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Relationships Between Pell And Pell-Lucas Polynomials And Extended Hecke Groups*

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Abstract

In this paper, we consider Pell and Pell-Lucas polynomials. Using the roots of these polynomials, we obtain some astounding relationships between these polynomials and the extended modular group, extended Hecke groups, and extended generalized Hecke groups with geometric interpretations. We define new sequences derived from these relations.

1 Introduction

Fibonacci, Lucas, Pell, and Pell-Lucas polynomials are families of orthogonal polynomials and they may be expressed recursively. These polynomials are widely used in the study of many topics such as number theory, combinatorics, algebra, approximation theory, geometry, and graph theory (see [1, 2, 3, 4]). These polynomials satisfy the following properties (see e.g. [1, 2, 5, 6]). It is well-known that Fibonacci polynomials, $F_n(x)$ are defined by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 3 \tag{1}$$

where $F_1(x) = 1$ and $F_2(x) = x$. Lucas polynomials are defined as

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), \quad n \geq 3 \tag{2}$$

where $L_1(x) = x$ and $L_2(x) = x^2 + 2$. Pell polynomials, $P_n(x)$, are defined as follows.

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \quad n \geq 2 \tag{3}$$

where $P_0(x) = 0$ and $P_1(x) = 1$. Pell-Lucas polynomials are defined by

$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x), \quad n \geq 2 \tag{4}$$

where $Q_0(x) = 2$ and $Q_1(x) = 2x$. The sum of the coefficients of $F_n(x)$ is n^{th} Fibonacci number F_n. That is, $F_n(1) = F_n$. Similarly, Lucas, Pell, and Pell-Lucas numbers obtained via $L_n(1) = L_n$, $P_n(1) = P_n$, and $Q_n(1) = Q_n$. For all $n \geq 1$, $\deg[F_n(x)] = n - 1$, $\deg[L_n(x)] = n$, $\deg[P_n(x)] = n - 1$, and $\deg[Q_n(x)] = n$. Some well-known relationships among these polynomials are recalled as follows.

$$L_n(x) = F_{n-1}(x) + F_{n+1}(x), \tag{5}$$

$$(x^2 + 4)F_n^2(x) = L_{n-1}(x) + L_{n+1}(x), \tag{6}$$

$$Q_n(x) = P_{n-1}(x) + P_{n+1}(x), \tag{7}$$

$$4(x^2 + 1)P_n^2(x) = Q_{n-1}(x) + Q_{n+1}(x). \tag{8}$$

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Using generating functions and solving recurrences, these polynomials are explicitly given by the Binet-type formulas

$$F_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \ , \ L_n(x) = \alpha^n(x) + \beta^n(x) \tag{9}$$

where $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$.

$$P_n(x) = \frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)} \ , \ Q_n(x) = \gamma^n(x) + \delta^n(x) \tag{10}$$

where $\gamma(x) = x + \sqrt{x^2+1}$ and $\delta(x) = x - \sqrt{x^2+1}$.

The ratio of two successive polynomials of Fibonacci and Lucas families converges to the Golden Section, which appears in many fields in the literature, such as nature, art, architecture, biology, physics, chemistry, cosmos, theology, finance, and so on (see, e.g., [1, 7, 8, 9, 10, 11]). Furthermore, the ratio of two consecutive polynomials of Pell and Pell-Lucas families converges to Silver Mean. The ratio is another member of the class of metallic means defined by Spinadel, apart from the Golden Mean. Other metallic means with special naming are Bronze Mean and Cooper Mean (see [12]). There are many attractive studies on different aspects related to the number sequences, polynomials, and metallic means mentioned above (see [2, 6, 13, 14, 15, 16, 17, 18, 19] for more details). In [20], V. E. Hoggat and M. Bicknell obtain the roots of large classes of Fibonacci and Lucas polynomials using hyperbolic trigonometric functions. Therefore, the general root formulas for the polynomials have been obtained. This contribution is quite remarkable considering the Abel-Ruffini theorem. There are many papers on the topic from different viewpoints (see [1, 5, 21, 22, 23, 24, 25, 26, 27]). Also, in [28], P. F. Byrd studied hyperbolic function representations of Pell polynomials called Fibonacci polynomials at that time. Pell and Pell-Lucas polynomials have been extensively studied from many points of view in the literature. See [2, 3, 4, 6, 16] for more details. Furthermore, in [29], [30], Birol et al. obtain the general root formula of Pell and Pell-Lucas polynomials using hyperbolic trigonometric functions.

