

# Quasi $p$ -Convex Functions\*

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## Abstract

In this paper, the concept of quasi  $p$ -convex function is given. Then fundamental characterizations and some operational properties of quasi  $p$ -convex functions which are analogue results to quasiconvexity are shown. Furthermore, the relations between quasi  $p$ -convex functions for different  $p$  values are obtained.

## 1 Introduction

Convex functions always attract special interest from different disciplines as well as mathematics. The property of a local extremum to be a global minimum for these functions makes them very requested instrument in handling optimization problems, which are encountered in almost every discipline of science. Overtime, the advances in science and curiosity of mathematicians lead to the appearance of different convexity types such as pseudoconvexity,  $p$ -convexity,  $s$ -convexity,  $B$ -convexity, etc. [1, 2, 14, 4, 5, 8, 9, 16, 18].

The most prominent of these generalizations are undoubtedly quasiconvex (also quasiconcave) functions. Quasiconvex functions are more general than convex functions. Although convex functions have convex sublevel sets, there are many nonconvex functions having convex sublevel sets. These functions comprise the class of the quasiconvex functions. The first tracks of quasiconvex functions are seen on the studies of Finetti in 1949 in an attempt to recognize some of characteristics of the functions having convex level sets, including utility functions [11]. Then, Fenchel was one of the pioneers in naming "quasi", formalizing and developing quasiconvex and quasiconcave functions [10]. Arrow-Enthoven in [3] presented an important study on application of quasiconcavity in utility functions in which he asserts that the fundamental minimal property of the utility function in the theory of consume demand is quasiconcavity. Some of the important functions in economics such as the constant elasticity of substitution function, the Leontief production function and the Cobb-Dougllass functions are quasiconcave functions. This convexity type has been one of the indispensable part of the mathematical economics theory. The historical development of the relation between quasiconcavity and economics and quick review of quasiconcave functions can be seen in [13] and [12], respectively.

In this study, a generalization of quasiconvex functions defined via  $p$ -convexity, namely, quasi  $p$ -convex function, is introduced. Then some characterizations are given.

The basic notion of  $p$ -convexity stems from  $p$ -convex sets, which has their origins in  $p$ -normed spaces [6, 7, 15, 17]. Then  $p$ -convex functions are defined on  $p$ -convex sets. Some properties and characterizations can be seen in [18]. Similarly, since quasi  $p$ -convex functions will be defined on  $p$ -convex set, let us recall some facts on  $p$ -convex sets and functions.

**Definition 1** ([6]) *Let  $U$  be a subset of  $n$ -dimensional Euclidean space,  $\mathbb{R}^n$  and  $0 < p \leq 1$ . If for each  $x, y \in U$ ,  $\lambda, \mu \geq 0$  such that  $\lambda^p + \mu^p = 1$ ,  $\lambda x + \mu y \in U$ , then  $U$  is called a  $p$ -convex set in  $\mathbb{R}^n$ .*

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The definition of  $p$ -convexity of  $U$  can also be given as

$$\lambda^{\frac{1}{p}}x + (1 - \lambda)^{\frac{1}{p}}y \in U$$

for all  $x, y \in U$  and  $\lambda \in [0, 1]$ .

In case of  $p = 1$  the definition of convex set is obtained. Contrary to convexity,  $p$ -convex combination of two points may not be a  $p$ -convex set, which does not form a line segment connecting  $x$  and  $y$ . In case that  $U$  is subset of real numbers  $\mathbb{R}$ , it is clear from the definition that the only singleton  $p$ -convex subset of  $\mathbb{R}$  is  $\{0\}$  ( $\{(0, 0, \dots, 0)\}$  for  $n$ -dimensional case) and also a  $p$ -convex subset of real numbers is an interval which includes zero or accepts zero as an endpoint of the interval.

**Definition 2** ([18]) *Let  $U \subseteq \mathbb{R}^n$  be a  $p$ -convex set. The function  $f : U \rightarrow \mathbb{R}$  is called a  $p$ -convex function if the inequality*

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y) \quad (1)$$

*is satisfied for all  $\lambda, \mu \geq 0$  such that  $\lambda^p + \mu^p = 1$  and for each  $x, y \in U$ . If the symbol " $\leq$ " is replaced with " $<$ " in the inequality (1) then  $f$  is called a strictly  $p$ -convex function.*

Throughout the paper, unless otherwise stated,  $U \subseteq \mathbb{R}^n$  is a  $p$ -convex set.  $\mathbb{R}_+^n$  and  $\mathbb{R}_{++}^n$  will denote nonnegative and positive orthants, respectively.

## 2 Main Results

In this section, the concept of quasi  $p$ -convex function is given, then characterization of quasi  $p$ -convex function, Jensen inequality for quasi  $p$ -convex function and the relations between quasi  $p$ -convex functions for different  $p$  values are obtained. Furthermore, preservation of quasi  $p$ -convexity on some functional operations such as composition and infimum are investigated.

