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Radius Of Starlikeness Of Functions Defined By Ratios Of Analytic Functions*

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Abstract

Let f and g be analytic functions on the open unit disk $\mathbb{D} \subset \mathbb{C}$ with f/g belonging to the class $\mathcal{P}$ of functions with positive real part (consisting of all functions p analytic in $\mathbb{D}$ with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$) or to its subclass consisting of functions p with $|p(z) - 1| < 1$. We obtain the sharp radius constants for the function f to be starlike of order α, parabolic starlike, or to belong to few other related classes when $g/k \in \mathcal{P}$ where k denotes the Koebe function defined by $k(z) = z/(1 - z)^2$.

1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions f on the open unit disk $\mathbb{D} = \mathcal{D}_1$ normalized by $f(0) = 0$ and $f'(0) = 1$, and let $\mathcal{S}$ be the subclass of all univalent function in $\mathcal{A}$ where $\mathcal{D}_r = \{z \in \mathcal{C} : |z| < r\}$. It is well-known that every convex (or starlike) function f maps $\mathbb{D}_r$ onto a convex (or respectively starlike) domain. Though every convex univalent function is starlike (of order 1/2), the converse is not true in general. However, every starlike function $f \in \mathcal{A}$ maps each $\mathbb{D}_r$ onto a convex domain for $r \leq 2 - \sqrt{3}$. This number $2 - \sqrt{3}$ is called the radius of convex of starlike functions. This idea can be extended to any two arbitrary subclasses $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$. The $\mathcal{G}$ radius of $\mathcal{F}$, denoted by $\mathcal{R}_{\mathcal{G}}(\mathcal{F})$, is defined as the largest number $\mathcal{R}_{\mathcal{G}}$ such that $r^{-1}f(rz) \in \mathcal{G}$ for all r with $0 < r < \mathcal{R}_{\mathcal{G}}$, and for all $f \in \mathcal{F}$. Whenever the class $\mathcal{G}$ is characterized by a geometric property $\mathbf{P}$ the number $\mathcal{R}_{\mathcal{G}}$ is called as the radius of the property $\mathbf{P}$ of the class $\mathcal{F}$. Although there are variety of radius problems considered in literature (see [5, 9, 10, 11, 14, 13, 26, 27]), we investigate the functions f characterized by the ratio of f with another function $g \in \mathcal{A}$; these types of problems were considered by MacGregor [16, 17, 18]. Ali et al. [1] determined various radii results for functions f satisfying the following conditions:

(i) $\operatorname{Re}(f(z)/g(z)) > 0$ where $\operatorname{Re}(g(z)/z) > 0$ or $\operatorname{Re}(g(z)/z) > 1/2$.

(ii) $|(f(z)/g(z) - 1) - 1| < 1$ where $\operatorname{Re}(f(z)/g(z)) > 0$ or g is convex.

All these classes are associated to class of functions with positive real part; this class, denoted by $\mathcal{P}$, consists of all analytic functions $p : \mathbb{D} \to \mathbb{C}$ with $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$ for all $z \in \mathbb{D}$. Asha and Ravichandran [21] investigated several radii for the functions $f/g \in \mathcal{P}$ and $(1 + z)g/z \in \mathcal{P}$, belonging to some subclasses of starlike functions (see [7, 8] for further works). For $0 \leq \alpha < 1$, we let $\mathcal{P}(\alpha) := \{p \in \mathcal{P} : \operatorname{Re} p(z) > \alpha\}$. Let k be the Koebe function defined by $k(z) = z/(1 - z)^2$. In this paper, we consider the two subclasses Π_1 and Π_2 of analytic functions given below:

$$\Pi_1 =: \{f \in \mathcal{A} : f/g \in \mathcal{P} \text{ for some } g \in \mathcal{A} \text{ with } g/k \in \mathcal{P}\},$$

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and

$$\Pi_2 =: \{f \in \mathcal{A} : |f/g - 1| < 1 \text{ for some } g \in \mathcal{A} \text{ with } g/k \in \mathcal{P}\}.$$

We determine radii for functions in these two classes Π_1 and Π_2 to belong to several subclasses of starlike functions which we discuss below. In 1985, Padmanabhan and Parvatham [20] used the Hadamard product (convolution) and subordination to introduce the class of functions $f \in \mathcal{A}$ satisfying $z(k_\alpha * f)'/(k_\alpha * f) \prec h$ where $k_\alpha(z) = z/(1-z)^\alpha$, $\alpha \in \mathbb{R}$, and h is convex. This class reduces to the usual classes of starlike and convex functions respectively for $\alpha = 1$ and $\alpha = 2$ when h is the normalized mapping of $\mathbb{D}$ onto the right half-plane. In 1989, Shanmugam [22] studied the class

$$\mathcal{S}^*_g(\varphi) =: \{f \in \mathcal{A} : z(f * g)'/(f * g) \prec \varphi\}$$