In [29], the roots of Pell polynomials $P_n(x)$ are given as follows:

$$x = i \cos \frac{k\pi}{n}, \ k = 1, 2, \ldots, n-1. \tag{11}$$

In [30], the roots of Pell-Lucas polynomials $Q_n(x)$ are given as follows:

$$x = i \cos \frac{(2k+1)\pi}{2n}, \ k = 0, 1, \ldots, n-1. \tag{12}$$

On the other hand, in [31], extended Hecke groups $\overline{H}(\lambda_q)$ are defined analogously with the extended modular group. The groups are generated by three Möbius transformations

$$T(z) = -\frac{1}{z} \ , \ U(z) = z + \lambda \text{ and } R(z) = -\frac{1}{\overline{z}}$$

where $\lambda_q = 2\cos\frac{\pi}{q}, q \in \mathbb{N}$. Let $S = TU$, i.e.

$$S(z) = -\frac{1}{z+\lambda}.$$

The Möbius transformation

$$T(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0, \quad a, b, c, d \in \mathbb{C}),$$

where $\mathbb{C}$ denotes the set of complex numbers. Here the restriction $ad - bc \neq 0$ is necessary because, if $ad - bc = 0$, it is easy to see that $T(z) = \frac{az+b}{cz+d}$ reduces to a constant. Let

$$G = \left\{ T(z) = \frac{az+b}{cz+d} : ad - bc \neq 0, \quad a, b, c, d \in \mathbb{C} \right\}.$$

Then $(G,\circ)$ becomes a group where the operation $\circ$ is a function composition. On the other hand, let

$$M_{2x2}=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad-bc\neq 0,\quad a,b,c,d\in\mathbb{C}\right\}.$$

Now it is easy to see that these two groups are isomorphic, that is, $(G,\circ)\cong(M_{2x2},\bullet)$.

Identifying the transformation $\frac{az+b}{cz+d}$ with the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is related to a multiplicative group of 2×2 matrices in which a matrix is identified with its negative. Notice that T, S, and R have matrix representations

$$T=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ S=\begin{bmatrix} 0 & -1 \\ 1 & \lambda \end{bmatrix} \text{ and } R=\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},$$

respectively. The extended Hecke group $\overline{H}_q=\overline{H}(\lambda_q)$ has the presentation

$$\overline{H}(\lambda_q)=\langle T,S,R \mid T^2=S^q=R^2=(TR)^2=(SR)^2=I\rangle\cong D_2 *_{\mathbb{Z}_2} D_q.$$

Several of these groups are $\overline{H}(\lambda_3)=\overline{\Gamma}=PGL(2,\mathbb{Z})$ (the extended modular group), $\overline{H}(\sqrt{2})$, $\overline{H}(\frac{1+\sqrt{5}}{2})$, and $\overline{H}(\sqrt{3})$.