**Definition 3** *Let  $p \in (0, 1]$ .*

(i) *A function  $f : U \rightarrow \mathbb{R}$  is called quasi  $p$ -convex function if*

$$f(\lambda x + \mu y) \leq \max\{f(x), f(y)\}$$

*for each  $x, y \in U$ ;  $\lambda, \mu \geq 0$  such that  $\lambda^p + \mu^p = 1$  or, equivalently*

$$f(y) \leq f(x) \Rightarrow f((1 - \lambda)^{\frac{1}{p}}x + \lambda^{\frac{1}{p}}y) \leq f(x)$$

*for every  $x, y \in U$  and for every  $\lambda \in [0, 1]$ .*

(ii) *A function  $f$  is called quasi  $p$ -concave function if  $-f$  is quasi  $p$ -convex, i.e., for each  $x, y \in U$ ;  $\lambda, \mu \geq 0$  such that  $\lambda^p + \mu^p = 1$ ,  $\min\{f(x), f(y)\} \leq f(\lambda x + \mu y)$ , or, equivalently*

$$f(y) \leq f(x) \Rightarrow f(y) \leq f((1 - \lambda)^{\frac{1}{p}}x + \lambda^{\frac{1}{p}}y) \quad (2)$$

*for every  $x, y \in U$  and for every  $\lambda \in [0, 1]$ .*

**Example 1** *Let  $n \in \mathbb{N}$ . If we define,  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that  $f(x) = x^n$ , then  $f$  is a quasi  $p$ -convex function. Because, using the fact that  $\lambda + \mu \leq 1$  from  $\lambda^p + \mu^p = 1$ , we can write*

$$\begin{aligned} f(\lambda x + \mu y) &= (\lambda x + \mu y)^n \leq (\lambda M + \mu M)^n \\ &= (\lambda + \mu)^n M^n \leq M^n = \max\{x^n, y^n\} = \max\{f(x), f(y)\} \end{aligned}$$

*where  $M = \max\{x, y\}$ . If we define,  $g : \mathbb{R}_+ \rightarrow \mathbb{R}$  such that  $g(x) = -x^n$  for  $n \in \mathbb{N}$ , then  $g$  is a quasi  $p$ -concave function.*

**Example 2** Let us define the function  $f$  such that

$$f : U \rightarrow \mathbb{R}, \quad f(x_1, x_2, \dots, x_n) = \sum_{i=1}^n |kx_i|,$$

for  $k \in \mathbb{R}$ . Then,  $f$  is a quasi  $p$ -convex function. Because, using the fact that  $\lambda + \mu \leq 1$ , we have

$$\begin{aligned} f(\lambda x + \mu y) &= f(\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, \dots, \lambda x_n + \mu y_n) \\ &= \sum_{i=1}^n |k(\lambda x_i + \mu y_i)| \leq \lambda \sum_{i=1}^n |kx_i| + \mu \sum_{i=1}^n |ky_i| \\ &\leq \begin{cases} (\lambda + \mu) \sum_{i=1}^n |kx_i|, & \text{if } \sum_{i=1}^n |ky_i| \leq \sum_{i=1}^n |kx_i| \\ (\lambda + \mu) \sum_{i=1}^n |ky_i|, & \text{if } \sum_{i=1}^n |kx_i| \leq \sum_{i=1}^n |ky_i| \end{cases} \\ &\leq \max\{f(x), f(y)\} \end{aligned}$$

for  $x = (x_1, x_2, \dots, x_n)$ ,  $y = (y_1, y_2, \dots, y_n) \in U$  and  $\lambda, \mu \geq 0$  such that  $\lambda^p + \mu^p = 1$ .

**Theorem 1** Let  $f : U \rightarrow \mathbb{R}$  be a quasi  $p$ -convex function. If  $t = \inf_{x \in U} f(x)$ , then the set

$$E = \{x \in U : f(x) = t\}$$

is a  $p$ -convex set.

**Proof.** Let  $x, y \in E$  and  $\lambda, \mu \geq 0$  such that  $\lambda^p + \mu^p = 1$ . Then, we can write

$$t \leq f(\lambda x + \mu y) \leq \max\{f(x), f(y)\} = t.$$

This shows that  $\lambda x + \mu y \in E$  and so  $E$  is a  $p$ -convex set. ■

**Theorem 2** The function  $f : U \rightarrow \mathbb{R}$  is a quasi  $p$ -convex function if and only if the lower level set  $L_\alpha$  defined by

$$L_\alpha = \{x \in U : f(x) \leq \alpha\}$$

is a  $p$ -convex set for all  $\alpha \in \mathbb{R}$ .

