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# Characterization Of Quasi Lindley Distribution By Truncated Moments And Conditional Expectation Of Order Statistics*

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**Abstract**

In this paper, three characterization results of Quasi Lindley distribution are obtained. The first two characterization results are based on relation between truncated moments and failure rate functions. The third characterization result is based on conditional expectation of adjacent order statistics. Further, some of its important deductions are also discussed.

## 1 Introduction

Characterization of a probability distribution plays an important role in probability and statistics. Before a particular probability distribution model is applied to fit the real world data, it is necessary to confirm whether the given probability distribution satisfies the underlying requirements by its characterization. A probability distribution can be characterized through various methods. There has been a great interest, in recent years, in the characterizations of probability distributions by truncated moments (see, for example Hamedani (2011), Hamedani and Ghosh (2015), Ahsnullah and Shakil (2015), Ahsnullah *et al.* (2016), Shakil *et al.* (2016) amongst others).

Lindley distribution was introduced by Lindley (1958). A random variable $X$ is said to have Lindley distribution with parameter $\theta$ if its probability density function (*pdf*) is of the form

$$f(x) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x};\ x > 0,\ \theta > 0. \tag{1}$$

Its distribution function (*df*) is

$$F(x) = 1 - \frac{\theta + 1 + \theta x}{1 + \theta x}e^{-\theta x};\ x > 0,\ \theta > 0. \tag{2}$$

Ghitany *et al.* (2008) have discussed various properties of this distribution and showed that in many ways (1) provides a better model for some applications than the exponential distribution. Shankar and Mishra (2013) introduced a two-parameter distribution called quasi Lindley distribution (QLD) with parameters $\alpha$ and $\theta$. Its *pdf* is given by

$$f(x;\alpha,\theta) = \frac{\theta(\alpha + \theta x)}{\alpha + 1}e^{-\theta x};\ x > 0,\ \theta > 0,\ \alpha > -1 \tag{3}$$

and the *df* is

$$F(x;\alpha,\theta) = 1 - \frac{1 + \alpha + \theta x}{\alpha + 1}e^{-\theta x};\ x > 0,\ \theta > 0,\ \alpha > -1. \tag{4}$$

It can easily be seen that at $\alpha = \theta$, the QLD reduces to the Lindley distribution and at $\alpha = 0$, it reduces to the gamma distribution with parameters $(2, \theta)$. Shanker and Mishra (2013) have discussed its various

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properties and showed that this QLD is a better model than the Lindley distribution for modeling waiting and survival times data. The failure rate function $(frf)$ of QLD is given by

$$r(x) = \frac{f(x)}{1-F(x)} = \frac{\theta(\alpha+\theta x)}{1+\alpha+\theta x}. \tag{5}$$

The reverse failure rate function $(rfrf)$ of QLD is given by

$$\eta(x) = \frac{f(x)}{F(x)} = \frac{\theta(\alpha+\theta x)}{(1+\alpha)e^{\theta x}-(1+\alpha+\theta x)}. \tag{6}$$

The $k$-th moment (about the origin) of QLD is given by [Shankar and Mishra (2013)]

$$E[X^k] = \frac{k!(k+1+\alpha)}{(\alpha+1)\theta^k}. \tag{7}$$

Ahsanullah *et al.* (2017) have given the characterization results for Lindley distribution. They characterized Lindley distribution through left and right truncated moments. Kilany (2017) characterized Lindley distribution by conditional expectation of order statistics. In this paper, we have obtained characterization results for quasi Lindley distribution.

## 2 Characterizations Through Truncated Moments

First, we give the following two lemmas which are used to prove Theorems 1 and 2 respectively.