In [32], a more general class $\overline{H}_{p,q}$ of extended Hecke groups $\overline{H}(\lambda_q)$ is introduced by taking

$$X(z)=\frac{-1}{z-\lambda_p},\ V(z)=z+\lambda_p+\lambda_q \text{ and } R(z)=-\frac{1}{\overline{z}}$$

where $2\leq p\leq q,\ p+q>4$. Here if we take $Y=XV=\frac{-1}{z+\lambda_q}$, then the group presentation is

$$\overline{H}_{p,q}=\langle X,Y,R \mid X^p=Y^q=R^2=(XR)^2=(YR)^2=I\rangle\cong D_p *_{\mathbb{Z}_2} D_q.$$

The group $\overline{H}_{p,q}$ is named as extended generalized Hecke group. Also $\overline{H}_{2,q}=\overline{H}_q$. Furthermore, all extended Hecke groups $\overline{H}_q$ are included in extended generalized Hecke groups $\overline{H}_{2,q}$. The extended modular group, extended Hecke groups, and extended generalized Hecke groups have been studied considerably (see [31, 32, 33, 34, 35, 36, 37] for more details). Also, there are many notable studies on $2\cos\frac{\pi}{q}$ and $\cos\frac{2\pi}{q}$ in the literature. Moreover, finding the minimal polynomial of $\cos\frac{2\pi}{q}$ is an old problem due to its connection to the cyclotomic polynomials. The algebraic numbers are examined in many papers related to Chebyshev polynomials, Gaussian periods, Dickson polynomials, Ramanujan sums, and Möbius inversion (see [38, 39, 40, 41] for more details).

There are strong connections between extended Hecke groups and the recurrence number sequences that are Fibonacci, Lucas, Pell, and Pell-Lucas, see [33, 34, 36].

In this study, we focus on the roots of Pell and Pell-Lucas polynomials. We get strong relationships between the roots of Pell and Pell-Lucas polynomials and the extended modular group, extended Hecke groups, extended generalized Hecke groups.

2 Relationships Between The Roots of Pell Polynomials and The Extended Modular Group, Extended Hecke Groups and Extended Generalized Hecke Groups

In this section, we study the complex numbers as vectors in the complex plane. All the roots of Pell polynomials are pure imaginary complex numbers. The norm of each root of a Pell polynomial is smaller than one. We clarify the roots in the complex plane as related to the parameter of the extended modular group, extended Hecke groups, and extended generalized Hecke groups.

Corollary 1 *We clarify the relationship between the parameter of the extended modular group and roots of Pell polynomial in the complex plane geometrically. The parameter of the extended modular group is* $\lambda_3 = 2\cos\frac{\pi}{3}$*. All of the roots of Pell polynomial* $P_3(x)$ *are known as* $i\cos\frac{\pi}{3}$ *and* $i\cos\frac{2\pi}{3}$ *from Equation 11 for* $k = 1, 2$*. Firstly, we consider of the roots as vectors and we rotate the root 270 degrees counterclockwise around the origin in the complex plane. Thus, we get* $\cos\frac{\pi}{3}$ *via the first root. Later, we apply to double the norm of the vector. Consequently, here* $2\cos\frac{\pi}{3}$ *is obtained. Therefore, we can state that the Pell polynomial* $P_3(x)$ *generates a parameter for the extended modular group as a geometric interpretation.*

The parameter of extended Hecke group as Fuchsian group of first kind is $\lambda_q = 2\cos\frac{\pi}{q}$ for $q \geq 3$ and all of the roots of Pell polynomial $P_q(x)$ known as $i\cos\frac{k\pi}{q}$ from Equation 11 for $k = 1, 2, ...q-1$. If the first root $i\cos\frac{\pi}{q}$ of the Pell polynomial $P_q(x)$ is rotated 270 degrees counterclockwise around the origin in the complex plane and double the norm of the vector the parameter of the extended Hecke group $\overline{H}(\lambda_q)$ is obtained. Therefore, we can state geometrically that the Pell polynomial $P_q(x)$ generates a parameter for the extended Hecke group $\overline{H}(\lambda_q)$ as Fuchsian group of the first kind.