**Proof.** ( $\Rightarrow$ ): Suppose that  $f$  is a quasi  $p$ -convex function and let  $x, y \in L_\alpha$  for  $\alpha \in \mathbb{R}$ . From  $p$ -convexity of  $U$ , we have  $\lambda x + \mu y \in U$  for  $\lambda^p + \mu^p = 1$ . Using the quasi  $p$ -convexity of  $f$ , we have

$$f(\lambda x + \mu y) \leq \max\{f(x), f(y)\} \leq \alpha.$$

Hence,  $\lambda x + \mu y \in L_\alpha$  and therefore  $L_\alpha$  is a  $p$ -convex set.

( $\Leftarrow$ ): Suppose that  $L_\alpha$  is a  $p$ -convex set for each  $\alpha \in \mathbb{R}$ . Let  $x, y \in U$ . Then,  $x, y \in L_{\alpha^*}$  for  $\alpha^* = \max\{f(x), f(y)\}$ . By assumption,  $L_{\alpha^*}$  is  $p$ -convex set, so  $\lambda x + \mu y \in L_{\alpha^*}$ . Therefore,  $f(\lambda x + \mu y) \leq \alpha^* = \max\{f(x), f(y)\}$  for  $\lambda, \mu \geq 0$  such that  $\lambda^p + \mu^p = 1$ . Hence,  $f$  is a quasi  $p$ -convex function. ■

**Remark 1** The upper level set defined as  $L_\alpha^* = \{x \in U : f(x) \geq \alpha\}$  is a  $p$ -convex set if and only if  $f$  is a quasi  $p$ -concave function.

**Theorem 3** If the function  $f : U \rightarrow \mathbb{R}_+$  is a  $p$ -convex function, then  $f$  is a quasi  $p$ -convex function.

**Proof.** By the  $p$ -convexity of  $f$  and the inequality  $\lambda + \mu \leq 1$  for all  $\lambda, \mu \geq 0$  such that  $\lambda^p + \mu^p = 1$ , we have

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y) \leq (\lambda + \mu) \max\{f(x), f(y)\} \leq \max\{f(x), f(y)\}.$$

■

**Corollary 4** *If the function  $f : U \rightarrow \mathbb{R}_+$  is a strictly  $p$ -convex function, then  $f$  is a strictly quasi  $p$ -convex function.*

**Proof.** It is clear from Theorem 3. ■

**Example 3** *The functions  $h, \varphi$  defined by*

$$h(x) = \|x\|_q = \left( \sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} \quad \text{where } 1 \leq q$$

and

$$\varphi(x) = \|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$$

for  $x = (x_1, x_2, \dots, x_n) \in U$  are quasi  $p$ -convex functions from Example 3.5 in [18] and Theorem 3.

One of the basic properties of the convex function is the Jensen inequality. The following theorem gives the Jensen inequality for quasi  $p$ -convex functions:

**Theorem 5** *Let  $f : U \rightarrow \mathbb{R}_+$  be a quasi  $p$ -convex function. Let  $x_1, \dots, x_m \in U$  and  $\lambda_1, \dots, \lambda_m \geq 0$  with  $\lambda_1^p + \dots + \lambda_m^p = 1$ . Then*

$$f(\lambda_1 x_1 + \dots + \lambda_m x_m) \leq \max\{f(x_1), f(x_2), \dots, f(x_m)\}.$$

**Proof.** We use induction on  $m$ . The inequality is trivially true when  $m = 2$ . Assume that it is true when  $m = k$ , where  $k > 2$ . Now we show the validity when  $m = k + 1$ . Let a real number  $x$  be defined by the equation  $x = \lambda_1 x_1 + \dots + \lambda_{k+1} x_{k+1}$  where  $x_1, \dots, x_{k+1} \in U$  and  $\lambda_1, \dots, \lambda_{k+1} \geq 0$  with  $\lambda_1^p + \dots + \lambda_{k+1}^p = 1$ . At least one of  $\lambda_1, \dots, \lambda_{k+1}$  must be less than 1. Let us say  $\lambda_{k+1} < 1$  and write  $\lambda_1^p + \dots + \lambda_k^p = 1 - \lambda_{k+1}^p$ . One can find  $\lambda_* < 1$  such that  $\lambda_1^p + \dots + \lambda_k^p = \lambda_*^p$ . Since

$$\left( \frac{\lambda_1}{\lambda_*} \right)^p + \dots + \left( \frac{\lambda_k}{\lambda_*} \right)^p = 1$$

and the assumption of hypothesis, we get

$$f\left(\frac{\lambda_1}{\lambda_*} x_1 + \dots + \frac{\lambda_k}{\lambda_*} x_k\right) \leq \max\{f(x_1), f(x_2), \dots, f(x_k)\}.$$