where g is fixed and φ a convex function, respectively; this class includes several classes defined by means of linear operator such as Ruscheweyh differential operator and Sălăgean operator. When $g(z) = z/(1-z)$ and $g(z) = z/(1-z)^2$, the subclass $\mathcal{S}^*_g(\varphi)$ is denoted respectively by $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$. In 1992, Ma and Minda [15] studied growth, distortion, covering theorems and coefficient problems for the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ when $\varphi \in \mathcal{P}$ is just a univalent function mapping unit disk $\mathbb{D}$ onto domain symmetric with respect to the real line and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. For $\varphi(z) = (1 + (1-2\alpha)z)/(1-z)$ with $0 \le \alpha < 1$, the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ reduce to the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α and the class $\mathcal{K}(\alpha)$ of convex functions of order α respectively. For more work in this direction, see [3, 6]. When φ equals $1 + (2/\pi)^2(\log((1+\sqrt{z})/(1-\sqrt{z})))^2$, $\sqrt{1+z}$, e^z, $1 + (4/3)z + (2/3)z^2$, $\sin z$, $z + \sqrt{1+z^2}$ and $1 + (zk + z^2/(k^2 - kz))$ where $k = \sqrt{2} + 1$, we denote the class $\mathcal{S}^*(\varphi)$ respectively by $\mathcal{S}_P$, $\mathcal{S}^*_L$, $\mathcal{S}^*_e$, $\mathcal{S}^*_c$, $\mathcal{S}^*_{\sin}$, $\mathcal{S}^*_{\mathbb{C}}$, and $\mathcal{S}^*_R$. The class $\mathcal{S}^*_L$ was introduced by Sokól and Stankiewicz [25]. We refer to [21, 7, 8] for information about the other classes.

2 Radius Results for the Class Π_1

Recall that Π_1 is defined by

$$\Pi_1 =: \{f \in \mathcal{A} : f/g \in \mathcal{P} \text{ for some } g \in \mathcal{A} \text{ with } g/k \in \mathcal{P}\}.$$

The functions $f_0, f_1 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_0(z) = \frac{z(1+z)^2}{(1-z)^4} \quad \text{and} \quad f_1(z) = \frac{z}{(1+z)^2} \tag{1}$$

belong to the class Π_1 and therefore the class Π_1 is non-empty. They satisfy the required conditions with the functions $g_0, g_1 : \mathbb{D} \to \mathbb{C}$ defined by

$$g_0(z) = \frac{z(1+z)}{(1-z)^3} \quad \text{and} \quad g_1(z) = \frac{z}{1-z^2};$$

indeed, we have

$$\operatorname{Re}\frac{f_i(z)}{g_i(z)} > 0 \quad \text{and} \quad \operatorname{Re}\frac{(1-z)^2 g_i(z)}{z} > 0$$

for $i = 0, 1$. The function f_0 is an extremal function for the radius problem that we consider. The function f_1 is univalent, but the function f_0 is not univalent as the coefficients of the Taylor's series $f_0(z) = z + 6z^2 + 19z^3 + 44z^4 + \cdots$ do not satisfy the de Branges theorem. The derivative of f_0 is given by

$$f_0'(z) = \frac{(1+6z+z^2)(1+z)}{(1-z)^5}.$$

Since $f_0'(-3+2\sqrt{2}) = 0$, the function f_0 is a non-univalent functions and therefore the radius of univalence cannot exceed $3 - 2\sqrt{2}$. By Theorem 1(1), the radius of starlikeness of the class Π_1 is $3 - 2\sqrt{2}$ and it follows that the radius of univalence of this class is also $3 - 2\sqrt{2} \approx 0.171573$.

The other radius results for the class Π_1 are given in the following theorem.

Theorem 1 *The following radius results hold for the class Π_1:*

(1) The $\mathcal{S}^*(\alpha)$ radius is $R_{\mathcal{S}^*(\alpha)} = (1-\alpha)/(3+\sqrt{8+\alpha^2}), \quad 0 \le \alpha < 1$.

(2) The $\mathcal{S}^*_L$ radius is $R_{\mathcal{S}^*_L} = (\sqrt{2}-1)(\sqrt{10}-3) \approx 0.067217$.

(3) The $\mathcal{S}_P$ radius is $R_{\mathcal{S}_P} = (6-\sqrt{33})/3 \approx 0.0851$.

(4) The $\mathcal{S}^*_e$ radius is $R_{\mathcal{S}^*_e} = (e-1)/(3e+\sqrt{8e^2+1}) \approx 0.1080$.

(5) The $\mathcal{S}^*_c$ radius is $R_{\mathcal{S}^*_c} = (9-\sqrt{73})/4 \approx 0.1140$.

(6) The $\mathcal{S}^*_{\sin}$ radius is $R_{\mathcal{S}^*_{\sin}} = \sin(1)/(\sqrt{9+\sin^2(1)+2\sin(1)}+3) \approx 0.1320$.

(7) The $\mathcal{S}^*_{☾}$ radius is $R_{\mathcal{S}^*_{☾}} = 3/\sqrt{2} - \sqrt{1/2\,(11-2\sqrt{2})} \approx 0.09999$.

(8) The $\mathcal{S}^*_R$ radius $R_{\mathcal{S}^*_R} = (3-2\sqrt{5-2\sqrt{2}})/(2\sqrt{2}-1) \approx 0.0289$.

We need the following lemmas to prove our results.

Lemma 1 ([1, Lemma 2.2, p.4]) *For $0 < \alpha < \sqrt{2}$, let $\mathbf{r}_a$ be given by*

$$\mathbf{r}_a = \begin{cases} (\sqrt{1-a^2} - (1-a^2))^{\frac{1}{2}}, & 0 < a \le 2\sqrt{2}/3, \\ \sqrt{2} - a, & 2\sqrt{2}/3 \le a < \sqrt{2}. \end{cases}$$

Then $\{\omega : |\omega - a| < r_a\} \subseteq \{\omega : |\omega^2 - 1| < 1\}$.