**Lemma 1 (Ahsanullah *et al.* (2017))** *Suppose that the random variable $X$ has an absolutely continuous df $F(x)$ with $F(0)=0$, $F(x)>0$ for all $x$, pdf $f(x)=F'(x)$, $frf$ $r(x)=\frac{f(x)}{1-F(x)}$. Let $g(x)$ be a continuous function in $x>0$ and $0<E[g(X)]<\infty$. If*

$$E[g(x)|X>x] = h(x)r(x) \qquad x>0$$

*where $h(x)$ is a differentiable function in $x>0$, then*

$$f(x) = K\exp\left(-\int_0^x \frac{g(y)+h'(x)}{h(y)}dy\right), \qquad x>0$$

*where $K>0$ is a normalizing constant.*

**Lemma 2 (Ahsanullah *et al.* (2017))** *Suppose that the random variable $X$ has an absolutely continuous df $F(x)$ with $F(0)=0$, $F(x)>0$ for all $x$, pdf $f(x)=F'(x)$, $rfrf$ $\eta(x)=\frac{f(x)}{F(x)}$. Let $g(x)$ be a continuous function in $x>0$ and $0<E[g(X)]<\infty$. If*

$$E[g(x)|X>x] = w(x)\eta(x) \qquad x>0$$

*where $w(x)$ is a differentiable function in $x>0$, then*

$$f(x) = K\exp\left(-\int_0^x \frac{w'(x)-g(y)}{w(y)}dy\right), \qquad x>0$$

*where $K>0$ is a normalizing constant.*

**Theorem 1** *Suppose that the random variable $X$ has absolutely continuous distribution with the pdf $f(x)$ and the df $F(x)$ with $F(0)=0$, $F(x)>0$ for all $x$ and the $frf$, $r(x)=\frac{f(x)}{1-F(x)}$. Assume that $0<E[X^k]<\infty$, then for a given positive integer $k$, $X$ has pdf given in (3) if and only if*

$$E[X^k|X\geq x] = \frac{r(x)}{(\alpha+\theta x)}\sum_{j=0}^{k+1}\mu_j x^j \tag{8}$$

*where $\mu_0=\frac{k!(k+1+\alpha)}{\theta^{k+1}}$, $\mu_j=\frac{k!(k+1+\alpha)}{j!\theta^{k-j+1}}$, $\mu_{j+1}=\frac{\theta}{j+1}\mu_j$, $j=0,1,2,...k-1$ and $\mu_{k+1}=1$.*

**Proof.** First we will prove (3) implies (8). Suppose $X$ has *pdf* given in (3), we have

$$E[X^k|X \geq x] = \frac{1}{1-F(x)} \int_x^\infty y^k f(y)dy. \tag{9}$$

From relation among and *pdf*, *df* and $frf$, we have

$$E[X^k|X \geq x] = \frac{r(x)}{f(x)} \int_x^\infty y^k f(y)dy. \tag{10}$$

Using (3), we have

$$E[X^k|X \geq x] = \frac{r(x)}{(\alpha+\theta x)e^{-\theta x}} \left[\int_x^\infty (\alpha+\theta y)y^k e^{-\theta y}dy\right]$$

or

$$E[X^k|X \geq x] = \frac{r(x)}{(\alpha+\theta x)e^{-\theta x}} \left[\alpha \int_x^\infty y^k e^{-\theta y}dy + \theta \int_x^\infty y^{k+1} e^{-\theta y}dy\right]. \tag{11}$$

Gradshteyn and Ryzhik (2007) (pg-340) have given the following result

$$\int_u^\infty x^n e^{-\mu x}dx = e^{-\mu u} \sum_{k=0}^{n} \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}}. \tag{12}$$

Using (12) in (11), we have

$$E[X^k|X \geq x] = \frac{r(x)}{(\alpha+\theta x)e^{-\theta x}} \left[\alpha e^{-\theta x} \sum_{j=0}^{k} \frac{k!}{j!} \frac{x^j}{\theta^{k-j+1}} + \theta e^{-\theta x} \sum_{j=0}^{k+1} \frac{(k+1)!}{j!} \frac{x^j}{\theta^{k+1-j+1}}\right]$$

which reduces to

$$E[X^k|X \geq x] = \frac{r(x)}{(\alpha+\theta x)} \left[\alpha \sum_{j=0}^{k} \frac{k!(k+1+\alpha)}{j!} \frac{x^j}{\theta^{k-j+1}} + x^{k+1}\right]. \tag{13}$$