By using quasi  $p$ -convexity of  $f$ ,

$$\begin{aligned} f(x) &= f\left(\lambda_* \left(\frac{\lambda_1}{\lambda_*} x_1 + \dots + \frac{\lambda_k}{\lambda_*} x_k\right) + \lambda_{k+1} x_{k+1}\right) \\ &\leq \max\left\{f\left(\frac{\lambda_1}{\lambda_*} x_1 + \dots + \frac{\lambda_k}{\lambda_*} x_k\right), f(x_{k+1})\right\} \\ &\leq \max\{f(x_1), f(x_2), \dots, f(x_k)\}, f(x_{k+1})\} \\ &= \max\{f(x_1), f(x_2), \dots, f(x_{k+1})\} \end{aligned}$$

is obtained. This completes the proof. ■

**Theorem 6** Let  $U \subseteq \mathbb{R}^n$  be a cone and  $p$ -convex set. Let  $f : U \rightarrow \mathbb{R}_+$  be a homogeneous function. If

$$f\left(\frac{x+y}{2^{\frac{1}{p}}}\right) \leq \frac{f(x)+f(y)}{2^{\frac{1}{p}}} \text{ for } x, y \in U,$$

then  $f$  is a quasi  $p$ -convex function.

**Proof.** Let the inequality hold for  $x, y \in U$ . Let  $\lambda, \mu > 0$  such that  $\lambda^p + \mu^p = 1$ .

$$f(\lambda x + \mu y) = f\left(2^{\frac{1}{p}} \frac{\lambda x + \mu y}{2^{\frac{1}{p}}}\right) \leq \frac{f(2^{\frac{1}{p}} \lambda x) + f(2^{\frac{1}{p}} \mu y)}{2^{\frac{1}{p}}} = \lambda f(x) + \mu f(y) \leq \max\{f(x), f(y)\}.$$

■

**Theorem 7** Let  $f : U \rightarrow \mathbb{R}$ . For any  $x, y \in U$ , the function  $\varphi : [0, 1] \rightarrow \mathbb{R}$  defined by  $\varphi(\lambda) = f(\lambda x + (1 - \lambda^p)^{\frac{1}{p}} y)$  is a quasi  $p$ -convex function, then  $f$  is also a quasi  $p$ -convex function.

**Proof.** Let  $x, y \in U$  and  $\lambda \in [0, 1]$ . Then, we get

$$f\left(\lambda x + (1 - \lambda^p)^{\frac{1}{p}} y\right) = \varphi(\lambda) = \varphi\left(\lambda \cdot 1 + (1 - \lambda^p)^{\frac{1}{p}} \cdot 0\right) \leq \max\{\varphi(1), \varphi(0)\} = \max\{f(x), f(y)\}.$$

■

**Theorem 8** Let  $0 < p < 1$ . A function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a quasi  $p$ -convex function if and only if  $f$  is an increasing function.

**Proof.** ( $\Leftarrow$ ) According to Theorem 2, it is enough to show that  $L_\alpha$  is  $p$ -convex for all  $\alpha \in \mathbb{R}$ . Consider  $L_\alpha$  for some  $\alpha \in \mathbb{R}$ . If  $L_\alpha = \emptyset$  or  $L_\alpha = \{0\}$  then it is clear that  $L_\alpha$  is  $p$ -convex set. Other case is that  $L_\alpha$  contains at least two elements  $x_1, x_2$  with  $x_1 < x_2$ . So  $f(x_1) \leq \alpha$  and  $f(x_2) \leq \alpha$ . For  $\lambda \in [0, 1]$ ,  $g(\lambda) = \lambda^{\frac{1}{p}} x_1 + (1 - \lambda)^{\frac{1}{p}} x_2$ . Then it is clear from the first derivative that  $g(\lambda)$  is unimodal function, i.e., there exists a  $\lambda_* \in [0, 1]$  such that  $g$  is decreasing on  $[0, \lambda_*]$  and  $g$  is increasing on  $[\lambda_*, 1]$ . So, it attains maximum value at either  $\lambda = 0$  or  $\lambda = 1$ . For  $\lambda \in [0, 1]$ ,  $\lambda^{\frac{1}{p}} x_1 + (1 - \lambda)^{\frac{1}{p}} x_2 \leq \max\{x_1, x_2\} = x_2$  and, hence,  $f(\lambda^{\frac{1}{p}} x_1 + (1 - \lambda)^{\frac{1}{p}} x_2) \leq f(x_2) \leq \alpha$ . So,  $\lambda^{\frac{1}{p}} x_1 + (1 - \lambda)^{\frac{1}{p}} x_2 \in L_\alpha$ . Since it can be done for all  $\alpha \in \mathbb{R}$ ,  $L_\alpha$  is  $p$ -convex set.

( $\Rightarrow$ ) Suppose  $f$  is not an increasing function. Then, there exists  $x_1, x_2 \in \mathbb{R}_+$  such that  $x_1 < x_2$  and  $f(x_1) > f(x_2)$ . Let us choose  $\alpha = \min\{f(x_1), f(x_2)\}$ . Since,  $L_\alpha$  does not include  $x_1$ ,  $L_\alpha$  is not  $p$ -convex set. Therefore,  $f$  is not quasi  $p$ -convex function from Theorem 2. ■

Since, the negative of a quasi  $p$ -concave function is a quasi  $p$ -convex function, we can obtain the following result:

**Corollary 9** Let  $0 < p < 1$ . A function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a quasi  $p$ -concave function if and only if  $f$  is a decreasing function.