Lemma 2 ([23, Lemma 1, p. 321]) *For $a > \frac{1}{2}$, let $\mathbf{r}_a$ be given by*

$$\mathbf{r}_a = \begin{cases} a - \frac{1}{2}, & \frac{1}{2} < a \le \frac{3}{2}, \\ \sqrt{2a-2}, & a \ge \frac{3}{2}. \end{cases}$$

Then $\{w : |w-a| < r_a\} \subseteq \{w : Re w > |w-1|\}$.

Lemma 3 ([19, Lemma 2.2, p.368]) *For $e^{-1} < a < e$, let $\mathbf{r}_a$ be given by*

$$\mathbf{r}_a = \begin{cases} a - e^{-1}, & e^{-1} < a \le \frac{e+e^{-1}}{2}, \\ e - a, & \frac{e+e^{-1}}{2} \le a \le e. \end{cases}$$

Then $\{w : |w-a| < \mathbf{r}_a\} \subseteq \{w : |\log w| < 1\} = \Omega_e$.

Lemma 4 ([24, Lemma 2.2, p. 926]) *For $\frac{1}{3} < a < 3$, let $\mathbf{r}_a$ be given by*

$$\mathbf{r}_a = \begin{cases} a - \frac{1}{3}, & \frac{1}{3} < a \le \frac{5}{3}, \\ 3 - a, & \frac{5}{3} \le a \le 3. \end{cases}$$

Then $\{w : |w-a| < \mathbf{r}_a\} \subseteq \Omega_c$, where Ω_c is the region bonded by the cardioid given

$$\left\{x+iy : (9x^2+9y^2-18x+5)^2 - 16(9x^2+9y^2-6x+1) = 0\right\}.$$

Lemma 5 ([2, Lemma 3.3, p.7]) *For $1-\sin 1 < a < 1+\sin 1$, let $r_a = \sin 1 - |a-1|$. Then $\{w : |\omega - a| < \mathbf{r}_a\} \subseteq \Omega_s$; Ω_s is the image of the unit disk $\mathbb{D}$ under $1+\sin z$.*

Lemma 6 ([4, Lemma 2.1, p. 3]) *For $\sqrt{2}-1 < a < \sqrt{2}+1$, let $\mathbf{r}_a = 1 - |\sqrt{2} - a|$. Then*

$$\{w : |w-a| < \mathbf{r}_a\} \subseteq \left\{w : |w^2-1| < 2|w|\right\}.$$

Lemma 7 ([12, Lemma 2.2, p. 202]) *For $2(\sqrt{2}-1) < a < 2$, let $\mathbf{r}_a$ be given by*

$$\mathbf{r}_a = \begin{cases} a - 2\left(\sqrt{2}-1\right), & 2\left(\sqrt{2}-1\right) < a \leq \sqrt{2}, \\ 2-a, & \sqrt{2} \leq a \leq 2. \end{cases}$$

Then $\{w : |w-a| < \mathbf{r}_a\}$, where Ω_r is the image of the disk $\mathbb{D}$ under the function $1 + (zk+z^2)/\left(k^2 - kz\right)$, $k = \sqrt{2}+1$.

Proof of Theorem 1. Let the function $f \in \Pi_1$. Then there is a function $g : \mathbb{D} \to \mathbb{C}$ satisfying

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0 \quad \text{and} \quad \operatorname{Re}\left(\frac{(1-z)^2 g(z)}{z}\right) \quad \forall z \in \mathbb{D}. \tag{2}$$

Define functions $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ as the following.

$$p_1(z) = \frac{(1-z)^2 g(z)}{z} \quad \text{and} \quad p_2(z) = \frac{f(z)}{g(z)}. \tag{3}$$

By using (2) and (3), we have $p_1, p_2 \in \mathcal{P}$, and $f(z) = zp_1(z)p_2(z)/(1-z)^2$. Then it follows that

$$\frac{zf'(z)}{f(z)} = \frac{zp_1'(z)}{p_2(z)} + \frac{zp_2'(z)}{p_2(z)} + \frac{1+z}{1-z}. \tag{4}$$

The bilinear transformation $(1+z)/(1-z)$ maps the disk $|z| \leq r$ onto the disk

$$\left|\frac{1+z}{1-z} - \frac{1-r^2}{1+r^2}\right| \leq \frac{2r}{1-r^2}. \tag{5}$$

For $p \in \mathcal{P}(\alpha)$, we have

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2(1-\alpha)r}{(1-r)(1+(1-2\alpha)r)}, \quad |z| \leq r. \tag{6}$$

Using (4), (5) and (6), we see that the function f maps disk $|z| \leq r$ into disk

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+r^2}{1-r^2}\right| \leq \frac{6r}{1-r^2}. \tag{7}$$

From (7), it follows that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-6r+r^2}{1-r^2} \geq 0, \tag{8}$$

for all $0 \leq r \leq 3-2\sqrt{2}$. Therefore, the function $f \in \Pi_1$ is starlike in $|z| \leq 3-2\sqrt{2} \approx 0.171573$. Hence, all our radii found here must be less or equal to $3-2\sqrt{2}$.