Parameters of the extended generalized Hecke groups $\overline{H}_{p,q}$ are $\lambda_p = 2\cos\frac{\pi}{p}$ and $\lambda_q = 2\cos\frac{\pi}{q}$. Also, all roots of Pell polynomial $P_p(x)$ known as $i\cos\frac{k\pi}{p}$ for $k = 1, 2, ...p-1$. And all roots of Pell polynomial $P_q(x)$ known as $i\cos\frac{k\pi}{q}$ for $k = 1, 2, ...q-1$ from Equation 11. If the first roots $i\cos\frac{\pi}{p}$ of the Pell polynomial $P_p(x)$ and $i\cos\frac{\pi}{q}$ of the Pell polynomial $P_q(x)$ are rotated 270 degrees counterclockwise around the origin in the complex plane and double the norm of these vectors, the parameters of the extended generalized Hecke groups $\overline{H}_{p,q}$ are obtained. Therefore, we can state geometrically that the Pell polynomial $P_p(x)$ and $P_q(x)$ generate parameters for the extended generalized Hecke groups.

Remark 1 *The parameter of the extended Hecke group can not be derived from a unique Pell polynomial. For instance, the parameter of the extended Hecke group* $\overline{H}_3 = \overline{\Gamma}$ *named the extended modular group is obtained from the Pell polynomial* $P_3(x)$*. Here the first root of the polynomial* $P_3(x)$ *is obtained as* $i\cos\frac{\pi}{3}$ *from Equation 11 for* $k = 1$*. Also, the parameter is obtained from another Pell polynomial* $P_6(x)$ *using Equation 11 for* $k = 2$*. Therefore, we can state the parameter of the extended Hecke group* $\overline{H}_3$ *related to the Pell polynomials* $P_{3m}(x)$ *when* $m \in \mathbb{N}$*. More generally, the parameter of the extended Hecke group* $\overline{H}_l$ *can be derived from Pell polynomials* $P_{ls}(x)$ *when* $s \in \mathbb{N}$*.*

Remark 2 *Every Pell polynomial* $P_n(x)$ *for* $n \geq 3$ *generates at least one parameter for the extended Hecke group. For example, the Pell polynomial* $P_7(x)$ *generates one parameter as* $2\cos\frac{\pi}{7}$ *via the root* $i\cos\frac{\pi}{7}$ *rotated* 270 *degrees counterclockwise around the origin in the complex plane and double the norm of the vector. The Pell polynomial* $P_8(x)$ *generates two parameters as* $2\cos\frac{\pi}{8}$ *and* $2\cos\frac{\pi}{4}$ *via the roots* $i\cos\frac{\pi}{8}$ *and* $i\cos\frac{2\pi}{8}$ *rotated* 270 *degrees counterclockwise around the origin in the complex plane and double the norm of these vectors. The Pell polynomial* $P_{16}(x)$ *generates three parameters as* $2\cos\frac{\pi}{16}$*,* $2\cos\frac{\pi}{8}$*, and* $2\cos\frac{\pi}{4}$ *via the roots* $i\cos\frac{\pi}{16}$*,* $i\cos\frac{2\pi}{16}$*, and* $i\cos\frac{4\pi}{16}$ *rotated* 270 *degrees counterclockwise around the origin in the complex plane and double the norm of these vectors.*

We set a general way to get the relationship between the parameter of the extended Hecke group as Fuchsian group of the first kind and Pell polynomial $P_n(x)$. All the roots of Pell polynomial $P_n(x)$ known as $i\cos\frac{k\pi}{n}$ for $k = 1, 2, ...n-1$. $P_n(x)$ generates parameter for the extended Hecke group every provided condition that k divides n except for $k = \frac{n}{2}$ and $k = n$. For instance, $P_6(x) = 32x^5 + 32x^3 + 6x$ generates exactly two parameters for extended Hecke groups denoted by $\overline{H}_6$ and $\overline{H}_3 = \overline{\Gamma}$.