**Theorem 10** Let  $U \subseteq \mathbb{R}_{++}^n$  be a  $p$ -convex set and  $f : U \rightarrow \mathbb{R}_{++}$  be homogeneous function. If  $f$  is quasi  $p$ -concave function then,  $f$  is a  $p$ -concave function.

**Proof.** From the homogeneity of  $f$ , we have

$$(1 - \lambda)^{\frac{1}{p}} f(x) + \lambda^{\frac{1}{p}} f(y) = f\left(\frac{(1 - \lambda)^{\frac{1}{p}} x + \lambda^{\frac{1}{p}} y}{1}\right)$$

for  $\lambda \in [0, 1]$ . Since  $f$  is positive valued function, there exists a positive number  $\theta$  such that  $f\left(\frac{(1 - \lambda)^{\frac{1}{p}} x}{1}\right) = \theta^{\frac{1}{p}} f(\lambda^{\frac{1}{p}} y)$ . Then, we get

$$(1 - \lambda)^{\frac{1}{p}} f(x) + \lambda^{\frac{1}{p}} f(y) = (1 + \theta^{\frac{1}{p}}) f(\lambda^{\frac{1}{p}} y) \tag{3}$$

and

$$\left(\frac{\theta}{1+\theta}\right)^{\frac{1}{p}} \left((1-\lambda)^{\frac{1}{p}}x + \lambda^{\frac{1}{p}}y\right) = \left(\frac{\theta}{1+\theta}\right)^{\frac{1}{p}} (1-\lambda)^{\frac{1}{p}}x + \left(\frac{1}{1+\theta}\right)^{\frac{1}{p}} \theta^{\frac{1}{p}} \lambda^{\frac{1}{p}}y. \quad (4)$$

Hence, it is seen that the right side of (4) is  $p$ -convex combination of  $(1-\lambda)^{\frac{1}{p}}x$  and  $\theta^{\frac{1}{p}}\lambda^{\frac{1}{p}}y$ . Thus, using (2), we have

$$f\left(\left(\frac{\theta}{1+\theta}\right)^{\frac{1}{p}} \left((1-\lambda)^{\frac{1}{p}}x + \lambda^{\frac{1}{p}}y\right)\right) \geq f(\theta^{\frac{1}{p}}\lambda^{\frac{1}{p}}y).$$

By using homogeneity and simplifying the expression,

$$f\left((1-\lambda)^{\frac{1}{p}}x + \lambda^{\frac{1}{p}}y\right) \geq (1+\theta)^{\frac{1}{p}}f(\lambda^{\frac{1}{p}}y).$$

From  $(1+\theta)^{\frac{1}{p}} \geq (1+\theta^{\frac{1}{p}})$  and (3),

$$f\left((1-\lambda)^{\frac{1}{p}}x + \lambda^{\frac{1}{p}}y\right) \geq (1-\lambda)^{\frac{1}{p}}f(x) + \lambda^{\frac{1}{p}}f(y)$$

is obtained. ■

Some properties for quasi  $p$ -convex functions are given below:

**Theorem 11** *If  $f_i : U \rightarrow \mathbb{R}$  are quasi  $p$ -convex functions for  $i = 1, 2, \dots, m$ , then  $f = \max_{1 \leq i \leq m} \{f_i\}$  is a quasi  $p$ -convex function.*

**Proof.** For fixed  $x, y \in U$  and  $\lambda, \mu \geq 0$  with  $\lambda^p + \mu^p = 1$ , there exists  $t \in \{1, 2, \dots, m\}$  such that  $f(\lambda x + \mu y) = \max_{1 \leq i \leq m} \{f_i(\lambda x + \mu y)\} = f_t(\lambda x + \mu y)$ . Using quasi  $p$ -convexity of  $f_t$ , we have

$$f_t(\lambda x + \mu y) \leq \max\{f_t(x), f_t(y)\} \leq \max\left\{\max_{1 \leq i \leq m} \{f_i(x)\}, \max_{1 \leq i \leq m} \{f_i(y)\}\right\} = \max\{f(x), f(y)\}.$$

Since this inequality can be made for each  $x, y \in U$ , the proof is complete. ■

**Theorem 12** *If the function  $f : U \rightarrow \mathbb{R}$  is a quasi  $p$ -convex function and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing function, then  $g \circ f : U \rightarrow \mathbb{R}$  is a quasi  $p$ -convex function.*