1. The number $\rho = R_{\mathcal{S}^*(\alpha)}$, is the smallest positive root of the equation $(1+\alpha)r^2 - 6r + 1 - \alpha = 0$ in $[0,1]$. For $0 < r \leq \mathrm{R}_{\mathcal{S}^*(\alpha)}$, from (8), it follows that

   $$\operatorname{Re}\frac{zf'(z)}{f(z)} \geq \frac{1-6r+r^2}{1-r^2} \geq \frac{1-6\rho+\rho^2}{1-\rho^2} = \alpha.$$

   This shows that the radius of starlikeness of order α is at least $R_{\mathcal{S}^*(\alpha)}$. To show that it is sharp, consider the function $f_0 \in \Pi_1$ given in (1). For this function f_0, we have

   $$\frac{zf_0'(z)}{f_0(z)} = \frac{1+6z+z^2}{1-z^2}.$$

   At $z = -\rho$, we have

   $$Re\frac{zf_0'(z)}{f_0(z)} = \frac{1-6\rho+\rho^2}{1-\rho^2} = \alpha,$$

   proving the sharpness of the radius.

2. We can give a proof using Lemma 1 but we give a different proof here. The number $\rho := R_{\mathcal{S}_L}$ is the smallest positive root of the equation $(1+\sqrt{2})r^2+6r+1-\sqrt{2}=0$ in interval $(0,1)$, and, from (7), it is clear that, for $0 \le r \le \rho$,

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \left|\frac{zf'(z)}{f(z)}-\frac{1+r^2}{1-r^2}\right| + \frac{2r^2}{1-r^2} \le \frac{6r+2r^2}{1-r^2} \le \frac{6\rho+\rho^2}{1-\rho^2} = \sqrt{2}-1 \tag{9}$$

and

$$\left|\frac{zf'(z)}{f(z)}+1\right| \le 2+\left|\frac{zf'(z)}{f(z)}-1\right| \le \sqrt{2}+1. \tag{10}$$

Thus, from (9) and (10), it follows that, for $0 \le r \le \rho$,

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2-1\right| = \left|\frac{zf'(z)}{f(z)}-1\right|\left|\frac{zf'(z)}{f(z)}+1\right| \le (\sqrt{2}+1)(\sqrt{2}-1)=1.$$

For the function $f_0 \in \Pi_1$ given in (1), we have, at $z=\rho$,

$$\frac{zf_0'(z)}{f_0(z)} = 1+\frac{6\rho+2\rho^2}{1-\rho^2} = \sqrt{2}$$

and so, at $z=\rho$,

$$\left|\left(\frac{zf_0'(z)}{f_0(z)}\right)^2-1\right| = 1.$$

This proves the sharpness.

3. For $\rho := R_{\mathcal{S}_P} = (6-\sqrt{33})/3$, we have

$$\frac{1}{2} < 1 \le a = \frac{1+r^2}{1-r^2} \le \frac{1+\rho^2}{1-\rho^2} = \frac{3\sqrt{33}-1}{16} \approx 1.0146 < 3/2.$$

Also, for $\rho = R_{\mathcal{S}_P}$, we have

$$\frac{6\rho}{(1-\rho^2)} \le \frac{1+\rho^2}{1-\rho^2} - \frac{1}{2}$$

and the disk in (7) for $r=\rho$ becomes

$$\left|\frac{zf'(z)}{f(z)}-a\right| = \left|\frac{zf'(z)}{f(z)}-\frac{1+\rho^2}{1-\rho^2}\right| \le \frac{1+\rho^2}{1-\rho^2}-\frac{1}{2} = a-\frac{1}{2}.$$

By Lemma 2, it follows that the disk in (7) lies inside region Ω_{PAR}. This proves that the radius of parabolic starlikeness is at least $R_{\mathcal{S}_P}$.

The radius is sharp for the function $f_0 \in \Pi_1$. At the point $z=-\rho=-R_{\mathcal{S}_P}$, we have

$$Re\left(\frac{zf_0'(z)}{f_0(z)}\right) = \frac{1-6\rho+\rho^2}{1-\rho^2} = \frac{1}{2} = \frac{6\rho-2\rho^2}{1-\rho^2} = \left|\frac{zf_0'(z)}{f_0(z)}-1\right|.$$

4. For $e^{-1} < a \le \frac{e+e^{-1}}{2}$, Lemma 3 gives

$$\left\{w \in \mathbb{C} : |w-a| < a-e^{-1}\right\} \subseteq \{w \in \mathbb{C} : |\log w| < 1\} =: \Omega_e, \tag{11}$$

For $\rho = R_{\mathcal{S}_e^*}$, we have

$$e^{-1} < a := \frac{1+\rho^2}{1-\rho^2} = \frac{1+9e^2}{e(1+3\sqrt{1+8e^2})} \approx 1.0236 \le \frac{e+e^{-1}}{2} \approx 1.5430$$

and, ρ being smallest positive root of the equation $(1+e)r^2 - 6er + e - 1 = 0$,

$$\frac{6\rho}{1-\rho^2} \leq \frac{1+\rho^2}{1-\rho^2} - \frac{1}{e} = a - e^{-1}.$$

Consequently, the disk in (7) for $r = \rho$ becomes

$$\left|\frac{zf'(z)}{f(z)} - a\right| = \left|\frac{zf'(z)}{f(z)} - \frac{1+\rho^2}{1-\rho^2}\right| \leq \frac{1+\rho^2}{1-\rho^2} - \frac{1}{e} = a - e^{-1}.$$