And after simplification, we have

$$E[X^k|X \geq x] = \frac{r(x)}{(\alpha+\theta x)} \sum_{j=0}^{k+1} \mu_j x^j$$

where $\mu_0 = \frac{k!(k+1+\alpha)}{\theta^{k+1}}$, $\mu_j = \frac{k!(k+1+\alpha)}{j!\theta^{k-j+1}}$, $\mu_{j+1} = \frac{\theta}{j+1}\mu_j$, $j = 0, 1, 2, ...k-1$ and $\mu_{k+1} = 1$ and hence the necessary part. For sufficiency part let

$$g(x) = x^k \quad and \quad h(x) = \frac{1}{(\alpha+\theta x)} \sum_{j=0}^{k+1} \mu_j x^j. \tag{14}$$

Using the recurrence relations of the $\mu_j$'s, we have

$$\theta \sum_{j=0}^{k+1} \mu_j x^j - \sum_{j=}^{k+1} j\mu_j x^{j-1} = \sum_{j=0}^{k11} [\theta\mu_j - (j+1)\mu_{j+1}]x^j + [\theta\mu_k - (k+1)\mu_{k+1}]x^k + \theta\mu_{k+1}x^{k+1} = (\alpha+\theta x)x^k. \tag{15}$$

From equations (14) and (15), we have

$$\frac{g(x)}{h(x)} = \frac{(\alpha+\theta x)x^k}{\sum_{j=0}^{k+1} \mu_j x^j} = \frac{\theta \sum_{j=0}^{k+1} \mu_j x^j - \sum_{j=}^{k+1} j\mu_j x^{j-1}}{\sum_{j=0}^{k+1} \mu_j x^j} = \theta - \frac{\sum_{j=}^{k+1} j\mu_j x^{j-1}}{\sum_{j=0}^{k+1} \mu_j x^j}. \tag{16}$$

Also from equation (14), we have

$$\ln[h(x)] = -\ln[\alpha+\theta x] + \ln[\sum_{j=0}^{k+1}\mu_j x^j]. \tag{17}$$

Differentiating (17) with respect to $x$, we have

$$\frac{h'(x)}{h(x)} = \frac{\sum_{j=1}^{k+1} j\mu_j x^{j-1}}{\sum_{j=0}^{k+1}\mu_j x^j} - \frac{\theta}{(\alpha+\theta x)}. \tag{18}$$

Using (16) and (18), we have

$$\frac{h'(x)+g(x)}{h(x)} = -\frac{\theta}{(\alpha+\theta x)} + \theta. \tag{19}$$

Integrating equation (19) over $(0, x)$, we have

$$\int_0^x \frac{h'(y)+g(y)}{h(y)}dy = \int_0^x \left(-\frac{\theta}{(\alpha+\theta y)} + \theta\right)dy = \ln[\alpha+\theta x] + \ln\alpha + \theta x. \tag{20}$$

From Lemma 1 and (20), we have

$$f(x) = C\exp\left(\int_0^x \frac{h'(y)+g(y)}{h(y)}dy\right) = C\exp[-\{-\ln[\alpha+\theta x] + \ln\alpha + \theta x\}] = \frac{C}{\alpha}(\alpha+\theta x)e^{-\theta x} \tag{21}$$

where C is normalizing constant such that $\int_0^\infty f(x)dx = 1$ which gives $C = \frac{\alpha\theta}{(\alpha+1)}$. Therefore, from (21), we get

$$f(x) = \frac{\theta(\alpha+\theta x)}{\alpha+1}e^{-\theta x}$$

which is the *pdf* of QLD given in (3) and hence sufficiency part. ■

**Remark 1** *Putting $\alpha = \theta$ in Theorem 1, we get the characterization result for Lindley distribution as obtained by Ahsanullah et al. (2017).*

**Remark 2** *Putting $\alpha = 0$ in Theorem 1, we get the characterization result for gamma $(2, \theta)$ distribution.*