**Proof.** Let  $x, y \in U$  and  $\lambda, \mu \geq 0$  such that  $\lambda^p + \mu^p = 1$ . We can write

$$\begin{aligned} (g \circ f)(\lambda x + \mu y) &= g(f(\lambda x + \mu y)) \leq g(\max\{f(x), f(y)\}) \\ &= \max\{g(f(x)), g(f(y))\} = \max\{(g \circ f)(x), (g \circ f)(y)\}. \end{aligned}$$

Hence,  $g \circ f$  is a quasi  $p$ -convex function. ■

**Remark 2** *Using an increasing function and a quasi  $p$ -convex function, new quasi  $p$ -convex functions can be obtained from Theorem 12. For example, the function  $h : U \rightarrow \mathbb{R}$  defined by  $h(x) = e^{f(x)}$  is a quasi  $p$ -convex function where  $f$  is a quasi  $p$ -convex function.*

**Theorem 13** *If the function  $f : U \rightarrow \mathbb{R}$  is a quasi  $p$ -concave function and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing function, then  $g \circ f : U \rightarrow \mathbb{R}$  is a quasi  $p$ -concave function.*

**Proof.** Let  $x, y \in U$  and  $\lambda, \mu \geq 0$  such that  $\lambda^p + \mu^p = 1$ . Then, we have

$$\begin{aligned} (g \circ f)(\lambda x + \mu y) &= g(f(\lambda x + \mu y)) \geq g(\min\{f(x), f(y)\}) \\ &= \min\{g(f(x)), g(f(y))\} = \min\{(g \circ f)(x), (g \circ f)(y)\}. \end{aligned}$$

So,  $g \circ f$  is a quasi  $p$ -concave function. ■

**Remark 3** Using an increasing function and a quasi  $p$ -concave function, new quasi  $p$ -concave functions can be obtained from Theorem 13. The Gaussian distribution  $h : \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by  $h(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$  is a quasi  $p$ -concave function. This is clearly seen by using quasi  $p$ -concavity of  $f$  defined by  $f(x) = \frac{-x^2}{2}$  and increment of the function  $g$  defined by  $g(x) = \frac{e^x}{\sqrt{2\pi}}$ .

More generally, the function  $h : U \rightarrow \mathbb{R}$  defined by  $h(x) = e^{f(x)}$  is a quasi  $p$ -concave function where  $f$  is a quasi  $p$ -concave function.

**Theorem 14** If the function  $f : U \rightarrow \mathbb{R}$  is a  $p$ -convex function and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing quasi  $p$ -convex function, then  $g \circ f : U \rightarrow \mathbb{R}$  is a quasi  $p$ -convex function.

**Proof.** Let  $f$  be a  $p$ -convex function and  $g$  be an increasing quasi  $p$ -convex function. Then, for each  $x, y \in U$  and  $\lambda, \mu \geq 0$  such that  $\lambda^p + \mu^p = 1$ , we have

$$\begin{aligned} (g \circ f)(\lambda x + \mu y) &= g(f(\lambda x + \mu y)) \leq g(\lambda f(x) + \mu f(y)) \\ &\leq \max\{g(f(x)), g(f(y))\} = \max\{(g \circ f)(x), (g \circ f)(y)\}. \end{aligned}$$

Therefore,  $g \circ f$  is a quasi  $p$ -convex function. ■

**Remark 4** By using a  $p$ -convex function and an increasing quasi  $p$ -convex function, new quasi  $p$ -convex functions can be obtained from Theorem 14. For example, by considering the functions  $f, h$  and  $\varphi$  defined in Example 2 and Example 3, it is obtained quasi  $p$ -convex functions  $e^{f(x)}, e^{h(x)}, e^{\varphi(x)}$ .

**Lemma 15** ([18]) Let  $U \subseteq \mathbb{R}^n$  be a  $p$ -convex set and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear transformation. Then,  $V = g(U) \subseteq \mathbb{R}^m$  is a  $p$ -convex set.

**Theorem 16** Let  $g : U \rightarrow \mathbb{R}^m$  be a linear transformation. Assume that  $f : V \rightarrow \mathbb{R}$  is a quasi  $p$ -convex function where  $V = g(U)$ . Then,  $f \circ g : U \rightarrow \mathbb{R}$  is a quasi  $p$ -convex function.

**Proof.** Let  $f$  be a quasi  $p$ -convex function,  $g$  be a linear transformation and  $\lambda, \mu \geq 0$  such that  $\lambda^p + \mu^p = 1$ . From Lemma 15,  $V$  is a  $p$ -convex set. Thus, for  $x, y \in U$ , we get

$$\begin{aligned} (f \circ g)(\lambda x + \mu y) &= f(g(\lambda x + \mu y)) = f(\lambda g(x) + \mu g(y)) \\ &\leq \max\{f(g(x)), f(g(y))\} = \max\{(f \circ g)(x), (f \circ g)(y)\}. \end{aligned}$$

So,  $f \circ g$  is a quasi  $p$ -convex function. ■

It is important to know when the condition  $\lambda^p + \mu^p = 1$  in Definition 3 can be equivalently replaced by the condition  $\lambda^p + \mu^p \leq 1$ .