Theorem 1 (Birol-Extended Hecke-Pell Theorem) *The number of the parameters for the extended Hecke groups generated by* $P_n(x)$ *is calculated by the formula*

$$T(n) = \begin{cases} \prod_{i=1}^{t}(a_i+1) - 2 & \textit{if } n \textit{ even}, \\ \prod_{i=1}^{t}(a_i+1) - 1 & \textit{if } n \textit{ odd}, \end{cases}$$

where $n = \prod_{i=1}^{t} p_i^{a_i}$ *for* p_i *distinct prime numbers and* a_i *positive integers.*

Proof. It can be proved using the fundamental theorem of arithmetic, the formula for the total number of divisors of a number considering the root formula of Pell polynomials and the parameter of the extended Hecke group as $\lambda_q = 2\cos\frac{\pi}{q}, q \in \mathbb{N}, q \geq 3$. ■

Considering the polynomial space, the $\{P_n(x) : n \geq 3\}$ set of Pell polynomial is a relation with the ability to generate any common parameter for the extended Hecke groups. This relation has reflection and symmetry properties.

Remark 3 *We call the above relation as* θ*. Notice that,* θ *is not reflexive relation. We give a counterexample to prove that.* $\Big(P_3(x), P_{33}(x)\Big) \in \theta$ *via* $\overline{\Gamma}$ *and* $\Big(P_{33}(x), P_{11}(x)\Big) \in \theta$ *via* $\overline{H}_{11}$ *but* $\Big(P_3(x), P_{11}(x)\Big) \notin \theta$*. Although* $P_3(x)$ *and* $P_{11}(x)$ *generate one parameter for the extended Hecke groups, these polynomials do not generate a common parameter for any extended Hecke group.* $P_3(x)$ *and* $P_{11}(x)$ *generate a parameter for the extended Hecke groups* $\overline{\Gamma}$ *and* $\overline{H}_{11}$*, respectively.*

(Birol-Extended Hecke-Pell Number Sequence) We define a new number sequence derived from Theorem 1. This sequence shows the relationship between the root of the Pell polynomial and the extended Hecke groups interestingly. The third, the fourth and the thirteenth terms of the number sequence are obtained as 1, 2, and 3 from Pell polynomials $P_5(x)$, $P_6(x)$, and $P_{15}(x)$, respectively. This sequence is as follows

$$1,\ 1,\ 1,\ 2,\ 1,\ 2,\ 2,\ 2,\ 1,\ 4,\ 1,\ 2,\ 3,\ 3,\ 1,\ 4,\ 1,\ 4,\ 3, ...$$

3 Relationships Between The Roots of Pell-Lucas Polynomials and The Extended Modular Groups, Extended Hecke Groups and Extended Generalized Hecke Groups

In this section, we consider the complex numbers as vectors in the complex plane. All the roots of Pell-Lucas polynomials are pure imaginary complex numbers. Each norm of the roots of a Pell-Lucas polynomial is smaller than one. We interpret the roots in the complex plane as related to the parameter of the extended modular group, extended Hecke groups, and extended generalized Hecke groups.

We analyze the relationship between parameter of the extended modular group and roots of Pell-Lucas polynomial in the complex plane geometrically. The parameter of the extended modular group is $\lambda_3 = 2\cos\frac{\pi}{3}$. The roots of Pell-Lucas polynomial $Q_n(x)$ known as $i\cos\frac{(2k+1)\pi}{2n}$ for $k = 0, 1, ..., n-1$ from Equation 12. Firstly, we consider of these roots as vectors and we rotate the roots 270 degrees counterclockwise around the origin in the complex plane. Thus, we get $\cos\frac{(2k+1)\pi}{2n}$ for $k = 0, 1, ..., n-1$. Later, we apply to double the norm of these vectors. Consequently, here $2\cos\frac{(2k+1)\pi}{2n}$ for $k = 0, 1, ..., n-1$ are obtained. Now, we examine whether these vectors can coincide with the parameter of the extended modular group parameter. For any of these vectors to coincide with the extended modular group parameter must be $2n - 6k = 3$. However, it is clear that the equality is not possible considering the Equation 12. Therefore, we can state the Pell-Lucas polynomials $Q_n(x)$ do not generate a parameter for the extended modular group as a geometric interpretation.