By (11) the above disk is inside Ω_e proving that the $\mathcal{S}_e^*$ radius for the class Π_1 is at least $R_{\mathcal{S}_e^*}$. The result is sharp for the function f_0 given in (1). Indeed, at $z = -\rho$ where $\rho = R_{\mathcal{S}_e^*}$, we have

$$\left|\log\left(\frac{zf_0'(z)}{f_0(z)}\right)\right| = \left|\log\left(\frac{1-6\rho+\rho^2}{1-\rho^2}\right)\right| = 1.$$

5. For $\frac{1}{3} < a \leq \frac{5}{3}$, by an application of Lemma 4, it follows that

$$\left\{w \in \mathbb{C} : |w - a| < a - \frac{1}{3}\right\} \subseteq \Omega_c,$$

where Ω_c is the domain bounded by the cardioid $\{x+iy : (9x^2+9y^2-18x+5)^2 - 16(9x^2+9y^2-6x+1) = 0\}$. For $\rho = R_{\mathcal{S}_c^*}$, we have

$$\frac{1}{3} < a := \frac{1+\rho^2}{1-\rho^2} = \frac{3\sqrt{73}-1}{24} \approx 1.0263 \leq \frac{5}{3}$$

and, ρ being the smallest positive root of the equation $2r^2 - 9r + 1 = 0$,

$$\frac{6\rho}{1-\rho^2} = \frac{1+\rho^2}{1-\rho^2} - \frac{1}{3}.$$

Therefore, the disk in (7) becomes

$$\left|\frac{zf'(z)}{f(z)} - a\right| = \left|\frac{zf'(z)}{f(z)} - \frac{1+\rho^2}{1-\rho^2}\right| \leq= \frac{1+\rho^2}{1-\rho^2} - \frac{1}{3} = a - \frac{1}{3}$$

and this disk is inside Ω_c. This shows that $\mathcal{S}_c^*$ radius is at least $R_{\mathcal{S}_c^*}$.
For the function f_0 given in (1), at $z = \rho = R_{\mathcal{S}_c^*}$, we have

$$\frac{zf_0'(z)}{f_0(z)} = \frac{1-6\rho+\rho^2}{1-\rho^2} = \frac{1}{3} = \varphi_c(-1) \in \partial\varphi_c(\mathbb{D})$$

where $\varphi_c(z) = 1 + 4z/3 + 2z^2/3$.

6. For $\rho = R_{\mathcal{S}^*_{\sin}}$, and $a := (1+r^2)/(1-r^2)$, we have

$$|a-1| = \frac{2\rho^2}{1-\rho^2} \approx 0.13199 < \sin 1 \approx 0.8414.$$

and

$$\frac{6\rho}{1-\rho^2} \leq \sin 1 - \frac{2\rho^2}{1-\rho^2}.$$

The disk in (7) for $r = \rho$ becomes

$$\left|\frac{zf'(z)}{f(z)} - a\right| = \left|\frac{zf'(z)}{f(z)} - \frac{1+\rho^2}{1-\rho^2}\right| \leq \sin 1 - \frac{2\rho^2}{1-\rho^2} = \sin 1 - |1-a|.$$

Lemma 5 shows that the disk in (7) is inside Ω_s where $\Omega_s =: \varphi_s(\mathbb{D})$ is the image of the unit disk $\mathbb{D}$ under the mapping $\varphi_s(z) = 1+\sin z$. This proves that the $\mathcal{S}^*_{\sin}$ radius is at least $R_{\mathcal{S}^*_{\sin}}$. For the function f_0 given in (1), with $\rho = R_{\mathcal{S}^*_{\sin}}$, we have

$$\left(\frac{zf'(z)}{f(z)}\right) = \frac{1+6\rho+\rho^2}{1-\rho^2} = 1+\sin 1 \in \varphi_s(1) \in \partial\varphi_s(\mathbb{D}).$$

7. For $\rho = R_{\mathcal{S}^*_{\leftmoon}}$, we have

$$a := \frac{1+\rho^2}{1-\rho^2} \approx 1.0202 \in (\sqrt{2}-1, \sqrt{2}+1)$$

and

$$\frac{1-6\rho+\rho^2}{1-\rho^2} = \sqrt{2}-1.$$

The disk in (7) becomes

$$\left|\frac{zf'(z)}{f(z)} - a\right| \leq 1 - |\sqrt{2}-a|$$

and by Lemma 6 it lies inside $\{w : |w^2-1| < 2|w|\}$. This shows that $\mathcal{S}^*_{\leftmoon}$ radius is at least $R_{\mathcal{S}^*_{\leftmoon}}$. The sharpness follows as the function f_0 defined in (1) satisfies, at $z = \rho = R_{\mathcal{S}^*_{\leftmoon}}$,

$$\begin{aligned}\left|\left(\frac{zf_0'(z)}{f_0(z)}\right)^2 - 1\right| &= \left|\left(\frac{1-6\rho+\rho^2}{1-\rho^2}\right)^2 - 1\right| = 2(\sqrt{2}-1) \\ &= 2\frac{1-6\rho+\rho^2}{1-\rho^2} = 2\left|\frac{zf_0'(z)}{f_0(z)}\right|.\end{aligned}$$