**Theorem 17** Let  $f : U \rightarrow \mathbb{R}$  be a quasi  $p$ -convex function. Then, the inequality (1) holds for all  $x, y \in U$ , and all  $\lambda, \mu \geq 0$  with  $\lambda^p + \mu^p \leq 1$  if and only if  $f(0) \leq f(x)$  for all  $x \in U$ .

**Proof.** Necessity is obvious by taking  $\lambda = \mu = 0$  and  $x = y$  in the inequality (1). Therefore, assume that  $x, y \in U; \lambda, \mu \geq 0$  and  $0 < \gamma = \lambda^p + \mu^p < 1$ . Put  $\alpha = \lambda\gamma^{-\frac{1}{p}}$  and  $\beta = \mu\gamma^{-\frac{1}{p}}$ . Then

$$\alpha^p + \beta^p = \frac{\lambda^p}{\gamma} + \frac{\mu^p}{\gamma} = 1$$

and hence we have sufficiency:

$$\begin{aligned} f(\lambda x + \mu y) &= f\left(\alpha\gamma^{\frac{1}{p}}x + \beta\gamma^{\frac{1}{p}}y\right) \leq \max\left\{f\left(\gamma^{\frac{1}{p}}x\right), f\left(\gamma^{\frac{1}{p}}y\right)\right\} \\ &= \max\left\{f\left(\gamma^{\frac{1}{p}}x + (1-\gamma)^{\frac{1}{p}}.0\right), f\left(\gamma^{\frac{1}{p}}y + (1-\gamma)^{\frac{1}{p}}.0\right)\right\} \\ &\leq \max\{\max\{f(x), f(0)\}, \max\{f(y), f(0)\}\} \leq \max\{f(x), f(y)\}. \end{aligned}$$

■

**Theorem 18** Let  $0 < p_1 \leq p_2 \leq 1$ . If  $f$  is a quasi  $p_2$ -convex function and  $f(0) \leq f(x)$  for all  $x \in U$ , then  $f$  is a quasi  $p_1$ -convex function.

**Proof.** Assume that  $f$  is a  $p_2$ -convex function,  $x, y \in U$  and  $\lambda, \mu \geq 0$  with  $\lambda^{p_1} + \mu^{p_1} = 1$ . Then  $\lambda^{p_2} + \mu^{p_2} \leq \lambda^{p_1} + \mu^{p_1} = 1$  and according to Theorem 17, we have  $f(\lambda x + \mu y) \leq \max\{f(x), f(y)\}$ . ■

**Theorem 19** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an increasing function and  $g : U \rightarrow \mathbb{R}$  be a quasi  $p_2$ -convex function. If  $0 < p_1 \leq p_2 \leq 1$  and  $g(0) \leq g(x)$  for all  $x \in U$ , then  $f \circ g$  is a quasi  $p_1$ -convex function.

**Proof.** Let  $x, y \in U$  and  $\lambda, \mu \geq 0$  with  $\lambda^{p_1} + \mu^{p_1} = 1$ . Since  $\lambda^{p_2} + \mu^{p_2} \leq \lambda^{p_1} + \mu^{p_1} = 1$ , therefore, according to Theorem 17 and the assumptions, we have

$$\begin{aligned} (f \circ g)(\lambda x + \mu y) &= f(g(\lambda x + \mu y)) \leq f(\max\{g(x), g(y)\}) \\ &\leq \max\{f(g(x)), f(g(y))\} = \max\{(f \circ g)(x), (f \circ g)(y)\}, \end{aligned}$$

which means that  $f \circ g$  is a quasi  $p_1$ -convex function. ■

**Theorem 20** Let  $0 < p_1 \leq p_2 \leq 1$  and  $U \subseteq \mathbb{R}$ . If  $f : U \rightarrow \mathbb{R}_+$  is an increasing quasi  $p_1$ -convex function and  $g : U \rightarrow \mathbb{R}_+$  is an increasing quasi  $p_2$ -convex function such that  $g(0) \leq g(x)$  for all  $x \in U$ , then  $fg$  is a quasi  $p_1$ -convex function.

**Proof.** Let  $\lambda, \mu \geq 0$  with  $\lambda^{p_2} + \mu^{p_2} \leq \lambda^{p_1} + \mu^{p_1} = 1$ . Using by Theorem 17 and since the functions  $f$  and  $g$  are increasing on  $U$ , we have

$$\begin{aligned} (fg)(\lambda x + \mu y) &= f(\lambda x + \mu y)g(\lambda x + \mu y) \\ &\leq \max\{f(x), f(y)\} \cdot \max\{g(x), g(y)\} \\ &\leq \max\{f(x)g(x), f(y)g(y)\} \\ &\leq \max\{fg(x), fg(y)\}, \end{aligned}$$

which means that  $fg$  is a quasi  $p_1$ -convex function. ■

**Theorem 21** Let  $0 < p_1 \leq p_2 \leq 1$ ,  $f : U \rightarrow \mathbb{R}_+$  be a quasi  $p_1$ -convex function and  $g : U \rightarrow \mathbb{R}_+$  be a quasi  $p_2$ -convex function. If  $g(0) = 0$  and

$$\max\{f(x)g(y), f(y)g(x)\} \leq \max\{f(x)g(x), f(y)g(y)\} \quad (5)$$

for all  $x, y \in U$ , then  $fg$  is a quasi  $p_1$ -convex function.

