z^n\Big\}'_{|z=z_k}\Big\{K_n^*P_n^{(\alpha,\beta)'}\Big(\frac{1+z_k^2}{2z_k}\Big)z_k^{n-1}\Big\}}\right|.\end{aligned}$$

Since $P_n^{(\alpha,\beta)}(x_k)=0$ and $|z_k|=1$, we get

$$I_1=2\sum_{k=1}^{2n}\frac{\left|(z^2-1)^{r+1}\right|\left|P_n^{(\alpha,\beta)}(x)\right|\left|P_n^{(\alpha,\beta)'}(x)\right|}{\left|(z_k^2-1)^{r+2}\right|\left|z-z_k\right|\left|P_n^{(\alpha,\beta)'}(x_k)\right|^2}.$$

Using (5) and (6), we get

$$\begin{aligned}
I_1 &= 2\sum_{k=1}^{2n}\frac{2^{r+1}(1-x^2)^{\frac{r+1}{2}}\left|P_n^{(\alpha,\beta)}(x)\right|\left|P_n^{(\alpha,\beta)'}(x)\right|}{2^{r+2}(1-x_k^2)^{\frac{r+2}{2}}\sqrt{2}\sqrt{1-xx_k-\sqrt{1-x^2}\sqrt{1-x_k^2}}\left|P_n^{(\alpha,\beta)'}(x_k)\right|^2}\\
&= \frac{1}{\sqrt{2}}\sum_{k=1}^{2n}\frac{(1-x^2)^{\frac{r+1}{2}}\left|P_n^{(\alpha,\beta)}(x)\right|\left|P_n^{(\alpha,\beta)'}(x)\right|\sqrt{1-xx_k+\sqrt{1-x^2}\sqrt{1-x_k^2}}}{(1-x_k^2)^{\frac{r+2}{2}}\sqrt{(1-xx_k)^2-(1-x^2)(1-x_k^2)}\left|P_n^{(\alpha,\beta)'}(x_k)\right|^2}\\
&= \frac{1}{\sqrt{2}}\sum_{k=1}^{2n}\frac{(1-x^2)^{\frac{r+1}{2}}\left|P_n^{(\alpha,\beta)}(x)\right|\left|P_n^{(\alpha,\beta)'}(x)\right|\sqrt{1-xx_k+\sqrt{(1-xx_k)^2-(x-x_k)^2}}}{(1-x_k^2)^{\frac{r+2}{2}}\left|x-x_k\right|\left|P_n^{(\alpha,\beta)'}(x_k)\right|^2}\\
&\leq \sum_{k=1}^{2n}\frac{(1-x^2)^{\frac{r+1}{2}}\left|P_n^{(\alpha,\beta)}(x)\right|\left|P_n^{(\alpha,\beta)'}(x)\right|\sqrt{1-xx_k}}{(1-x_k^2)^{\frac{r+2}{2}}\left|x-x_k\right|\left|P_n^{(\alpha,\beta)'}(x_k)\right|^2}.
\end{aligned}$$

For $|x-x_k|\geq\frac{1}{2}\left|1-x_k^2\right|$, we have

$$\begin{aligned}
I_1 &\leq \sum_{k=1}^{2n}\frac{2(1-x^2)^{\frac{r+1}{2}}\left|P_n^{(\alpha,\beta)}(x)\right|\left|P_n^{(\alpha,\beta)'}(x)\right|\sqrt{1-xx_k}}{(1-x_k^2)^{\frac{r+2}{2}}(1-x_k^2)\left|P_n^{(\alpha,\beta)'}(x_k)\right|^2}\\
&\leq \sum_{k=1}^{2n}\frac{2\sqrt{2}(1-x^2)^{\frac{r}{2}}\left\{\left(\sqrt{(1-x^2)}\left|P_n^{(\alpha,\beta)}(x)\right|\right)\right\}\left|P_n^{(\alpha,\beta)'}(x)\right|}{(1-x_k^2)^{\frac{r+4}{2}}\left|P_n^{(\alpha,\beta)'}(x_k)\right|^2}.
\end{aligned}$$