The parameter of the extended Hecke group as Fuchsian group of first kind is $\lambda_q = 2\cos\frac{\pi}{q}$ for $q \geq 3$ and all roots of Pell-Lucas polynomial $Q_q(x)$ known as $i\cos\frac{(2k+1)\pi}{2q}$ for $k = 0, 1, ..., q-1$ from Equation 12. If the first root $i\cos\frac{\pi}{2q}$ of the Pell-Lucas polynomial $Q_q(x)$ is rotated 270 degrees counterclockwise around the origin in the complex plane and double the norm of these vectors the parameter of the extended Hecke group $\overline{H}(\lambda_{2q})$ is obtained. Therefore, we can state geometrically that the Pell-Lucas polynomial $Q_q(x)$ generates a parameter for the extended Hecke group $\overline{H}(\lambda_{2q})$ as Fuchsian group of first kind.

Parameters of the extended generalized Hecke groups $\overline{H}_{p,q}$ are $\lambda_p = 2\cos\frac{\pi}{p}$ and $\lambda_q = 2\cos\frac{\pi}{q}$. Also, all roots of Pell-Lucas polynomial $Q_p(x)$ known as $i\cos\frac{(2k+1)\pi}{2p}$ for $k = 0, 1, ..., p-1$. And all roots of Pell-Lucas polynomial $Q_q(x)$ known as $i\cos\frac{(2k+1)\pi}{2q}$ for $k = 0, 1, ..., q-1$. If the first roots $i\cos\frac{\pi}{2p}$ of the Pell-Lucas polynomial $Q_p(x)$ and $i\cos\frac{\pi}{2q}$ of the Pell-Lucas polynomial $Q_q(x)$ are rotated 270 degrees counterclockwise around the origin in the complex plane and double the norm of these vectors, the parameters of the extended generalized Hecke groups $\overline{H}_{2p,2q}$ are obtained. Therefore, we can state geometrically that the Pell-Lucas polynomial $Q_p(x)$ and $Q_q(x)$ generate parameters for the extended generalized Hecke groups $\overline{H}_{2p,2q}$.

Remark 4 *Every Pell-Lucas polynomial $Q_n(x)$ for $n \geq 2$ generates at least one parameter for the extended Hecke group. For example, the Pell-Lucas polynomial $Q_8(x)$ generates one parameter as $2\cos\frac{\pi}{16}$ via the root $i\cos\frac{\pi}{16}$ rotate 270 degrees counterclockwise around the origin in the complex plane and double the norm of the vector. The Pell-Lucas polynomial $Q_9(x)$ generates two parameters as $2\cos\frac{\pi}{18}$ and $2\cos\frac{\pi}{6}$ via the roots $i\cos\frac{\pi}{18}$ and $i\cos\frac{3\pi}{18}$ rotate 270 degrees counterclockwise around the origin in the complex plane and double the norm of these vectors. The Pell-Lucas polynomial $Q_{15}(x)$ generates three parameters as $2\cos\frac{\pi}{30}$, $2\cos\frac{\pi}{10}$, and $2\cos\frac{\pi}{6}$ via the roots $i\cos\frac{\pi}{30}$, $i\cos\frac{3\pi}{30}$, and $i\cos\frac{5\pi}{30}$ rotate 270 degrees counterclockwise around the origin in the complex plane and double the norm of these vectors.*

We set a general way to get the relationship between the parameter of the extended Hecke group as Fuchsian group of first kind and Pell-Lucas polynomial $Q_n(x)$. All roots of Pell-Lucas polynomial $Q_n(x)$ known as $i\cos\frac{(2k+1)\pi}{2n}$ for $k = 0, 1, ...n-1$. $Q_n(x)$ generates parameter for extended Hecke group every provided condition that $2k+1$ divides n except for $2k+1 = n$. For instance, $Q_7(x) = 128x^7 + 224x^5 + 112x^3 + 14x$ generates exactly one parameter for the extended Hecke group denoted by $\overline{H}_{14}$.