8. For $\rho = R_{\mathcal{S}^*_R}$, we have

$$2(\sqrt{2}-1) < a := \frac{1+\rho^2}{1-\rho^2} \approx 1.00167 \leq \sqrt{2} < 2,$$

and

$$\frac{1-6\rho+\rho^2}{1-\rho^2} = 2-2\sqrt{2}.$$

The disk (7) becomes

$$\left|\frac{zf'(z)}{f(z)} - a\right| < a - 2(\sqrt{2}-1)$$

and this disk, by Lemma 7, lies inside the domain Ω_r. This proves that $\mathcal{S}^*_R$ radius is at least $R_{\mathcal{S}^*_R}$. To prove sharpness, consider the function $f_0 \in \Pi_1$ given in (1). At $z = -\rho = -R_{\mathcal{S}^*_R}$, we have

$$\frac{zf'(z)}{f(z)} = \frac{1-6\rho+\rho^2}{1-\rho^2} = 2(\sqrt{2}-1) = \varphi_r(-1) \in \partial\varphi_r(\mathbb{D})$$

where $\varphi_r(z) = 1 + (kz+z^2)/(k^2-kz)$, $k = \sqrt{2}+1$.

■

3 Radius Results for the Class Π_2

The functions $f_2, f_3 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_2(z) = \frac{z}{1-z} \quad \text{and} \quad f_3(z) = \frac{z(1+z)^2}{(1-z)^3}, \tag{12}$$

satisfy the conditions $|f_i(z)/g_i(z) - 1| < 1$ and $\mathrm{Re}((1-z)^2 g_i(z)) > 0$ for $i = 2, 3$ with $g_2, g_3 : \mathbb{D} \to \mathbb{C}$ defined by

$$g_2(z) = \frac{z}{1-z^2} \quad \text{and} \quad g_3(z) = \frac{z(1+z)}{(1-z)^3},$$

and hence $f_2, f_3 \in \Pi_2$. This proves that the class Π_2 is non-empty. The Taylor series $f_3(z) = z + 5z^2 + 13z^3 + 25z^4 + \cdots$ shows that it is not univalent. It is an extremal function for the radius problems we consider. The derivative of f_3 is given by

$$f_3'(z) = \frac{(1+5z)(1+z)}{(1-z)^4}.$$

Since $f_3'(-1/5) = 0$ and, by Theorem 2(1), the radius of starlikeness of the class Π_1 is $1/5$, it follows that the radius of univalence of this class is also $1/5$. The other radius results for class Π_2 are given in the following theorem.

Theorem 2 *The following radius results hold for the class Π_2:*

(1) The $\mathcal{S}^*(\alpha)$ radius is $R_{\mathcal{S}^*(\alpha)} = 2(1-\alpha)/(5 + \sqrt{4\alpha^2 - 4\alpha + 25})$, $\quad 0 \le \alpha < 1$.

(2) The $\mathcal{S}^*_L$ radius is at least $R_{\mathcal{S}^*_L} = (\sqrt{4\sqrt{2} + 25} - 5)/(2(\sqrt{2} + 2)) \approx 0.0786$.

(3) The $\mathcal{S}_p$ radius is $R_{\mathcal{S}_p} = 5 - 2\sqrt{6} \approx 0.1010$.

(4) The $\mathcal{S}^*_e$ radius is $\mathcal{S}^*_e = (2(e-1))/(5e + \sqrt{25e^2 - 4e + 4}) \approx 0.1276$.

(5) The $\mathcal{S}^*_c$ radius is $R_{\mathcal{S}^*_c} = \left(15 - \sqrt{217}\right)/2 \approx 0.1345$.

(6) The $\mathcal{S}^*_{\sin}$ radius is at least $\mathcal{S}^*_{\sin} = (\sqrt{25 + 4(3 + \sin(1))\sin(1)} - 5)/(2(3 + \sin(1))) \approx 0.1508$.

(7) The $\mathcal{S}^*_{\leftmoon}$ radius is $R_{\mathcal{S}^*_{\leftmoon}} = (5 - \sqrt{41 - 12\sqrt{2}})/(2\left(\sqrt{2} - 1\right)) \approx 0.1183$.

(8) The $\mathcal{S}^*_R$ radius is $R_{\mathcal{S}^*_R} = (5 - \sqrt{81 - 40\sqrt{2}})/(4\left(\sqrt{2} - 1\right)) \approx 0.0345$.

It is worth to point out that $R_{\mathcal{S}^*_P} = R_{\mathcal{S}^*(1/2)}$ and $R_{\mathcal{S}^*_e} = R_{\mathcal{S}^*(1/e)}$ in both theorems.

Proof. Since $|w - 1| < 1$ is equivalent to $\mathrm{Re}(1/w) > 1/2$, the condition $|f(z)/g(z) - 1| < 1$ is the same as the condition $\mathrm{Re}(g(z)/f(z)) > 1/2$. Let the function $f \in \Pi_2$. Let $g : \mathbb{D} \to \mathbb{C}$ be chosen such that

$$\mathrm{Re}\left(\frac{g(z)}{f(z)}\right) > \frac{1}{2} \quad \text{and} \quad \mathrm{Re}\left(\frac{(1-z)^2}{z} g(z)\right). \tag{13}$$

Define $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ by

$$p_1(z) = \frac{(1-z)^2}{z} g(z), \quad \text{and} \quad p_2(z) = \frac{g(z)}{f(z)}. \tag{14}$$

From (13) and (14), it follows that the function $p_1 \in \mathcal{P}$, the function $p_2 \in \mathcal{P}(1/2)$, and $f(z) = (z/(1-z)^2)p_1(z)/p_2(z)$. A calculation shows that

$$\frac{zf'(z)}{f(z)} = \frac{zp_1'(z)}{p_1(z)} - \frac{zp_2'(z)}{p_2(z)} + \frac{1+z}{1-z}. \tag{15}$$