**Proof.** Let  $\lambda, \mu \geq 0$  with  $\lambda^{p_2} + \mu^{p_2} \leq \lambda^{p_1} + \mu^{p_1} = 1$ . Using by Theorem 17 and the inequality (5), we can write

$$\begin{aligned} (fg)(\lambda x + \mu y) &= f(\lambda x + \mu y)g(\lambda x + \mu y) \leq \max\{f(x), f(y)\} \cdot \max\{g(x), g(y)\} \\ &\leq \max\{f(x)g(x), f(y)g(y)\} \leq \max\{fg(x), fg(y)\}. \end{aligned}$$

This shows that  $fg$  is a quasi  $p_1$ -convex function. ■

**Theorem 22** Let  $f : U \rightarrow \mathbb{R}_+$  and  $g : U \rightarrow \mathbb{R}_+$  are quasi  $p$ -concave functions. If

$$\min\{f(x)g(x), f(y)g(y)\} \leq \min\{f(x)g(y), f(y)g(x)\} \quad (6)$$

for all  $x, y \in U$ , then  $fg$  is a quasi  $p$ -concave function.



**Proof.** Let  $\lambda, \mu \geq 0$  with  $\lambda^p + \mu^p = 1$ . Using by the inequality (6), we can write

$$\begin{aligned} (fg)(\lambda x + \mu y) &= f(\lambda x + \mu y)g(\lambda x + \mu y) \geq \min\{f(x), f(y)\} \cdot \min\{g(x), g(y)\} \\ &\geq \min\{f(x)g(x), f(y)g(y)\} \geq \min\{fg(x), fg(y)\}. \end{aligned}$$

This shows that  $fg$  is a quasi  $p$ -concave function. ■

In case  $U = \mathbb{R}_+$  in theorem above, since  $f$  and  $g$  are decreasing functions as per Corollary 9, the inequality condition in (6) is satisfied naturally. So, we can state the following corollary.

**Corollary 23** *Let  $0 < p < 1$  and  $f, g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  are quasi  $p$ -concave functions, then  $fg$  is a quasi  $p$ -concave function.*

**Theorem 24** *Let  $U \subseteq \mathbb{R}^n$  be a  $p$ -convex set. If  $f : U \rightarrow \mathbb{R}_+$ ,  $g : U \rightarrow \mathbb{R}_{++}$  are  $p$ -convex and  $p$ -concave functions, respectively, then  $\frac{f}{g}$  is a quasi  $p$ -convex function.*

**Proof.** Let us call  $h(x) = \frac{f(x)}{g(x)}$  for  $x \in U$ . And show that for all  $x, y \in U$  such that  $h(x) \leq h(y)$ , the inequality  $h((1 - \lambda)^{\frac{1}{p}}x + \lambda^{\frac{1}{p}}y) \leq h(y)$  holds true for all  $\lambda \in [0, 1]$ . Since  $h(x) \leq h(y)$ ,  $f(x) \leq \frac{f(y)}{g(y)}g(x)$ . Considering the convexity of  $f$  and the concavity of  $g$ , together with their sign, we have

$$\begin{aligned} f\left((1 - \lambda)^{\frac{1}{p}}y + \lambda^{\frac{1}{p}}x\right) &\leq (1 - \lambda)^{\frac{1}{p}}f(y) + \lambda^{\frac{1}{p}}f(x) \leq (1 - \lambda)^{\frac{1}{p}}f(y) + \lambda^{\frac{1}{p}}\frac{f(y)}{g(y)}g(x) \\ &= \frac{f(y)}{g(y)}\left((1 - \lambda)^{\frac{1}{p}}g(y) + \lambda^{\frac{1}{p}}g(x)\right) \leq \frac{f(y)}{g(y)}g\left((1 - \lambda)^{\frac{1}{p}}x + \lambda^{\frac{1}{p}}y\right). \end{aligned}$$

Thus, we obtain

$$h\left((1 - \lambda)^{\frac{1}{p}}x + \lambda^{\frac{1}{p}}y\right) = \frac{f\left((1 - \lambda)^{\frac{1}{p}}y + \lambda^{\frac{1}{p}}x\right)}{g\left((1 - \lambda)^{\frac{1}{p}}x + \lambda^{\frac{1}{p}}y\right)} \leq \frac{f(y)}{g(y)} = h(y).$$

■

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