Using (7), (9), (11) and (13), we get

$$I_1=\mathbf{O}\left((1-x^2)^{r/2}n^{r+1}\sum_{k=1}^{2n}\frac{1}{k^{r-2\alpha+1}}\right).$$

From $r-2\alpha+1\geq 1$, we get

$$I_1=\mathbf{O}\left((1-x^2)^{r/2}n^{r+1}\log n\right)\qquad\left\{-1<\alpha\leq\frac{r}{2}\right\}. \tag{36}$$

The reader can verify that estimate remains the same in the case where $|x-x_k|<\frac{1}{2}\left|1-x_k^2\right|$. Following similar scheme as above gives

$$I_2=\mathbf{O}\left((1-x^2)^{r/2}n^{r}\log n\right)\qquad\left\{-1<\alpha\leq\frac{r}{2}\right\}, \tag{37}$$

and

$$I_3=\mathbf{O}\left((1-x^2)^{r/2}n^{r}\log n\right)\qquad\left\{-1<\alpha\leq\frac{r}{2}\right\}. \tag{38}$$

Combining (36), (37) and (38) gives Lemma 1. ■

Proof of Lemma 2. Consider (23), we have

$$|B_k(z)| = \Big|z^{-n}W(z)\big[b_kJ_k(z) + b_{0k}J_{10}(z) + b_{1k}J_{11}(z)\big]\Big|, \tag{39}$$

$$\sum_{k=1}^{2n-2}|B_k(z)| \le \underbrace{\sum_{k=1}^{2n-2}\Big|z^{-n}W(z)b_kJ_k(z)\Big|}_{M_1} + \underbrace{\sum_{k=1}^{2n-2}\Big|z^{-n}W(z)((b_{0k}J_{10}(z) + b_{1k}J_{11}(z))\Big|}_{M_2} \tag{40}$$

and

$$\sum_{k=1}^{2n-2}|B_k(z)| \le \ M_1 + M_2. \tag{41}$$

Using (24) and (21), we have

$$M_1 = \sum_{k=1}^{2n-2}\Big|z^{-n}W(z)\Big\{\frac{1}{W(t_k)t_k(t_k^2-1)^r}\Big\}\int_0^z z^{n+1}(z^2-1)^r\, l_k^*(z)dz\Big|.$$

Using (4), we get

$$\begin{aligned} M_1 &= \sum_{k=1}^{2n-2}\Big|z^{-n}W(z)\Big\{\frac{1}{W(t_k)t_k(t_k^2-1)^r}\Big\}\int_0^z z^{n+1}(z^2-1)^r\left\{\frac{W_1(z)}{(z-t_k)W_1'(t_k)}\right\}dz\Big| \\ &\le \sum_{k=1}^{2n-2}\left|\frac{z^{-n}W(z)}{W(t_k)t_k(t_k^2-1)^r}\right|\max_{|z|=1}\left|\int_0^z z^{n+1}(z^2-1)^r\left\{\frac{W_1(z)}{(z-t_k)W_1'(t_k)}\right\}dz\right| \\ &\le \sum_{k=1}^{2n-2}\left|\frac{z^{-n}W(z)(z^2-1)^rW_1(z)}{W(t_k)t_k(t_k^2-1)^r(z-t_k)W_1'(t_k)}\right|\left|\int_0^z z^{n+1}\,dz\right|. \end{aligned}$$

Using (1) and (2), we get

$$M_1 \le \sum_{k=1}^{2n-2}\left|\frac{z^{-n}\Big\{K_nP_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n\Big\}(z^2-1)^r\Big\{K_n^*P_n^{(\alpha,\beta)'}\left(\frac{1+z^2}{2z}\right)z^{n-1}\Big\}}{\Big\{K_nP_n^{(\alpha,\beta)}\left(\frac{1+t_k^2}{2t_k}\right)t_k^n\Big\}t_k(t_k^2-1)^r(z-t_k)\Big\{K_n^*P_n^{(\alpha,\beta)'}\left(\frac{1+z^2}{2z}\right)z^{n-1}\Big\}'_{|z=t_k}}\right|\frac{|\ z^{n+2}\ |}{n+2}$$