The bilinear transformation $\omega = (1+z)/(1-z)$ maps the disk $|z| \le r$ onto disk

$$\left| \frac{1+z}{1-z} - \frac{1+r^2}{1-r^2} \right| \le \frac{2r}{1-r^2}. \tag{16}$$

Using (16) and (6) in (15), we get

$$\left| \frac{zf'(z)}{f(z)} - \frac{1+r^2}{1-r^2} \right| \le \frac{5r+r^2}{1-r^2}. \tag{17}$$

From (17), it follows that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) \ge \frac{1-5r}{1-r^2} \ge 0 \tag{18}$$

for $0 \le r \le 1/5$. For the function f_3 given in (12), we have

$$\frac{zf_3'(z)}{f_3(z)} = \frac{1+5z}{1-z^2} = 0$$

for $z = -1/5$. Thus, the radius of starlikeness of the class Π_2 is $1/5$. All radius values to be computed here will be less or equal to $1/5$.

1. The number $\rho := R_{\mathcal{S}^*(\alpha)}$ is the smallest positive root of the equation $\alpha r^2 - 5r + 1 - \alpha = 0$. For $0 < r \le R_{\mathcal{S}^*(\alpha)}$, from (18), we have

$$Re\left(\frac{zf'(z)}{f(z)} \right) \ge \frac{1-5r}{1-r^2} \ge \frac{1-5\rho}{1-\rho^2} = \alpha.$$

For the function $f_3 \in \Pi_2$ given in (12), we have, at $z = -\rho = -R_{\mathcal{S}^*(\alpha)}$,

$$\frac{zf_3'(z)}{f_3(z)} = \frac{1-5\rho}{1-\rho^2} = \alpha.$$

This proves that the radius of starlikeness of order α is $R_{\mathcal{S}^*(\alpha)}$.

2. From (17), it follows that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le \left| \frac{zf'(z)}{f(z)} - \frac{1+r^2}{1-r^2} \right| + \frac{2r^2}{1-r^2} \le \frac{5r+3r^2}{1-r^2}. \tag{19}$$

The number $\rho = R_{\mathcal{S}_L^*}$ is the positive root of the equation

$$5r + 3r^2 - (1-r^2)(\sqrt{2}-1) = 0.$$

For $0 < r \le \rho = R_{\mathcal{S}_L^*}$, we have

$$\frac{5r+3r^2}{1-r^2} \le \frac{5\rho+3\rho^2}{1-\rho^2} = \sqrt{2}-1. \tag{20}$$

Therefore, by (19) and (20), it follows for $0 < r \le \rho = R_{\mathcal{S}_L^*}$ that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le \sqrt{2}-1, \tag{21}$$

and

$$\left| \frac{zf'(z)}{f(z)} + 1 \right| \le \sqrt{2}+1. \tag{22}$$

The last two inequalities (21) and (22) immediately yield

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| \leq \left|\frac{zf'(z)}{f(z)} + 1\right| \left|\frac{zf'(z)}{f(z)} - 1\right| \leq (\sqrt{2}+1)(\sqrt{2}-1) = 1.$$

This proves that $\mathcal{S}_L^*$ is at least $R_{\mathcal{S}_L^*}$.

3. For $0 \leq r \leq \rho := R_{\mathcal{S}_P} = 5 - 2\sqrt{6}$, we have for

$$\frac{1}{2} < 1 \leq a = \frac{1+\rho^2}{1-\rho^2} = \frac{5\sqrt{6}}{12} < 3/2$$

and, ρ being the smallest positive root of the equation $r^2 - 10r + 1 = 0$,

$$\frac{5\rho+\rho^2}{(1-\rho^2)} \leq \frac{1+\rho^2}{1-\rho^2} - \frac{1}{2}.$$

The disk in (17) becomes

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+\rho^2}{1-\rho^2}\right| \leq \frac{1+\rho^2}{1-\rho^2} - \frac{1}{2}.$$

By Lemma 2, the disk in (17) is inside the region Ω_{PAR}. Thus, the radius of parabolic starlikeness of the class Π_2 is at least $R_{\mathcal{S}_p}$.

For the function f_3 given in (12) at $z = -\rho$ where $\rho = R_{\mathcal{S}_p}$, we have

$$Re\left(\frac{zf_3'(z)}{f_3(z)}\right) = \frac{1-5\rho}{1-\rho^2} = \frac{5\rho-\rho^2}{1-\rho^2} = \left|\frac{zf_3'(z)}{f_3(z)} - 1\right|.$$

4. For $\rho = R_{\mathcal{S}_e^*}$, we have $1/e < a := (1+\rho^2)/(1-\rho^2) \approx 1.0331 \leq (e+e^{-1})/2$ and

$$\frac{5\rho+\rho^2}{1-\rho^2} = \frac{1+\rho^2}{1-\rho^2} - \frac{1}{e}.$$

The disk in (17) becomes

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+\rho^2}{1-\rho^2}\right| \leq \frac{1+\rho^2}{1-\rho^2} - \frac{1}{e}.$$

By Lemma 3, this disk is inside the region Ω_e, proving that $\mathcal{S}_e^*$ radius is at least $R_{\mathcal{S}_e^*}$.