Remark 5 *The parameter of the extended Hecke group can not be derived from a unique Pell-Lucas polynomial. For instance, the parameter of the extended Hecke group $\overline{H}_6$ is obtained from the Pell-Lucas polynomial $Q_3(x)$. Here the first root of the polynomial $Q_3(x)$ obtained as $i\cos\frac{\pi}{6}$ from Equation 12 for $k = 0$. Also, the parameter is obtained from another Pell-Lucas polynomial $Q_9(x)$ using Equation 12 for $k = 1$. Therefore, we can state the parameter of the extended Hecke group $\overline{H}_6$ related to the Pell-Lucas polynomials $Q_{6s+3}(x)$ when s is a nonnegative integer. More generally, the parameter of the extended Hecke group $\overline{H}_m$ can be derived from Pell-Lucas polynomials $Q_{lm+\frac{m}{2}}$ when l is a whole number and m is an even positive integer.*

Theorem 2 (Turan & Extended Hecke & Pell-Lucas Theorem) *The number of parameters for extended Hecke groups generated by $Q_n(x)$ calculated by the formula*

$$K(n) = \begin{cases} \prod_{i=1}^{t}(a_i+1) & \text{if } n \text{ even}, \\ \prod_{i=1}^{t}(a_i+1) - 1 & \text{if } n \text{ odd}, \end{cases}$$

where $n = \prod_{i=1}^{t} 2^b.p_i^{a_i}$ for p_i distinct odd prime numbers, a_i positive integers and b a nonnegative integer, for $n \geq 2$.

Proof. It can be proven using the fundamental theorem of arithmetic, the formula for the total number of odd divisors of a number considering the root formula of Pell-Lucas polynomials and the parameter of the extended Hecke group as $\lambda_q = 2\cos\frac{\pi}{q}, q \in \mathbb{N}, q \geq 3$. ■

Considering the polynomial space, the $\{Q_n(x) : n \geq 2\}$ set of Pell-Lucas polynomial is a relation with the ability to generate any common parameter for the extended Hecke groups. This relation has reflection and symmetry properties.

Remark 6 *We call the above relation as ω. Notice that ω is not reflexive relation. We give a counterexample to prove that. $\Big(Q_3(x), Q_{21}(x)\Big) \in \omega$ via for $\overline{H}_6$ and $\Big(Q_{21}(x), Q_7(x)\Big) \in \omega$ via $\overline{H}_{14}$ but $\Big(Q_3(x), Q_7(x)\Big) \notin \omega$.*

Although $Q_3(x)$ and $Q_7(x)$ generate one parameter for the extended Hecke groups, these polynomials do not generate a common parameter for any extended Hecke group. $Q_3(x)$ and $Q_7(x)$ generate a parameter for the extended Hecke groups $\overline{H}_6$ and $\overline{H}_{14}$, respectively.

(Pell-Lucas Number Sequence) We define a new number sequence derived from the Theorem 2. This sequence shows the relationship between the root of the Pell-Lucas polynomial and the extended Hecke groups interestingly. The eighteenth, the nineteenth and the twentieth terms of the number sequence are obtained as 1, 2 and 3 from Pell-Lucas polynomials $Q_{19}(x)$, $Q_{20}(x)$ and $Q_{21}(x)$, respectively. This sequence is as follows:

$$1,\ 1,\ 1,\ 1,\ 2,\ 1,\ 1,\ 2,\ 2,\ 1,\ 2,\ 1,\ 2,\ 3,\ 1,\ 1,\ 3,\ 1,\ 2,\ 3,\ 2, \ldots$$

Remark 7 *Notice that each term of the extended Hecke & Pell-Lucas sequence coincides the number of odd proper divisors of n for $n \geq 2$.*

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