Since $P_n^{(\alpha,\beta)'}(x_k^*) = 0$ and $|t_k| = 1$, we get

$$M_1 \le \frac{2}{n+2}\sum_{k=1}^{2n-2}\frac{\left|P_n^{(\alpha,\beta)}(x)\right|\left|(z^2-1)^r\right|\left|P_n^{(\alpha,\beta)'}(x)\right|}{\left|P_n^{(\alpha,\beta)}(x_k^*)\right|\left|(t_k^2-1)^{r+1}\right||z-t_k|\left|P_n^{(\alpha,\beta)''}(x_k^*)\right|}.$$

Owing to (5) and (6), we have

$$M_1 \le \frac{1}{(n+2)}\sum_{k=1}^{2n-2}\frac{(1-x^2)^{r/2}\left|P_n^{(\alpha,\beta)}(x)\right|\left|P_n^{(\alpha,\beta)'}(x)\right|\sqrt{1-xx_k^*}}{(1-x_k^{*2})^{\frac{r+1}{2}}\left|P_n^{(\alpha,\beta)}(x_k^*)\right||x-x_k^*|\left|P_n^{(\alpha,\beta)''}(x_k^*)\right|}.$$

For $|x-x_k^*| \ \ge \frac{1}{2}\left|1-x_k^{*2}\right|$, we have

$$\begin{aligned} M_1 &\le \frac{2}{(n+2)}\sum_{k=1}^{2n-2}\frac{(1-x^2)^{r/2}\left|P_n^{(\alpha,\beta)}(x)\right|\left|P_n^{(\alpha,\beta)'}(x)\right|\sqrt{1-xx_k^*}}{(1-x_k^{*2})^{\frac{r+1}{2}}\left|P_n^{(\alpha,\beta)}(x_k^*)\right||1-x_k^{*2}|\left|P_n^{(\alpha,\beta)''}(x_k^*)\right|} \\ &\le \frac{2\sqrt{2}}{(n+2)}\sum_{k=1}^{2n-2}\frac{(1-x^2)^{r/2}\left|P_n^{(\alpha,\beta)}(x)\right|\left|P_n^{(\alpha,\beta)'}(x)\right|}{(1-x_k^{*2})^{\frac{r+3}{2}}\left|P_n^{(\alpha,\beta)}(x_k^*)\right|\left|P_n^{(\alpha,\beta)''}(x_k^*)\right|}. \end{aligned} \tag{42}$$

Using (8), (9), (11), (12) and (14), we get

$$M_1 = \mathbf{O}\left((1-x^2)^{r/2} n^r \sum_{k=1}^{2n} \frac{1}{k^{r-2\alpha}}\right).$$

From $r - 2\alpha \geq 1$, we get

$$M_1 = \mathbf{O}\Big((1-x^2)^{r/2} n^r \log n\Big) \qquad \left\{-1 < \alpha \leq \frac{r-1}{2}\right\}. \tag{43}$$

The estimate remains the same in the case, where $|x - x_k^*| < \frac{1}{2}\left|1 - x_k^{*2}\right|$. Similarly, we have

$$M_2 = \mathbf{O}\Big((1-x^2)^{r/2} n^r \log n\Big) \qquad \left\{-1 < \alpha \leq \frac{r-1}{2}\right\}. \tag{44}$$

Combining (43) and (44) give us desired Lemma 2 . ■

5 Convergence

Theorem 3 *Let $f(z)$ be continous for $|z| \leq 1$ and analytic for $|z| < 1$ and $f^{(r)} \epsilon\, Lip\, \nu$, $\nu = 1 + \delta$, $\delta > 0$. Let the arbitrary numbers β_k's be such that*

$$|\beta_k| = \mathbf{O}\big(n\, \omega_{r+1}(f, n^{-1})\big), \quad k = 1, 2, ..., (2n-2). \tag{45}$$

Then sequence $\{R_n(z)\}$ is defined by

$$R_n(z) = \sum_{k=1}^{2n} f(z_k) A_k(z) + \sum_{k=1}^{2n-2} \beta_k B_k(z), \tag{46}$$

satisfies the following relation for $|z| \leq 1$

$$|R_n(z) - f(z)| = \mathbf{O}\Big((1-x^2)^{r/2} n^{r+1} \omega_{r+1}(f, n^{-1}) \log n\Big), \tag{47}$$

where $\omega_{r+1}(f, n^{-1})$ be the $(r+1)^{th}$ modulus of continuity of $f(z)$.