The result is sharp for the function f_3 given in (12). For this function, we have, at $z = -\rho$ where $\rho = R_{\mathcal{S}_e^*}$,

$$\left|\log\left(\frac{zf_3'(z)}{f_3(z)}\right)\right| = \left|\log\left(\frac{1-5\rho}{1-\rho^2}\right)\right| = \left|\log(e^{-1})\right| = 1.$$

5. For $\rho = R_{\mathcal{S}_c^*}$, we have $1/3 < a := (1+\rho^2)/(1-\rho^2) = \frac{1}{72}(1+5\sqrt{217}) \approx 1.03686 \leq 5/2$ and, ρ being the smallest positive root of $r^2 - 15r + 2 = 0$,

$$\frac{5\rho+\rho^2}{1-\rho^2} = \frac{1+\rho^2}{1-\rho^2} - \frac{1}{3}.$$

The disk in (17) becomes

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+\rho^2}{1-\rho^2}\right| \leq \frac{1+\rho^2}{1-\rho^2} - \frac{1}{3}.$$

By Lemma 3, this disk is inside the region Ω_c, proving that $\mathcal{S}_c^*$ radius is at least $R_{\mathcal{S}_c^*}$.
The radius is sharp for the function f_3 given in (12). At $z = -\rho$ where $\rho = R_{\mathcal{S}_c^*}$, we have

$$\frac{zf_3'(z)}{f_3(z)} = \frac{1-5\rho}{1-\rho^2} = \frac{1}{3} = \varphi_c(-1) \in \partial\varphi_c(\mathbb{D})$$

where $\varphi_c(z) = 1 + 4z/3 + 2z^2/3$.

6. For $\rho = R_{\mathcal{S}_{\sin}^*}$, and $a := (1+\rho^2)/(1-\rho^2)$, we have

$$|a-1| = \frac{2\rho^2}{1-\rho^2} \approx 0.0465396 < \sin 1 \approx 0.8414.$$

and

$$\frac{5\rho+\rho^2}{1-\rho^2} \leq \sin 1 - \frac{2\rho^2}{1-\rho^2}.$$

The disk in (7) for $r = \rho$ becomes

$$\left|\frac{zf'(z)}{f(z)} - a\right| = \left|\frac{zf'(z)}{f(z)} - \frac{1+\rho^2}{1-\rho^2}\right| \leq \sin 1 - \frac{2\rho^2}{1-\rho^2} = \sin 1 - |1-a|.$$

Lemma 5 shows that the disk in (17) is inside Ω_s where $\Omega_s =: \varphi_s(\mathbb{D})$ is the image of the unit disk $\mathbb{D}$ under the mapping $\varphi_s(z) = 1 + \sin z$. This proves that the $\mathcal{S}_{\sin}^*$ radius is at least $R_{\mathcal{S}_{\sin}^*}$.

7. For $\rho = R_{\mathcal{S}_{☾}^*}$, we have

$$a := \frac{1+\rho^2}{1-\rho^2} \approx 1.02839 \in (\sqrt{2}-1, \sqrt{2}+1)$$

and

$$\frac{5\rho+\rho^2}{1-\rho^2} = \frac{1+\rho^2}{1-\rho^2} + 1 - \sqrt{2}.$$

The disk in (17) becomes

$$\left|\frac{zf'(z)}{f(z)} - a\right| \leq 1 - |\sqrt{2} - a|$$

and by Lemma 6 it lies inside $\{w : |w^2 - 1| < 2|w|\}$. This shows that $\mathcal{S}_{☾}^*$ radius is at least $R_{\mathcal{S}_{☾}^*}$. The sharpness follows as the function f_3 defined in (12) satisfies, at $z = \rho = R_{\mathcal{S}_{☾}^*}$,

$$\begin{aligned}\left|\left(\frac{zf_3'(z)}{f_3(z)}\right)^2 - 1\right| &= \left|\left(\frac{1-5\rho}{1-\rho^2}\right)^2 - 1\right| = 2(\sqrt{2}-1)\\ &= 2\frac{1-5\rho}{1-\rho^2} = 2\left|\frac{zf_3'(z)}{f_3(z)}\right|.\end{aligned}$$

8. For $\rho = R_{\mathcal{S}_R^*}$, we have

$$2(\sqrt{2}-1) < a := \frac{1+\rho^2}{1-\rho^2} \approx 1.00238 \leq \sqrt{2} < 2,$$

and

$$\frac{5\rho+\rho^2}{1-\rho^2} = \frac{1+\rho^2}{1-\rho^2} - 2(\sqrt{2}-1).$$

The disk (17) becomes

$$\left|\frac{zf'(z)}{f(z)} - a\right| < a - 2(\sqrt{2}-1).$$

By Lemma 7, this disk lies inside the domain Ω_r. This proves that $\mathcal{S}^*_R$ radius is at least $R_{\mathcal{S}^*_R}$.

To prove sharpness, consider the function $f_3 \in \Pi_2$ given in (12). At $z = -\rho = -R_{\mathcal{S}^*_R}$, we have

$$\frac{zf_3'(z)}{f_3(z)} = \frac{1-5\rho}{1-\rho^2} = 2(\sqrt{2}-1) = \varphi_r(-1) \in \partial\varphi_r(\mathbb{D})$$

where $\varphi_r(z) = 1 + (kz+z^2)/(k^2-kz)$, $k = \sqrt{2}+1$.

■

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