**Theorem 2** *Suppose that the random variable $X$ has absolutely continuous distribution with the pdf $f(x)$ and the df $F(x)$ with $F(0) = 0$, $F(x) > 0$ for all $x > 0$ and the reversed failure rate function $(rfrf)$,$\eta(x) = \frac{f(x)}{F(x)}$. Assume that $0 < E[X^k] < \infty$, then for a given positive integer $k$, $X$ has pdf given in (3) if and only if*

$$E[X^k|X \leq x] = \frac{\eta(x)}{(\alpha+\theta x)}\left(\mu_0 e^{\theta x} - \sum_{j=0}^{k+1}\mu_j x^j\right) \tag{22}$$

*where $\mu_0 = \frac{k!(k+1+\alpha)}{\theta^{k+1}}$, $\mu_j = \frac{k!(k+1+\alpha)}{j!\theta^{k-j+1}}$, $\mu_{j+1} = \frac{\theta}{j+1}\mu_j$, $j = 0, 1, 2, ...k-1$ and $\mu_{k+1} = 1$.*

**Proof.** First we will prove (3) implies (22). Suppose $X$ has *pdf* given in (3), we have

$$E[X^k|X \leq x] = \frac{1}{F(x)}\int_0^x y^k f(y)dy. \tag{23}$$

From relation among *pdf*, *df* and $rfrf$, we have

$$E[X^k|X \leq x] = \frac{\eta(x)}{f(x)}\int_0^x y^k f(y)dy. \tag{24}$$

Using (3) in (24), we have From relation among $pdf$, $df$ and $rfrf$, we have

$$E[X^k|X \leq x] = \frac{\eta(x)}{(\alpha+\theta x)e^{-\theta x}} \int_0^x (\alpha+\theta y) y^k e^{-\theta y} dy$$

or

$$E[X^k|X \leq x] = \frac{\eta(x)}{(\alpha+\theta x)e^{-\theta x}} \left[\alpha \int_0^x y^k e^{-\theta y} dy + \theta \int_0^x y^{k+1} e^{-\theta y} dy\right]. \tag{25}$$

Gradshteyn and Ryzhik (2007) (pg-340) have given the following result

$$\int_0^u x^n e^{-\mu x} dx = \frac{n!}{\mu^{n+1}} - e^{-\mu u} \sum_{k=0}^{n} \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}}. \tag{26}$$

From (25) and (26), we have

$$E[X^k|X \leq x] = \frac{\eta(x)}{(\alpha+\theta x)} \left[\frac{k!(k+1+\alpha)}{\theta^{k+1}} e^{\theta x} - \sum_{j=0}^{k} \frac{k!(k+1+\alpha)}{j!} \frac{x^j}{\theta^{k-j+1}} - x^{k+1}\right]$$

which reduces to

$$E[X^k|X \leq x] = \frac{\eta(x)}{(\alpha+\theta x)} \left(\mu_0 e^{\theta x} - \sum_{j=0}^{k+1} \mu_j x^j\right)$$

where $\mu_0 = \frac{k!(k+1+\alpha)}{\theta^{k+1}}$, $\mu_j = \frac{k!(k+1+\alpha)}{j!\theta^{k-j+1}}$, $\mu_{j+1} = \frac{\theta}{j+1}\mu_j$, $j = 0, 1, 2, ...k-1$ and $\mu_{k+1} = 1$ and hence if part. For sufficient part Let

$$g(x) = x^k \quad \text{and} \quad w(x) = \frac{1}{(\alpha+\theta x)} \left[\mu_0 e^{\theta x} - \sum_{j=0}^{k+1} \mu_j x^j\right]. \tag{27}$$

From (27) and (15), we have

$$\frac{g(x)}{w(x)} = \frac{(\alpha+\theta x)x^k}{\mu_0 e^{\theta x} - \sum_{j=0}^{k+1} \mu_j x^j} = \frac{\theta \sum_{j=0}^{k+1} \mu_j x^j - \sum_{j=1}^{k+1} j\mu_j x^{j-1}}{\mu_0 e^{\theta x} - \sum_{j=0}^{k+1} \mu_j x^j} = \frac{\theta \mu_0 e^{\theta x} - \sum_{j=1}^{k+1} j\mu_j x^{j-1}}{\mu_0 e^{\theta x} - \sum_{j=0}^{k+1} \mu_j x^j} - \theta. \tag{28}$$