Proof. Since $R_n(z)$ be the uniquely determined polynomial of degree $\leq 4n + 2r - 1$ and the polynomial $F_n(z)$ satisfying equation (15) can be expressed as

$$F_n(z) = \sum_{k=1}^{2n} F_n(z_k) A_k(z) + \sum_{k=1}^{2n-2} F_n^{'}(z_k) B_k(z), \tag{48}$$

we can write

$$|R_n(z) - f(z)| \leq |R_n(z) - F_n(z)| + |F_n(z) - f(z)|. \tag{49}$$

Using (46) and (48), we have

$$\begin{aligned} |R_n(z) - f(z)| \;&\leq\; \sum_{k=1}^{2n} |f(z_k) - F_n(z_k)|\, |A_k(z)| + \sum_{k=1}^{2n-2} |\beta_k - F_n'(z_k)|\, |B_k(z)| + |F_n(z) - f(z)| \\ &\leq\; \underbrace{\sum_{k=1}^{2n} |f(z_k) - F_n(z_k)|\, |A_k(z)|}_{N_1} + \underbrace{\sum_{k=1}^{2n-2} |\beta_k|\, |B_k(z)|}_{N_2} + \underbrace{\sum_{k=1}^{2n-2} |F_n'(z_k)|\, |B_k(z)|}_{N_3} \\ &\quad + \underbrace{|F_n(z) - f(z)|}_{N_4} \end{aligned}$$

and

$$|R_n(z) - f(z)| \leq N_1 + N_2 + N_3 + N_4, \tag{50}$$

where

$$N_1 = \sum_{k=1}^{2n} |f(z_k) - F_n(z_k)| \, |A_k(z)| \, .$$

From (15) and (33), we have

$$N_1 = \mathbf{O}\bigg(\omega_{r+1}(f, n^{-1})(1-x^2)^{r/2} n^{r+1} \log n\bigg) \tag{51}$$

and

$$N_2 = \sum_{k=1}^{2n-2} |\beta_k| \, |B_k(z)| \, .$$

From (45) and (34), we have

$$N_2 = \mathbf{O}\bigg(n \, \omega_{r+1}(f, n^{-1}))(1-x^2)^{r/2} n^{r} \log n\bigg) \tag{52}$$

and

$$N_3 = \sum_{k=1}^{2n-2} |F'_n(z_k)| \, |B_k(z)| \, .$$

From (16) and (34), we have

$$N_3 = \mathbf{O}\bigg(n \, \omega_{r+1}(f, n^{-1})(1-x^2)^{r/2} n^{r} \log n\bigg) \tag{53}$$

and

$$N_4 = |F_n(z) - f(z)| \, .$$

From (15), we have

$$N_4 = \mathbf{O}\bigg(\omega_{r+1}(f, n^{-1})\bigg). \tag{54}$$

Using (51)–(54) in (50), we get

$$|R_n(z) - f(z)| = \mathbf{O}\bigg((1-x^2)^{r/2} n^{r+1} \omega_{r+1}(f, n^{-1}) \log n\bigg).$$

Hence, Theorem 3 follows. ■

6 Conclusion

This research article poses a completely new problem by introducing the generalized Hermite-Fejér boundary conditions at the points ± 1. Since these additional nodes gradually increase the degree of the interpolatory polynomial. So, the order of convergence must also depend on that increment which can be seen in (47) as we require the $(r+1)^{th}$ modulus of continuity for the convergence purpose. Since the present problem is posed considering generalized Hermite-Fejér boundary conditions only at ± 1, a subtle open problem is to consider the generalized Hermite-Fejér boundary conditions at ± 1 as well as on all the nodal points, where Lagrange and Hermite data are prescribed (i.e $\pm 1 \cup Z_{2n} \cup T_{2n-2}$). This will provide a much broader aspect of convergence and comparisons to the present problem.

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