Also from equation (27), we have

$$\ln[w(x)] = -\ln(\alpha+\theta x) + \ln\left(\mu_0 e^{\theta x} - \sum_{j=0}^{k+1} \mu_j x^j\right). \tag{29}$$

Differentiating equation (29) $w.r.t.$ $x$, we have

$$\frac{w'(x)}{w(x)} = -\frac{\theta}{(\alpha+\theta x)} + \frac{\theta \mu_0 e^{\theta x} - \sum_{j=1}^{k+1} j\mu_j x^{j-1}}{\mu_0 e^{\theta x} - \sum_{j=0}^{k+1} \mu_j x^j} = -\frac{\theta}{(\alpha+\theta x)} + \frac{g(x)}{w(x)} + \theta. \tag{30}$$

From equations (28) and (30), we have

$$\frac{w'(x) - g(x)}{w(x)} = -\frac{\theta}{(\alpha+\theta x)} + \theta. \tag{31}$$

Integrating equation (31) over $(0, x)$, we have

$$\int_0^x \frac{w'(y) - g(y)}{w(y)} dy = \int_0^x \left(-\frac{\theta}{(\alpha+\theta y)} + \theta\right) dy = -\ln[(\alpha+\theta x)] + \ln\alpha + \theta x. \tag{32}$$

From Lemma 2 and (32), we have

$$f(x) = C\exp\left(-\int_0^x \frac{w'(y)-g(y)}{w(y)}dy\right) = C\exp\left(\ln\left[(\alpha+\theta x)\right] - \ln\alpha - \theta x\right) = \frac{C}{\alpha}(\alpha+\theta x)e^{-\theta x} \tag{33}$$

where $C$ is normalizing constant such that $\int_0^\infty f(x)dx = 1$ which gives $C = \frac{\alpha\theta}{(\alpha+1)}$. Therefore, from (33), we get

$$f(x) = \frac{\theta(\alpha+\theta x)}{(\alpha+1)}e^{-\theta x}$$

which is the *pdf* of QLD given in (3) and hence the sufficiency part. ■

**Remark 3** *Putting $\alpha = \theta$ in Theorem 2, we get the characterization result for Lindley distribution as obtained by Ahsanullah et al. (2017).*

**Remark 4** *Putting $\alpha = 0$ in Theorem 2, we get the characterization result for gamma $(2, \theta)$ distribution.*

## 3 Characterization Through Order Statistics

Let $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ be the order statistics from an absolutely continuous population with *df* $F(x)$ and *pdf* $f(x)$. Then the conditional *pdf* of $X_{r+1:n}$ given $X_{r:n}$ is equal to [David and Nagaraja (2003)]

$$(n-r)\left[\frac{1-F(y)}{1-F(x)}\right]^{n-r-1}\frac{f(y)}{1-F(x)},\ x \le y. \tag{34}$$

**Theorem 3** *Let $X$ be a continuous random variable with df $F(x)$ and pdf $f(x)$. Then for $r < n$*

$$E[X_{r+1:n}|X_{r:n} = x] = x + \frac{\Gamma[n-r+1,(n-r)(1+\alpha+\theta x)]}{\theta(n-r)^{n-r+1}(1+\alpha+\theta x)^{n-r}e^{-(n-r)(1+\alpha+\theta x)}},\ r = 1, 2, .., n-1 \tag{35}$$

*if and only if $X$ has the df given in (4).*

**Proof.** First we will prove (4) implies (35). We have from equation (34),

$$E[X_{r+1:n}|X_{r:n} = x] = \frac{(n-r)}{[1-F(x)]^{n-r}}\int_x^\beta y[1-F(y)]^{n-r-1}f(y)dy. \tag{36}$$

From (3) and (4), we have

$$E[X_{r+1:n}|X_{r:n} = x] = \frac{(n-r)\theta}{[(1+\alpha+\theta x)e^{-\theta x}]}\int_x^\infty y[(1+\alpha+\theta y)]^{n-r}(\alpha+\theta y)e^{-\theta y}dy. \tag{37}$$

Integrating equation (37) by parts and then rearranging, we have

$$E[X_{r+1:n}|X_{r:n} = x] = x + \frac{\int_x^\infty (1+\alpha+\theta y)e^{-\theta(n-r)y}dy}{[(1+\alpha+\theta x)e^{-\theta x}]^{n-r}}$$

which reduces to

$$E[X_{r+1:n}|X_{r:n} = x] = x + \frac{\Gamma[n-r+1,(n-r)(1+\alpha+\theta x)]}{\theta(n-r)^{n-r+1}(1+\alpha+\theta x)^{n-r}e^{-(n-r)(1+\alpha+\theta x)}} \tag{38}$$

and hence necessary part. For sufficiency part, Let

$$E[X_{r+1:n}|X_{r:n} = x] = \phi(x) = 1 + \omega(x) \tag{39}$$

where

$$\omega(x) = \frac{\Gamma[n-r+1,(n-r)(1+\alpha+\theta x)]}{\theta(n-r)^{n-r+1}(1+\alpha+\theta x)^{n-r}e^{-(n-r)(1+\alpha+\theta x)}}.$$

From (39), we have

$$\phi'(x) = 1 + \omega'(x).$$

Then, it is seen that

$$\int \frac{\phi'(t)}{\phi(t)-t}dt = \int \frac{1+\omega'(t)}{\omega(t)}dt = \int \frac{dt}{\omega(dt)} + \int \frac{\omega'(t)}{\omega(t)}dt.$$

From equation (39), we have

$$\int \frac{\phi'(t)}{\phi(t)-t}dt = \int \frac{\Gamma[n-r+1,(n-r)(1+\alpha+\theta t)]}{\theta(n-r)^{n-r+1}(1+\alpha+\theta t)^{n-r}e^{-(n-r)(1+\alpha+\theta t)}}dt + \ln[\omega(t)]. \tag{40}$$

Abramowitz and Stegun (1972) (pg-262) have given the result

$$\gamma(a,x) = \frac{\partial\Gamma(a,x)}{\partial x} = x^{a-1}e^{-x}. \tag{41}$$

From (40) and (41), we have

$$\int \frac{\phi'(t)}{\phi(t)-t}dt = -\ln\Gamma[n-r+1,(n-r)(1+\alpha+\theta t)] + \ln\Big[\frac{\Gamma[n-r+1,(n-r)(1+\alpha+\theta t)]}{\theta(n-r)^{n-r+1}(1+\alpha+\theta t)^{n-r}e^{-(n-r)(1+\alpha+\theta t)}}\Big]$$

or

$$\int \frac{\phi'(t)}{\phi(t)-t}dt = -\ln\big[\theta(n-r)^{n-r+1}(1+\alpha+\theta t)^{n-r}e^{-(n-r)(1+\alpha+\theta t)}\big]. \tag{42}$$

Taking the limits of integral in (42) from 0 to $x$, we have

$$\int_0^x \frac{\phi'(t)}{\phi(t)-t}dt = \ln\Big[\Big\{\frac{1+\alpha}{(1+\alpha+\theta x)e^{-\theta x}}\Big\}^{n-r}\Big]. \tag{43}$$

Using the result given by Khan and Abu-Salih (1989), (43) reduced to

$$[1-F(x)]^{n-r} = \exp\Bigg[-\ln\Big[\Big\{\frac{1+\alpha}{(1+\alpha+\theta x)e^{-\theta x}}\Big\}^{n-r}\Big]\Bigg] = \Big[\frac{1+\alpha+\theta x}{1+\alpha}e^{-\theta x}\Big]^{n-r}$$

which reduces to

$$F(x) = 1 - \frac{(1+\alpha+\theta x)}{1+\alpha}e^{-\theta x}$$

and hence the Theorem. ■

**Remark 5** *Putting $\alpha = \theta$ in Theorem 3, we get the characterization result for Lindley distribution as obtained by Kilany (2017).*

**Remark 6** *Putting $\alpha = 0$ in Theorem 3, we get the characterization result for gamma $(2,\theta)$ distribution.*

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